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Solution of the Vector Eigenmode Problem for Cylindrical Dielectric Waveguides Based on a Nonlocal Boundary Condition

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Abstract—The original problem in an unbounded domain is reduced to a problem in a disk convenient for numerical solution. The problem thus obtained is a parametric eigenvalue problem with a nonlocal boundary condition nonlinear in the spectral parameter. The analysis of the problem is based on the spectral theory of compact self-adjoint operators. The existence of a spectrum of the problem is proven, and the properties of the dispersion curves are studied.

INTRODUCTION

The study of spectral problems in the dielectric waveguide theory and the development of numerical methods for them have attracted considerable attention (see, for example, [1–3] and the references cited therein). The usual approach to an analysis of cylindrical waveguides is based on finding their eigenmodes, i.e., electromagnetic waves of the form $\Phi(\mathbf{x})\exp[i(\omega t-\beta x_3)]$, where $\mathbf{x}=(x_1, x_2)$ is the transverse coordinate vector, x_3 is the longitudinal coordinate, β is the longitudinal propagation constant, ω is the oscillation frequency, and t is time. Surface waves are of particular interest. Their amplitude $\Phi(x)$ decays exponentially as $|x| \longrightarrow \infty$, and the propagation constant β is a real number. The prime objective of a theoretical study is the analysis of dispersion curves. They describe the relation between the parameters β and ω for which eigenmodes exist.

The exact solution to the eigenmode problem for a uniform circular waveguide is well known (see, for example, [2], where a transcendental equation relating β and ω is obtained). Waveguides used in practice display a wide diversity (uniform fibers of arbitrary geometry, fibers with a variable permittivity over the cross section, waveguides consisting of two and more parallel fibers, etc.), and no exact solutions are available for them. There are approximate solutions for a number of particular waveguide structures (see, e.g., [2]). Different numerical methods have also been developed (see, e.g., [3]). An algorithm for computing uniform waveguides of arbitrary cross section was suggested and analyzed in [4–6]. In that algorithm, the method of contour integral equations is used to reduce the original problem to an equivalent nonlinear spectral problem for a Fredholm holomorphic operator-valued function.

Problems in unbounded domains frequently arise in various applications, such as acoustics, electromagnetic theory, aerodynamics, geophysics, underground hydromechanics, meteorology, etc. There are various approaches to the solution of such problems. One of the most widely used approaches is based on the introduction of an auxiliary boundary Γ that breaks the original domain into two portions: a bounded computational domain Ω and an unbounded domain Ω_∞. Boundary conditions are set on the boundary Γ (in particular, they provide the well-posedness of the problem in Ω). The problem in the bounded domain Ω is solved numerically. Subsequently, if needed, the solution in Ω_∞ is also determined. The second step is the most important in the approach: for the method to perform well, the auxiliary boundary condition must be simple and accurate. This is of particular importance for wave problems (in this context, it is the problem of setting accurate boundary conditions).

Most boundary conditions suggested on the auxiliary boundary are local and approximate [7]. More attractive are exact nonlocal conditions that reduce the problem in an unbounded domain to an equivalent problem in a bounded domain. In this class, we point out the "partial" boundary conditions introduced by A.G. Sveshnikov for the problem of diffraction by a locally nonuniform body [8]. Subsequently, they were systematically used for theoretical analysis and numerical solution of a wide range of diffraction problems with the help of projection methods (see [9] and the references cited therein). Generally, this condition has the form $L_\Gamma u = S_\Gamma u$, where L_Γ is the differential operator of the natural boundary condition generated by the equation of the problem and S_Γ is a nonlocal operator. With a special choice of the contour Γ (for example, a circle), this operator can be explicitly written using the separation of variables. The same approach was used in [7, 10] for solving various problems in unbounded domains by applying the finite element method. The method is easy to implement and is very efficient, as demonstrated by theoretical estimates and numerical experiments.

For an arbitrary contour Γ, the formulations used are based on integral equations. In that case, the operator S_Γ is calculated by solving an integral equation on Γ. In connection with this, we note papers [11–15], where the finite element method is combined with boundary integral equations. This method suffers from the lack of an explicit expression for S_Γ and from the complexity of its computation.

In this paper, the vector eigenmode problem for cylindrical dielectric waveguides is formulated as based on partial boundary conditions similar to those used in [9]. A theoretical analysis of the existence of surface waves is also presented here. The approach we suggest develops the method used in [16, 17], where an analogous scalar problem was studied. In our approach, the original problem is reduced to a parametric eigenvalue problem in a bounded domain with a nonlocal boundary condition nonlinear in the spectral parameter. The analysis of the latter problem is based on the spectral theory of bounded self-adjoint operators.

One way to analyze the existence of solutions to spectral problems in unbounded domains is to use the spectral theory of unbounded operators. In this direction, interesting results have been obtained recently (see, for example, [18–20]). Specifically, the same issues (related to the existence of solutions) as those considered in this paper were examined in [18]. The method we suggest can be viewed as an alternative approach to the solution of such problems. It seems more constructive, because the equations it produces are simpler to treat with numerical methods. Our method also yields new results. For example, we derive a significantly simpler equation (the cut-off equation) determining the number of solutions to the problem.

1. FORMULATION OF THE PROBLEM

We formulate the eigenmode problem for cylindrical dielectric waveguides following [18]. The spectral waveguide theory is based on Maxwell's homogeneous equations

$$\operatorname{curl}\mathcal{E} = -\mu_0\partial\mathcal{H}/\partial t, \quad \operatorname{curl}\mathcal{H} = \varepsilon\partial\mathcal{E}/\partial t, \tag{1}$$

where $\mathcal{E} = \mathcal{E}(x, x_3, t)$ and $\mathcal{H} = \mathcal{H}(x, x_3, t)$ are the electric and magnetic field strengths with Cartesian components $\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3$, and $\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3$, respectively; $\varepsilon = \varepsilon_0 n^2$ is the permittivity; ε_0 is the vacuum permittivity; n is the refractive index; and μ_0 is the vacuum permeability.

The dielectric waveguide is a cylinder whose refractive index $n = n(x)$ is constant along the cylinder generator and varies in the transverse cross section of the cylinder. Let $\mathbb{R}^2$ be the plane $\{x_3 = \text{const}\}$, Ω_i be a transverse cross section of the waveguide, i.e., a bounded, not necessarily simply connected domain of $\mathbb{R}^2$ containing the origin. The following is assumed about the refractive index: $n = n_\infty = \text{const} > 0$ at $x \notin \Omega_i$, n is continuous on $\bar{\Omega}_i$, and $n_+ = \max\{n(x), x \in \Omega_i\} > n_\infty$.

We study waveguide eigenmodes of the form

$$\begin{bmatrix}\mathcal{E}\\ \mathcal{H}\end{bmatrix}(x, x_3, t) = \operatorname{Re}\left(\begin{bmatrix}E\\ H\end{bmatrix}(x)\exp[i(\omega t - \beta x_3)]\right), \tag{2}$$

where $\beta, \omega > 0$. Substituting $\mathcal{E}$ and $\mathcal{H}$ of form (2) into Maxwell's equations (1), we obtain the following system of equations:

$$\operatorname{curl}_\beta E = -i\omega\mu_0 H, \quad \operatorname{curl}_\beta H = i\omega\varepsilon_0 n^2 E, \tag{3}$$

where

$$\operatorname{curl}_\beta E = \begin{bmatrix}\partial E_3/\partial x_2 + i\beta E_2\\ -i\beta E_1 - \partial E_3/\partial x_1\\ \partial E_2/\partial x_1 - \partial E_1/\partial x_2\end{bmatrix}.$$

We introduce the following notations. For a scalar field φ, vector fields $\mathbf{F} = (F_1, F_2)^\mathrm{T}$, $F = (\mathbf{F}^\mathrm{T}, F_3)^\mathrm{T}$, and

$D \subseteq \mathbb{R}^2$, we assume that

$$\mathrm{curl}\mathbf{F} = \partial F_2/\partial x_1 - \partial F_1/\partial x_2, \quad \mathrm{div}\mathbf{F} = \partial F_1/\partial x_1 + \partial F_2/\partial x_2,$$

$$\nabla\varphi = \begin{bmatrix} \partial\varphi/\partial x_1 \\ \partial\varphi/\partial x_2 \end{bmatrix}, \quad \mathrm{curl}\varphi = \begin{bmatrix} \partial\varphi/\partial x_2 \\ -\partial\varphi/\partial x_1 \end{bmatrix}, \quad \Delta\varphi = \frac{\partial^2\varphi}{\partial x_1^2} + \frac{\partial^2\varphi}{\partial x_2^2},$$

$$a_D(F, F') = \int\limits_D \left(\frac{1}{n^2}\mathrm{curl}\mathbf{F}\overline{\mathrm{curl}\mathbf{F}'} + \frac{1}{n_\infty^2}\mathrm{div}\mathbf{F}\overline{\mathrm{div}\mathbf{F}'} + \frac{1}{n^2}\nabla F_3 \cdot \overline{\nabla F_3'}\right)dx,$$

$$c_D(F, F') = i\int\limits_D \left(\frac{1}{n^2} - \frac{1}{n_\infty^2}\right)(\mathbf{F} \cdot \overline{\nabla F_3'} - \nabla F_3 \cdot \bar{\mathbf{F}}')dx,$$

$$b_D(F, F') = \int\limits_D \left(\frac{1}{n_\infty^2} - \frac{1}{n^2}\right)\mathbf{F} \cdot \bar{\mathbf{F}}'dx, \quad d_D(F, F') = \int\limits_D \mathbf{F} \cdot \bar{\mathbf{F}}'dx,$$

$$\mathrm{Div}_\beta F = \frac{\partial F_1}{\partial x_1} + \frac{\partial F_2}{\partial x_2} - i\beta F_3,$$

$$c(\beta; F, F') = \int\limits_{\mathbb{R}^2} \left(\frac{1}{n^2}\mathrm{curl}_\beta\mathbf{F} \cdot \overline{\mathrm{curl}_\beta\mathbf{F}'} + \frac{1}{n_\infty^2}\mathrm{Div}_\beta\mathbf{F}\overline{\mathrm{Div}_\beta\mathbf{F}'}\right)dx,$$

where the dot denotes the scalar product of vectors.

Let $V = [W_2^1(\mathbb{R}^2)]^3$ be the Sobolev space of complex-valued functions with the norm

$$\|F\|_{1,\mathbb{R}^2}^2 = \int\limits_{\mathbb{R}^2} (|\nabla F|^2 + |F|^2)dx,$$

where $|F|^2 = F \cdot \bar{F}$. Let $\mathbb{R}_+ = \{x \in \mathbb{R} : x > 0\}$.

It was proved in [18] that if, for some $(\beta, \omega) \in \mathbb{R}_+^2$, system (3) has a nontrivial solution $E, H \in [L_2(\mathbb{R}^2)]^3$, then the vector H belongs to $V\backslash\{0\}$ and solves the equation

$$c(\beta; H, H') = k^2 d_{\mathbb{R}^2}(H, H') \quad \forall H' \in V, \tag{4}$$

where $k^2 = \mu_0\varepsilon_0\omega^2$. The converse statement also holds: if $(\beta, \omega; H) \in \mathbb{R}_+^2 \times V\backslash\{0\}$ solves problem (4) and the field E is determined by the second equation in (3), then E, H belong to $[L_2(\mathbb{R}^2)]^3$ and satisfy the first equation in (3) as well. In this sense, problems (3) and (4) are equivalent.

It was proved in [18] that nontrivial solutions to problem (4) exist only for

$$(\beta, k) \in \Lambda = \{(\beta, k) : \beta/n_+ < k < \beta/n_\infty, \beta > 0\}.$$

Thus, the eigenmode problem for cylindrical dielectric waveguides is formulated as follows [18]: Find all $(\beta, k) \in \Lambda$ and $H \in V\backslash\{0\}$ that satisfy Eq. (4).

2. THE S_Γ OPERATOR

Let Ω be an open disk of radius R such that $\Omega_i \subset \Omega$; $\mathbf{v} = (\nu_1, \nu_2)$ be the outward unit normal to the boundary Γ of Ω, $\boldsymbol{\tau} = (-\nu_2, \nu_1)$ be the vector tangent to Γ, and $\Omega_\infty = \mathbb{R}^2\backslash\bar{\Omega}$. We introduce the spaces $V_\Omega = [W_2^1(\Omega)]^3$,

$V_\infty = [W_2^1(\Omega_\infty)]^3$, and $V_\infty^0 = \{H \in V_\infty : H|_\Gamma = 0\}$ of complex-valued functions. Let $(\cdot, \cdot)$ be the scalar product in V_Ω:

$$(H, H') = \int_\Omega (\nabla H \cdot \nabla \overline{H}' + H \cdot \overline{H}')dx.$$

Let $\sigma^2 = \beta^2 - k^2 n_\infty^2 > 0$. A vector $H_\sigma \in V_\infty$ is called the metaharmonic extension of $H \in [W_2^{1/2}(\Gamma)]^3$ to the domain Ω_∞ if $H_\sigma|_\Gamma = H$ and

$$s_{\Omega_\infty}(\sigma; H_\sigma, H') = 0 \quad \forall H' \in V_\infty^0,$$

where

$$s_{\Omega_\infty}(\sigma, H, H') = a_{\Omega_\infty}(H, H') + \frac{\beta^2}{n_\infty^2} d_{\Omega_\infty}(H, H') - k^2 d_{\Omega_\infty}(H, H')$$
$$= \int_{\Omega_\infty} \frac{1}{n_\infty^2}(\operatorname{curl} H \overline{\operatorname{curl} H'} + \operatorname{div} H \overline{\operatorname{div} H'} + \nabla H_3 \cdot \overline{\nabla H_3'} + \sigma^2 H \cdot \overline{H}')dx. \tag{5}$$

Integration by parts yields the following identity for the metaharmonic extension:

$$\int_{\Omega_\infty} (\nabla H_\sigma \cdot \overline{\nabla H'} + \sigma^2 H_\sigma \cdot \overline{H}')dx = 0 \quad \forall H' \in V_\infty^0. \tag{6}$$

Consequently, the extension exists and is unique. Moreover, it satisfies

$$\int_{\Omega_\infty} (|\nabla H_\sigma|^2 + \sigma^2 |H_\sigma|^2)dx = \inf_{H' \in V_\infty, H'|_\Gamma = H} \int_{\Omega_\infty} (|\nabla H'|^2 + \sigma^2 |H'|^2)dx \tag{7}$$

(see [21]). We define the operator $S_\Gamma(\sigma) : V_\Omega \longrightarrow V_\Omega$ by the formula

$$(S_\Gamma(\sigma)H, H') = s_{\Omega_\infty}(\sigma; H_\sigma, H'_\sigma), \tag{8}$$

where H and H' are any vectors of V_Ω, and H_σ and H'_σ are the metaharmonic extensions of the traces of H and H' on Γ to Ω_∞. Let us examine the properties of $S_\Gamma(\sigma)$.

Lemma 1. *For any $\sigma > 0$, it holds that $S_\Gamma(\sigma) = S_\Gamma^*(\sigma) \geq 0$.*

This implies that $S_\Gamma(\sigma)$ is a self-adjoint and nonnegative definite operator.

Proof. Since the form $s_{\Omega_\infty}(\sigma; \cdot, \cdot)$ is Hermitian on $V_\infty \times V_\infty$ and is nonnegative for any $\sigma > 0$, it suffices to demonstrate the boundedness of S_Γ. We use the following equivalent norms for the space $[W_2^{1/2}(\Gamma)]^3$ (see, for example, [21, p. 55]):

$$\|H\|_{1/2,\Gamma} = \inf_{H' \in V_\Omega, H'|_\Gamma = H} \|H'\|_{1,\Omega}, \quad \|H\|_{1/2,\Gamma} = \inf_{H' \in V_\infty, H'|_\Gamma = H} \|H'\|_{1,\Omega_\infty}.$$

In view of (5) and (7), it is easy to see that

$$(S_\Gamma(\sigma)H, H) \leq c \int_{\Omega_\infty} (|\nabla H_\sigma|^2 + \sigma^2 |H_\sigma|^2)dx = c \inf_{H' \in V_\infty, H'|_\Gamma = H} \int_{\Omega_\infty} (|\nabla H'|^2 + \sigma^2 |H'|^2)dx$$
$$\leq c_\sigma \inf_{H' \in V_\infty, H'|_\Gamma = H} \|H'\|_{1,\Omega_\infty}^2 = c_\sigma \|H\|_{1/2,\Gamma}^2 \leq c_\sigma \|H\|_{1,\Omega}^2.$$

The lemma is proved.

Since S_Γ is continuous and $[C^\infty(\Omega)]^3$ is dense in V_Ω, it suffices to define S_Γ on the functions from $[C^\infty(\Omega)]^3$.

Integration by parts in (8) yields

$$(S_\Gamma(\sigma)H, H') = \int\limits_{\Omega_\infty} \frac{1}{n_\infty^2}(-\Delta H_\infty + \sigma^2 H_\sigma)\cdot \overline{H'_\sigma}dx - \int\limits_\Gamma \frac{1}{n_\infty^2}\frac{\partial H_\sigma}{\partial \nu}\cdot \overline{H'}dx + \int\limits_\Gamma \frac{1}{n_\infty^2}\left(\frac{\partial H_1}{\partial \tau}\overline{H'_2} + H_2\frac{\partial \overline{H'_1}}{\partial \tau}\right)dx. \quad (9)$$

The separation of variables gives a solution to Eq. (6):

$$H_\sigma(x) = \sum_{n=-\infty}^{\infty}\frac{K_n(\sigma r)}{K_n(\sigma R)}a_n(H)e^{in\varphi}, \quad a_n(H) = \frac{1}{2\pi}\int\limits_0^{2\pi} H(R, \varphi)e^{-in\varphi}d\varphi. \quad (10)$$

Here, (r, φ) are the polar coordinates of x and $K_n(z)$ is the modified Bessel function of the nth order (see, for example, [22]). In view of (6) and (10), we find from (9) that

$$(S_\Gamma(\sigma)H, H') = \frac{2\pi}{n_\infty^2}\sum_{n=-\infty}^{\infty} A_n(\sigma R)a_n(H)\cdot\overline{a_n(H')} + \frac{2\pi i}{n_\infty^2}\sum_{n=-\infty}^{\infty} n(a_n(H_1)\overline{a_n(H'_2)} - a_n(H_2)\overline{a_n(H'_1)}),$$

where $A_n(z) = -zK_n'(z)/K_n(z)$.

It is well known that the modified Bessel functions $K_n(z)$ are positive for real $z > 0$ $(n \ge 0)$ and

$$K_n(z) = K_{-n}(z), \quad K'_n(z) = -K_{n-1}(z) - \frac{n}{z}K_n(z), \quad n \ge 1,$$

$$K'_n(z) = -K_{n+1}(z) + \frac{n}{z}K_n(z), \quad n \le -1, \quad K'_0(z) = -K_1(z).$$

This implies that

$$A_n(z) = |n| + zK_{|n|-1}(z)/K_{|n|}(z) > 0, \quad n = 0, \pm 1, \pm 2, \dots.$$

Define the operators $S(\sigma)$ and $S_0 : V_\Omega \longrightarrow V_\Omega$ by the formulas

$$\begin{gathered}(S(\sigma)H, H') = \frac{2\pi\sigma R}{n_\infty^2}\sum_{n=-\infty}^{\infty}\frac{K_{|n|-1}(\sigma R)}{K_{|n|}(\sigma R)}a_n(H)\cdot\overline{a_n(H')}, \quad \sigma > 0,\\ (S_0 H, H') = \frac{2\pi}{n_\infty^2}\sum_{n=-\infty}^{\infty}|n|a_n(H)\cdot\overline{a_n(H')} + \frac{2\pi i}{n_\infty^2}\sum_{n=-\infty}^{\infty} n(a_n(H_1)\cdot\overline{a_n(H'_2)} - a_n(H_2)\cdot\overline{a_n(H'_1)}).\end{gathered} \quad (11)$$

The operator $S_\Gamma(\sigma)$ is the sum of these two operators:

$$S_\Gamma(\sigma) = S(\sigma) + S_0 \quad \forall \sigma > 0. \quad (12)$$

Lemma 2. *For any $\sigma > 0$, the operator $S(\sigma)$ is compact, $S(\sigma) = S^*(\sigma) \ge 0$; $\|S(\sigma)\| \longrightarrow 0$ as $\sigma \longrightarrow 0$, the operator-valued function $S(\sigma)$ is continuously differentiable in $\sigma > 0$, the function $\sigma \longrightarrow (S(\sigma)H, H)$ is nondecreasing in $\sigma > 0$ for any fixed $H \in V_\Omega$, and $S_0 = S_0^* \ge 0$.*

Proof. The symmetry and nonnegativity of $S(\sigma)$ are obvious. Let us prove compactness. By virtue of the properties of Bessel functions, the functions $K_{|n|-1}(z)/K_{|n|}(z)$ $(n \ne 0)$ are continuous and monotonically increasing from zero to unity on [00, ∞], and the function $A_0(z) = zK_1(z)/K_0(z)$ is also continuous and monotonically increasing from zero on $[0, \infty]$. Let $c_1(\sigma) = 2\pi\max\{R\sigma, A_0(R\sigma)\}/n_\infty^2$. By the Parseval equality,

$$(S(\sigma)H, H) \le c_1(\sigma)\sum_{m=-\infty}^{\infty}|a_n(H)|^2 = c_1(\sigma)\|H\|^2_{[L_2(\Gamma)]^3}.$$

Since the embedding $V_\Omega \subset [L_2(\Gamma)]^3$ is compact and $\|H\|_{[L_2(\Gamma)]^3} \le c_\Gamma\|H\|_{1,\Omega}$ for any $H \in V_\Omega$, we conclude that $S(\sigma)$ is compact, and

$$\|S(\sigma)\| \le c_\Gamma^2 c_1(\sigma).$$

Note that $c_1(\sigma)$ is continuous on $[0, \infty)$ and behaves as $-\ln^{-1}(R\sigma)$ at the origin. Therefore, $\|S(\sigma)\| \longrightarrow 0$ as $\sigma \longrightarrow 0$.

It is easy to see that series (11) can be termwise differentiated with respect to $\sigma > 0$ for any $H, H' \in V_\Omega$. Consequently, $S(\sigma)$ is continuously differentiable with respect to $\sigma > 0$. Let us prove the monotonicity of $S(\sigma)$. To this end, it is convenient to norm $[W_2^{1/2}(\Gamma)]^3$ in the following manner:

$$\|H\|^2_{1/2,\Gamma} = \sum_{n=-1}^{\infty} (|n|+1)|a_n(H)|^2.$$

It is well known (see, for example, [23, p. 29]) that there exists a constant $c_{1/2}$ independent of H such that

$$\|H\|_{1/2,\Gamma} \le c_{1/2}\|H\|_{1,\Omega} \quad \forall H \in V_\Omega.$$

Straightforward computations show that

$$0 < A_n'(z) = \frac{A_n^2(z) - z^2 - n^2}{z} \le 2|n| \le 2(|n|+1), \quad A_0'(z) > 0,$$

for $z > 0$ and $n \ne 0$. Setting $c_2(\sigma) = 2\pi\max\{2, A_0'(R\sigma)\}/n_\infty^2$, we obtain

$$0 \le \frac{d}{d\sigma}(S(\sigma)H, H) = \frac{2\pi}{n_\infty^2}\sum_{n=-\infty}^{\infty} A_n'(R\sigma)|a_n(H)|^2$$

$$\le c_2(\sigma)\sum_{n=-\infty}^{\infty}(|n|+1)|a_n(H)|^2 = c_2(\sigma)\|H\|^2_{1/2,\Gamma} \le c_{1/2}^2 c_2(\sigma)\|H\|^2_{1,\Omega}.$$

The function $c_2(\sigma)$ is continuous in σ for $\sigma > 0$ and has a singularity of order $-(\sigma\ln\sigma)^{-1}$ at $\sigma \longrightarrow 0$. We conclude that $\sigma \longrightarrow (S(\sigma)H, H)$ is nondecreasing in $\sigma > 0$ for any fixed $H \in V_\Omega$.

The self-adjointness of S_0 is obvious. Let us check that it is nonnegative. Suppose that $a_{n,i} = a_n(H_i)$, $a'_{n,i} = \operatorname{Re} a_{n,i}$, and $a''_{n,i} = \operatorname{Im} a_{n,i}$. Then,

$$(S_0H, H) = \frac{2\pi}{n_\infty^2}\sum_{n\ne 0}|n|\left(|a_{n,3}|^2 + \left(a''_{n,1} - \frac{|n|}{n}a'_{n,2}\right)^2 + \left(a'_{n,1} + \frac{|n|}{n}a''_{n,2}\right)^2\right) \ge 0$$

for any $H \in V_\Omega$. The lemma is proved.

The lemma implies that $S(\sigma)$ can be extended by continuity to the half-line $[0, \infty)$ (it suffices to set $S(0) = 0$ to do this). The extended function is continuous: $\|S(\sigma) - S(\eta)\| \longrightarrow 0$ as $\sigma \longrightarrow \eta$; $\sigma, \eta \in [0, \infty)$. Hereafter, when using $S(\sigma)$ at $\sigma \in [0, \infty)$, we will always mean the above extension.

3. THE PROBLEM IN A BOUNDED DOMAIN

Let us reduce problem (4) to a problem in the disk Ω. The desired values in problem (4) are β and k. For convenience, we use $\sigma = \sqrt{\beta^2 - k^2 n_\infty^2}$ as a spectral parameter and seek (β, σ) instead of (β, k).

For any pair $(\beta, k) \in \Lambda$, we define the operators $A_0, B_0, C, B : V_\Omega \longrightarrow V_\Omega$ by the following identities:

$$(A_0H, H') = a_\Omega(H, H'), \quad (B_0H, H') = b_\Omega(H, H'),$$
$$(CH, H') = c_\Omega(H, H'), \quad (BH, H') = d_\Omega(H, H'),$$

where H and H' are arbitrary functions on V_Ω. Let $A = A_0 + S_0$.

Consider the problem of finding all $(\beta, \sigma) \in \mathbb{R}_+^2$ and $H \in V_\Omega \backslash \{0\}$ that satisfy the equation

$$[A + \beta C - \beta^2 B_0 + S(\sigma)]H = -(\sigma^2/n_\infty^2)BH. \tag{13}$$

Theorem 1. *Let* $(\beta, k; H^*) \in \Lambda \times V\backslash\{0\}$ *be a solution to problem* (4), *H be a restriction of H^* to Ω, and* $\sigma = \sqrt{\beta^2 - k^2 n_\infty^2}$. *Then,* $(\beta, \sigma; H)$ *is a solution to problem* (13). *Conversely, let* $(\beta, \sigma; H) \in \mathbb{R}_+^2 \times V_\Omega\backslash\{0\}$ *be a solution to problem* (13), *and let H^* coincide with H in Ω and be equal to the metaharmonic extension H_σ of the vector $H|_\Gamma$ in Ω_∞. Then,*

$$\sigma < \sqrt{1 - (n_\infty/n_+)^2}\beta \tag{14}$$

and the triple $(\beta, k; H^*)$, *with* $k = \sqrt{\beta^2 - \sigma^2}/n_\infty$, *is a solution to problem* (4).

Proof. Let $(\beta, k; H^*) \in \Lambda \times V\backslash\{0\}$ be a solution to problem (4). For all $H, H' \in V$, we have

$$c_{\Omega_\infty}(H, H') = 0, \quad b_{\Omega_\infty}(H, H') = 0,$$

$$c(\beta; H, H') = a_{\mathbb{R}^2}(H, H') + \beta c_{\mathbb{R}^2}(H, H') - \beta^2 b_{\mathbb{R}^2}(H, H') + \frac{\beta^2}{n_\infty^2} d_{\mathbb{R}^2}(H, H').$$

Identity (4) can be represented as

$$a_\Omega(H, H') + \beta c_\Omega(H, H') - \beta^2 b_\Omega(H, H') + (\beta^2/n_\infty^2) d_\Omega(H, H') + s_{\Omega_\infty}(\sigma; H, H') = k^2 d_\Omega(H, H') \quad \forall H' \in V. \tag{15}$$

Setting $H' = 0$ in Ω, we see that the solution to problem (4) is equal to H_σ in Ω_∞. Considering only those $H' \in V$ in (15) for which $H' = H'_\sigma$ in Ω_∞, we conclude that, if problem (4) has a solution, it satisfies the identity

$$a_\Omega(H, H') + \beta c_\Omega(H, H') - \beta^2 b_\Omega(H, H') + (\beta^2/n_\infty^2) d_\Omega(H, H') + (S_\Gamma(\sigma) H, H') = k^2 d_\Omega(H, H') \quad \forall H' \in V_\Omega. \tag{16}$$

Identity (16) is written in operator form

$$[A_0 + \beta C - \beta^2 B_0 + S_\Gamma(\sigma) + (\beta^2/n_\infty^2) B] H = k^2 B H. \tag{17}$$

Substituting (12) into (17) gives (13).

Let us prove the second statement of the theorem. Let $(\beta, \sigma; H) \in \mathbb{R}_+^2 \times V_\Omega\backslash\{0\}$ be a solution to problem (13). Define k by the equality $k = \sqrt{\beta^2 - \sigma^2}/n_\infty$. Then, (13) is equivalent to Eq. (17) and, consequently, to Eq. (16). Define the metaharmonic extension H_σ by using $H|_\Gamma$. Denote by H^* the vector coinciding with H in Ω and with H_σ in Ω_∞. Evidently, $H^* \in V\backslash\{0\}$. Using the definition of $S_\Gamma(\sigma)$ and noting that $s_{\Omega_\infty}(\sigma; H^*, H'_\sigma) = s_{\Omega_\infty}(\sigma; H^*, H')$ by the definition of the metaharmonic extension, we find that H^* satisfies (15) and (4). Therefore, the triple $(\beta, k; H^*)$ solves problem (4). Consequently, k is real and such that $k > \beta/n_+$, and (14) is valid. The theorem is proved.

Remark 1. Theorem 1 implies that, if problem (13) has a solution $(\beta, \sigma; H)$, then (β, σ) belongs to the subset

$$K = \{(\beta, \sigma) : \beta > 0, 0 < \sigma < \sqrt{1 - (n_\infty/n_+)^2}\beta\}.$$

Let $M(\beta) = (1/\beta)A + C - \beta B_0$ and $G(\beta, \sigma) = M(\beta) + (1/\beta)S(\sigma)$. Problem (13) is written in the form

$$(\beta, \sigma; H) \in \mathbb{R}_+^2 \times V_\Omega\backslash\{0\} : G(\beta, \sigma)H = -[\sigma^2/(\beta n_\infty^2)]BH$$

and is formulated as follows: Find all solutions $(\beta, \sigma) \in \mathbb{R}_+^2$ to the equation

$$\gamma(\beta, \sigma) = -\sigma^2/(\beta n_\infty^2),$$

where $\gamma(\beta, \sigma)$ are the eigenvalues of the linear spectral problem

$$(\gamma, H) \in \mathbb{R} \times V_\Omega\backslash\{0\} : G(\beta, \sigma)H = \gamma(\beta, \sigma)BH. \tag{18}$$

Such (β, σ) and the eigenvectors H corresponding to the eigenvalues $\gamma(\beta, \sigma)$ for these (β, σ) are the solutions to problem (13).

Lemma 3. *For any* $\beta > 0$ *and* $\sigma \geq 0$, *the operator* $G(\beta, \sigma)$ *is self-adjoint, the operator* B *is compact,* $B = B^* > 0$, *and*

$$M(\beta) + \lambda\beta B \geq (2\beta)^{-1}A \geq 0, \tag{19}$$

where $\lambda = \delta + 2\delta^2 n_\infty^2$ *and* $\delta = \max_{x \in \Omega}\left|1/n_\infty^2 - 1/n^2\right|$.

Proof. Evidently, for any $\beta > 0$, the operator $M(\beta)$ is self-adjoint. For any $\sigma \geq 0$, the operator $S(\sigma)$ is self-adjoint by Lemma 2. Thus, the first statement in the lemma is valid. The statements about the operator B are obvious. Let us prove inequality (19). It is easy to check that

$$|(CH, H)| \leq 2\delta\left(\int_\Omega |\nabla H_3|^2 dx\right)^{1/2}\left(\int_\Omega |\mathbf{H}|^2 dx\right)^{1/2} \leq \frac{1}{2n_\infty^2\beta}\int_\Omega |\nabla H_3|^2 dx + 2n_\infty^2\beta\delta^2(BH, H),$$

$$(B_0H, H) \leq \delta(BH, H), \quad (AH, H) \geq \frac{1}{2}(AH, H) + \frac{1}{2n_\infty^2}\int_\Omega |\nabla H_3|^2 dx,$$

where H is an arbitrary function on V_Ω. Consequently,

$$\begin{aligned}([M(\beta) + \lambda\beta B]H, H) &= ([(1/\beta)A + C - \beta B_0 + \lambda\beta B]H, H) \\ \geq (2\beta)^{-1}(AH, H) + (\lambda - \delta - 2n_\infty^2\delta^2)\beta(BH, H) &= (2\beta)^{-1}(AH, H) \geq 0.\end{aligned}$$

The lemma is proved.

By Lemma 3, problem (18) has a countable set of solutions (γ_k, H_k) for any fixed $\beta > 0$ and $\sigma \geq 0$. Let the eigenfunctions H_k be orthonormal:

$$(BH_k, H_l) = \delta_{kl}, \quad k, l = 1, 2, \ldots.$$

We arrange γ_k in increasing order (with their multiplicities taken into account):

$$-\lambda\beta \leq \gamma_1 \leq \ldots \leq \gamma_n \leq \ldots, \quad \gamma_n \longrightarrow \infty.$$

By the Riesz–Fischer theorem, for $k = 1, 2, \ldots$, we obtain

$$\gamma_k(\beta, \sigma) = \inf_{V_k \subset V_\Omega}\sup_{H \in V_k} R(\beta, \sigma; H), \quad R(\beta, \sigma; H) = \frac{(A(\beta, \sigma)H, H)}{(BH, H)}, \tag{20}$$

where the infimum is taken over all k-dimensional subspaces of V_Ω.

Lemma 4. *The equality* $(AH, H) = 0$ *is valid if and only if* $H = \text{const}$ *in* Ω.

Proof. Let $H = \text{const}$ in Ω. Then, evidently, $(AH, H) = 0$. Conversely, this equality and Lemma 2 imply that $(S_0H, H) = 0$, and

$$I \equiv \frac{1}{n_\infty^2}\int_\Omega (|\text{curl}\mathbf{H}|^2 + |\text{div}\mathbf{H}|^2 + |\nabla H_3|^2)dx = 0.$$

Integrating by parts, we easily find that

$$0 = I + (S_0H, H) = \frac{1}{n_\infty^2}\int_\Omega |\nabla H|^2 dx + \frac{2\pi}{n_\infty^2}\sum_{n=-\infty}^{\infty} |n||a_n(H)|^2.$$

Consequently, $H = \text{const}$ in Ω. The lemma is proved.

Lemma 5. *Each* $\gamma_k(\beta, \sigma)$ $(k = 1, 2, \ldots)$ *is a continuous nonincreasing function of* $\beta > 0$ *at any fixed* $\sigma \geq 0$ *and is a continuous nondecreasing function of* $\sigma \geq 0$ *at any fixed* $\beta > 0$. *Moreover, for any* $\sigma \geq 0$, *the following estimates are valid*:

$$\lim_{\beta \to \infty}[\gamma_k(\beta, \sigma)/\beta] \leq 1/n_+^2 - 1/n_\infty^2 < 0, \quad k = 1, 2, \ldots. \tag{21}$$

Proof. All assertions of the lemma, except for estimates (21), are directly implied by the definition (20) of the functions γ_k, the nonnegativity of A and B_0, and Lemma 2. Let us prove (21). Suppose that η is an arbitrary positive number. The properties of the function n imply that there exist a point $x_0 \in \Omega$ and a number ρ such that

$$\frac{1}{\rho^2}\int\limits_{\Omega_\rho}\left(\frac{1}{n^2}-\frac{1}{n_+^2}\right)dx \le \eta,$$

where $\Omega_\rho = \{x \in \Omega : |x - x_0| < \rho\}$. Let us denote by $\mu_\rho^{(k)}$ $(k = 1, 2, \ldots, m)$ the first m eigenvalues of the Laplace operator in Ω_ρ under the Dirichlet boundary condition, and let $w_\rho^{(k)}$ denote the corresponding eigenfunctions extended by zero beyond Ω_ρ and satisfying the conditions $\|w_\rho^k\|_{L_2(\Omega)} = 1$. Set

$$H_\rho^{(2k-1)} = (w_\rho^{(k)}, 0, 0)^{\mathrm{T}}, \quad H_\rho^{(2k)} = (0, w_\rho^{(k)}, 0)^{\mathrm{T}}, \quad k = 1, 2, \ldots, m.$$

Denote by $\tilde{W}_\rho$ the m-dimensional subspace of $W_2^1(\Omega)$ spanned by the vectors $w_\rho^{(1)}, \ldots, w_\rho^{(m)}$, and let W_ρ designate the $2m$-dimensional subspace of V_Ω spanned by $H_\rho^{(1)}, \ldots, H_\rho^{(2m)}$. For any vector $H \in W_\rho$, we have

$$(S_0H, H) = 0, \quad (S(\sigma)H, H) = 0, \quad (CH, H) = 0, \quad (A_0H, H) \le \frac{1}{n_\infty^2}\int\limits_\Omega |\nabla H|^2 dx,$$

$$-(B_0H, H) = \int\limits_\Omega\left(\frac{1}{n^2}-\frac{1}{n_+^2}\right)|H|^2 dx + \left(\frac{1}{n_+^2}-\frac{1}{n_\infty^2}\right).$$

Therefore,

$$\gamma_{2m}/\beta - (1/n_+^2 - 1/n_\infty^2) \le \alpha_m(\beta, \rho), \tag{22}$$

where

$$0 \le \alpha_m(\beta, \rho) = \sup_{H \in W_\rho}\left[\frac{1}{n_\infty^2\beta^2}\int\limits_\Omega |\nabla H|^2 dx + \int\limits_\Omega\left(\frac{1}{n^2}-\frac{1}{n_+^2}\right)|H|^2 dx\right] \le \frac{\mu_\rho^{(m)}}{n_\infty^2\beta^2} + \eta\rho^2 \sup_{v \in \tilde{W}_\rho,\, \|v\|_{L_2(\Omega)} = 1} \|v\|^2_{L_\infty(\Omega_\rho)}.$$

It is well known that

$$\mu_\rho^{(l)} = \mu_1^{(l)}/\rho^2, \quad \|w_\rho^{(l)}\|_{L_\infty(\Omega_\rho)} = \|w_1^{(l)}\|_{L_\infty(\Omega_1)}/\rho, \quad l = 1, 2, \ldots.$$

Consequently, for any $\beta > 1/(\rho\sqrt{\eta})$, we have the estimate

$$\alpha_m(\beta, \rho) \le c_m\eta,$$

where c_m is a constant independent of ρ, η, or β. Passage to the limit as $\eta \longrightarrow 0$ and $\beta \longrightarrow \infty$ in the last inequality and (22) yields (21) for any even k. Since $\gamma_{2m-1} \le \gamma_{2m}$, this inequality is valid for any integer k. The lemma is proved.

4. THE SOLVABILITY OF THE PROBLEM

As implied by Theorem 1, the original problem (4) is equivalent to problem (13). To solve problem (13) at a fixed $\beta > 0$, we have to find all solutions $\sigma \ge 0$ to the equations

$$\gamma_k(\beta, \sigma) = -\sigma^2/(\beta n_\infty^2), \quad k = 1, 2, \ldots, \tag{23}$$

where the functions $\gamma_k(\beta, \sigma)$ are defined by (20). Since the functions $\gamma_k(\beta, \sigma)$ are continuous and nondecreasing in $\sigma \ge 0$ at a fixed $\beta > 0$ (see Lemma 5), the number of equations in (23) that have a solution is

$$n(\beta) = \max\{k : \gamma_k(\beta, 0) < 0\}, \quad \beta > 0, \tag{24}$$

where $\gamma_k(\beta, 0)$ are the eigenvalues of problem (18) at $\sigma = 0$. Figure 1 illustrates the behavior of the functions $\sigma \longrightarrow \gamma_k(\beta, \sigma)$ as based on their properties proved in Lemma 5. If, for some $k = 1, 2, \ldots$ and $\beta > 0$, there

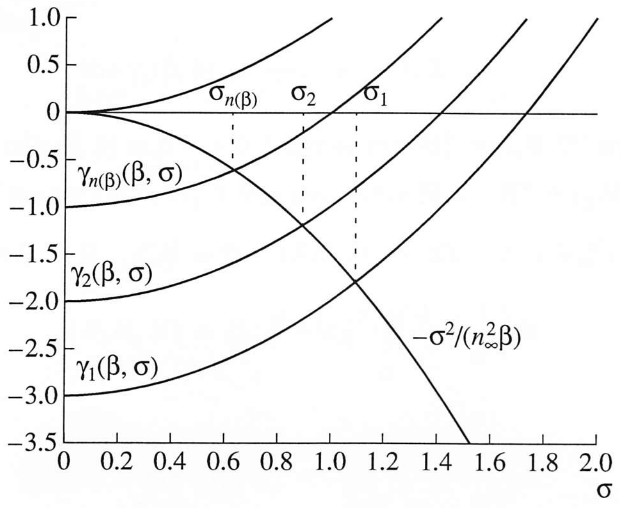

Fig. 1.

exists a solution to Eq. (23), that solution is evidently unique. We denote it as $\sigma_k(\beta)$. Thus, for a fixed $\beta > 0$, problem (13) has exactly $n(\beta)$ solutions $(\beta, \sigma_k(\beta))$, and

$$0 < \sigma_{n(\beta)}(\beta) \le \sigma_{n(\beta)-1}(\beta) \le \ldots \le \sigma_1(\beta) < \beta\sqrt{1-(n_\infty/n_+)^2}.$$

The last estimate follows from Theorem 1.

Let us describe a more convenient method for determining $n(\beta)$. Suppose that $H_3^* = (0, 0, 1)^\mathrm{T}$ at $x \in \Omega$, and

$$V_0 = \left\{ H \in V_\Omega : (BH, H_3^*) = \int_\Omega H_3 dx = 0 \right\}.$$

Consider the problem

$$(\beta, H) \in \bar{R}_+ \times V_0 \backslash \{0\} : (A + \beta C - \beta^2 B_0)H = 0. \tag{25}$$

Theorem 2. *Problem* (25) *has a countable set of solutions* $\{\beta_k^0\}_{k=1}^\infty$:

$$0 = \beta_1^0 = \beta_2^0 < \beta_3^0 \le \ldots \le \beta_k^0 \le \ldots, \quad \beta_k^0 \longrightarrow \infty, \quad k \longrightarrow \infty. \tag{26}$$

For a fixed $\beta > 0$, *the number of solutions of problem (13) is equal to*

$$n(\beta) = \max\{k : \beta_k^0 < \beta, k = 1, 2, \ldots\}. \tag{27}$$

Proof. By virtue of (24), the number of solutions to problem (13) equals the number of negative eigenvalues $\gamma_k(\beta, 0)$. Since $M(\beta)H_3^* = 0$ for any $\beta > 0$, there is a zero eigenvalue among these eigenvalues. It is eliminated by requiring that the eigenvectors H of problem (18) at $\sigma = 0$ belong to V_0. Then, as implied by (20), the eigenvalues $\gamma_k(\beta, 0)$ are determined by the equalities

$$\gamma_k(\beta, 0) = \inf_{V_k \subset V_0} \sup_{H \in V_k} R(\beta, H), \quad R(\beta, H) = \frac{(M(\beta)H, H)}{(BH, H)},$$

where the infimum is taken over all k-dimensional subspaces of V_0. All functions $\gamma_k(\beta, 0)$ are continuous for $\beta > 0$ and decrease monotonically. Indeed, for $H \in V_0$,

$$\frac{dR(\beta, H)}{d\beta} = -\frac{(1/\beta^2)(AH, H) + (B_0 H, H)}{(BH, H)} < 0.$$

It was shown in Lemma 5 that

$$\lim_{\beta \to \infty} \gamma_k(\beta, 0) = -\infty, \quad k = 1, 2, \dots. \tag{28}$$

Let us analyze the behavior of $\gamma_k(\beta, 0)$ as $\beta \longrightarrow 0$. Suppose that $H_1^* = (1, 0, 0)^T$ and $H_2^* = (0, 1, 0)^T$ for $x \in \Omega$ and V_2 is the linear span of the set $\{H_1^*, H_2^*\}$. For any vector $\mathbf{H} = c_1 H_1^* + c_2 H_2^* \in V_2$, we have

$$AH = 0, \quad CH = 0, \quad (BH, H) = |\Omega|(|c_1|^2 + |c_2|^2),$$

$$(B_0 H, H) = (|c_1|^2 + |c_2|^2)\int_\Omega \left(\frac{1}{n_\infty^2} - \frac{1}{n^2}\right)dx.$$

Consequently,

$$\gamma_2(\beta, 0) \le \sup_{H \in V_2} R(\beta, H) = -\frac{\beta}{|\Omega|}\int_\Omega \left(\frac{1}{n_\infty^2} - \frac{1}{n^2}\right)dx. \tag{29}$$

Next, for any $\beta > 0$, Lemma 3 implies the inequality

$$R(\beta, H) \ge \frac{1}{2\beta}\frac{(AH, H)}{(BH, H)} - \lambda\beta, \tag{30}$$

where H is any vector from V_0 and the parameter $\lambda > 0$ is defined in Lemma 3. Specifically, $R(\beta, H) \ge -\lambda\beta$. This implies that

$$\gamma_1(\beta, 0) = \inf_{H \in V_0} R(\beta, H) \ge -\lambda\beta. \tag{31}$$

Combining (29) and (31), we find for any $\beta > 0$ that

$$-\lambda\beta \le \gamma_1(\beta, 0) \le \gamma_2(\beta, 0) \le -c\beta,$$

where $c > 0$ is a constant independent of β. Thus, $\gamma_1(\beta, 0)$ and $\gamma_2(\beta, 0)$ can be extended by continuity to $\beta = 0$ by setting

$$\gamma_1(0, 0) = \gamma_2(0, 0) = 0. \tag{32}$$

Inequality (30) implies that

$$\gamma_3(\beta, 0) \ge \frac{1}{2\beta}\gamma_* - \lambda\beta, \quad \gamma_* = \inf_{V_3 \subset V_0} \sup_{H \in V_3} \frac{(AH, H)}{(BH, H)},$$

where the infimum is taken over all three-dimensional subspaces of V_0. By Lemma 4, it is evident that $\gamma_* > 0$. Thus,

$$\lim_{\beta \to 0} \gamma_3(\beta, 0) = +\infty.$$

For any $\beta > 0$ and $k = 4, 5, \dots$, we have $\gamma_k(\beta, 0) \ge \gamma_3(\beta, 0)$, and, consequently,

$$\lim_{\beta \to 0} \gamma_k(\beta, 0) = +\infty, \quad k = 3, 4, \dots. \tag{33}$$

Thus, we have shown that the functions $\gamma_k(\beta, 0)$ ($k = 1, 2, \dots$) are continuous and decrease monotonically for $\beta > 0$, and equalities (28), (32), and (33) are valid. Consequently, each of the equations

$$\gamma_k(\beta, 0) = 0, \quad k = 1, 2, \dots, \tag{34}$$

has a unique root $\beta_k^0 \in [0, \infty)$ (see Fig. 2, which shows the behavior of the functions $\gamma_k(\beta, 0)$ as based on their properties proved above). Therefore, the number of solutions to problem (13) can be determined by (27), where β_k^0 are the roots of Eqs. (34). The set of β_k^0 is countable, and they satisfy condition (26).

Let us now show that the numbers β_k^0 solve problem (25) and that there do not exist other solutions to this problem. Suppose that $H_k(\beta_k^0) \in V_0$ is the eigenvector of problem (18) at $\sigma = 0$ that corresponds to the

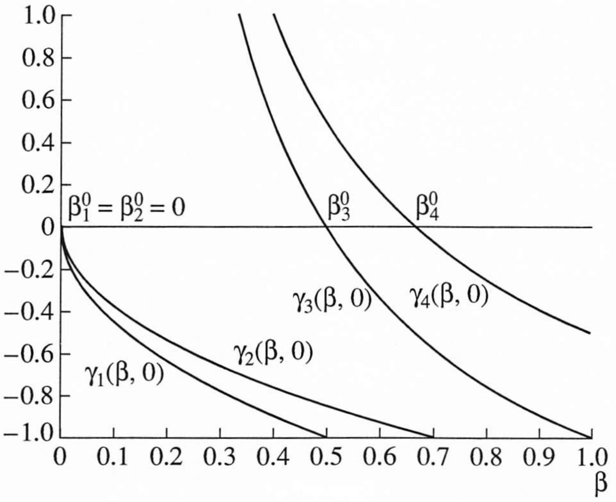

Fig. 2.

eigenvalue $\gamma_k(\beta_k^0, 0)$. Considering (18) at $\sigma = 0$, $\beta = \beta_k^0$, and $H = H_k(\beta_k^0)$, we deduce that the pair $(\beta_k^0, H_k(\beta_k^0))$ solves problem (25). Conversely, let (β_*, H_*) be a solution to problem (25), and let $\beta_* > 0$. This means that $A(\beta_*)H_* = 0$; i.e., there is either a zero solution to problem (18) when $\sigma = 0$ and $\beta = \beta_*$ or $\gamma_k(\beta_*, 0) = 0$ at some k. By the uniqueness of the roots of Eqs. (34), we see that $\beta_* = \beta_k^0$, and the pair $(\gamma_k(\beta_*, 0), H_*)$ solves problem (18) at $\sigma = 0$. When $\beta = 0$, problem (25) has only two linearly independent eigenfunctions $H_1 = c_1 H_1^*$ and $H_2 = c_2 H_2^*$, where c_1 and c_2 are any complex numbers. Therefore, when $\sigma = 0$ and $\beta = 0$, $(0, H_1)$ and $(0, H_2)$ are solutions to the problem obtained by multiplying both sides of Eq. (18) by β. The theorem is proved.

Remark 2. Theorem 2 implies that, for any $\beta > 0$, problem (13) (and, consequently, problem (4)) has at least two solutions. The number of solutions grows with β and goes to infinity as $\beta \longrightarrow \infty$. For any finite β, there exist no more than a finite number of solutions. This number equals $n(\beta)$ and is determined by the values $\beta = \beta_k^0$, which are called the cut-off points and solve Eq. (25). This equation is called the cut-off equation.

Let us describe a more convenient method for determining the cut-off points. We reduce problem (25) to an eigenvalue problem that is linear in the spectral parameter. To this end, define the operator C_0: $V_\Omega \longrightarrow V_\Omega$ by the rule

$$(C_0 H, H) = a_0(H_3)\overline{a_0(H_3)},$$

where $a_0(H_3)$ is the zero Fourier coefficient (10) of the function H_3. Evidently, C_0 is a self-adjoint nonnegative operator. Consider the problem of finding $\beta \geq 0$ and $H \in V_\Omega \backslash \{0\}$ such that

$$(A + C_0 + \beta C - \beta^2 B_0)H = 0. \tag{35}$$

It is easy to verify that problems (25) and (35) have the same set of solutions β_k^0. Indeed, let (β, H) be a solution to problem (25). Representing H as $H = cH_3^* + H^0$, where $H_3^* = (0, 0, 1)^T$ in Ω and $c = a_0(H_3)$, we evidently have $C_0 H^0 = 0$. Using this equality in conjunction with the fact that H_3^* belongs to the nullspace of $A + \beta C - \beta^2 B_0$ for any $\beta \geq 0$, we obtain the following sequence of equalities:

$$0 = (A + \beta C - \beta^2 B_0)H = (A + \beta C - \beta^2 B_0)H^0 = (A + C_0 + \beta C - \beta^2 B_0)H^0,$$

which implies that (β, H^0) solves problem (35). Let (β, H) be an eigenpair of problem (35). Take the scalar product of both sides of Eq. (35) with the vector H_3^*, we see that $a_0(H_3) = 0$; i.e., $(A + \beta C - \beta^2 B_0)H = 0$. Decompose H as

$$H = cH_3^* + H^0, \quad c = \int_\Omega H_3 dx/|\Omega|.$$

Then, $H_0 \in V_0$, and $(A + \beta C - \beta^2 B_0)H^0 = 0$; i.e., (β, H^0) solves problem (25).

Let $\hat{V}_\Omega = [W_2^1(\Omega)]^2$. We define the operators $\hat{A}$, $\hat{B}_0 : \hat{V}_\Omega \longrightarrow \hat{V}_\Omega$, $\hat{C}_0 : \hat{V}_\Omega \longrightarrow W_2^1(\Omega)$, $L : W_2^1(\Omega) \longrightarrow W_2^1(\Omega)$, and $\hat{C}_1 : W_2^1(\Omega) \longrightarrow \hat{V}_\Omega$ by the following identities, with the operator brackets understood in a suitable manner:

$$(\hat{A}\mathbf{H}, \mathbf{H}') = \int_\Omega \left(\frac{1}{n^2}\operatorname{curl}\mathbf{H}\overline{\operatorname{curl}\mathbf{H}'} + \frac{1}{n_\infty^2}\operatorname{div}\mathbf{H}\overline{\operatorname{div}\mathbf{H}'}\right)dx + \frac{2\pi}{n_\infty^2}\sum_{n=-\infty}^{\infty}|n|a_n(\mathbf{H})\cdot\overline{a_n(\mathbf{H}')}$$

$$+\frac{2\pi i}{n_\infty^2}\sum_{n=-\infty}^{\infty} n(a_n(H_1)\cdot\overline{a_n(H_2')} - a_n(H_2)\cdot\overline{a_n(H_1')}),$$

$$(\hat{B}_0\mathbf{H}, \mathbf{H}') = \int_\Omega (1/n_\infty^2 - 1/n^2)\mathbf{H}\cdot\overline{\mathbf{H}}'dx,$$

$$(\hat{C}_0\mathbf{H}, \eta) = i\int_\Omega\left(\frac{1}{n^2} - \frac{1}{n_\infty^2}\right)\mathbf{H}\cdot\overline{\nabla\eta}dx, \quad (\hat{C}_1\eta, \mathbf{H}) = -i\int_\Omega\left(\frac{1}{n^2} - \frac{1}{n_\infty^2}\right)\nabla\eta\cdot\overline{\mathbf{H}}dx,$$

$$(L\eta, \eta') = \int_\Omega \frac{1}{n^2}\nabla\eta\cdot\overline{\nabla\eta'}dx + \frac{2\pi}{n_\infty^2}\sum_{n=-\infty}^{\infty}|n|a_n(\eta)\overline{a_n(\eta')} + a_0(\eta)\overline{a_0(\eta')}.$$

Now, Eq. (35) can be written in the following block form:

$$\begin{pmatrix} \hat{A} & \beta\hat{C}_1 \\ \beta\hat{C}_0 & L \end{pmatrix}\begin{pmatrix} \mathbf{H} \\ H_3 \end{pmatrix} = \beta^2\begin{pmatrix} \hat{B}_0 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \mathbf{H} \\ H_3 \end{pmatrix}.$$

The operator L is positive definite. Eliminating H_3, we obtain a linear eigenvalue problem:

$$\hat{A}\mathbf{H} = \beta^2(\hat{B}_0 + \hat{C}_1 L^{-1}\hat{C}_0)\mathbf{H},$$

where the spectral parameter is β^2. Evidently, the eigenvalues of this problem are equal to the cut-off point values β_k^0 squared.

5. THE BEHAVIOR OF DISPERSION CURVES

The functions $\sigma_k(\beta)$ $(k = 1, 2, \ldots)$ defined on the intervals (β_k^0, ∞), where β_k^0 are the cut-off points, are called dispersion curves. For a given $\beta > \beta_k^0$, the value $\sigma_k(\beta)$ is defined as the unique solution to Eq. (23). Let us examine the properties of the functions $\sigma_k(\beta)$. As an illustration, Fig. 3 shows the dispersion curves for a uniform waveguide of circular cross section. These curves were constructed via solving the characteristic equation from [2].

Lemma 6. *For any* $k = 1, 2, \ldots$, *the function* $\beta\gamma_k(\beta, \sigma)$ *(where* $\gamma_k(\beta, \sigma)$ *is defined by* (20)) *is locally Lipschitzian in* β *on the set*

$$D = \{\beta \geq 0, \sigma \geq 0 : \gamma_k(\beta, \sigma) \leq 0\}.$$

More exactly, there exists a constant c independent of β *or* σ *and such that*

$$\left|\beta\gamma_k(\beta, \sigma) - \hat{\beta}\gamma_k(\hat{\beta}, \sigma)\right| \le c\max\{\beta, \hat{\beta}\}\left|\beta - \hat{\beta}\right| \tag{36}$$

for any β, $\hat{\beta}$, *and* σ *belonging to D.*

Proof. Let β, $\hat{\beta}$, and σ be arbitrary numbers in D. By definition (20), we have

$$\beta\gamma_k(\beta, \sigma) = \inf_{V_k \subset V_\Omega} \sup_{H \in V_k, (BH, H) = 1} R_\sigma(\beta, H), \tag{37}$$

where $R_\sigma(\beta, H) = ([A + \beta C - \beta^2 B_0 - S(\sigma)]H, H)$. The following equality is valid:

$$R_\sigma(\hat{\beta}, H) - R_\sigma(\beta, H) = (\hat{\beta} - \beta)(CH, H) - (\hat{\beta}^2 - \beta^2)(B_0 H, H).$$

Let $(BH, H) = 1$. By Lemma 3, we have $R_\sigma(\beta, H) + \lambda\beta^2 \ge (AH, H)/2$,

$$(B_0 H, H) \le \delta, \quad |(CH, H)| \le 2\delta\left(\int_\Omega |\nabla H_3|^2 dx\right)^{1/2} \le 2\delta n_\infty (AH, H)^{1/2},$$

where the parameters δ and λ are defined in Lemma 3. Using these inequalities, we derive

$$|(CH, H)| \le 4\delta n_\infty [R_\sigma(\beta, H) + \lambda\beta^2]^{1/2}.$$

Therefore,

$$R_\sigma(\hat{\beta}, H) \le R_\sigma(\beta, H) + \delta\left|\hat{\beta} - \beta\right|\{4n_\infty[R_\sigma(\beta, H) + \lambda\beta^2]^{1/2} + (\hat{\beta} + \beta)\}.$$

This inequality and equality (37) imply that

$$\hat{\beta}\gamma_k(\hat{\beta}, \sigma) \le \beta\gamma_k(\beta, \sigma) + \delta\left|\hat{\beta} - \beta\right|\{4n_\infty[\beta\gamma_k(\beta, \sigma) + \lambda\beta^2]^{1/2} + (\hat{\beta} + \beta)\}.$$

By assumption, $\gamma_k(\beta, \lambda) \le 0$. Consequently,

$$\hat{\beta}\gamma_k(\hat{\beta}, \sigma) - \beta\gamma_k(\beta, \sigma) \le \delta(4n_\infty\sqrt{\lambda} + 2)\max\{\beta, \hat{\beta}\}\left|\hat{\beta} - \beta\right|.$$

Interchanging β and $\hat{\beta}$ gives estimate (36). The lemma is proved.

Theorem 3. *For any* $k = 1, 2, \ldots$, *the following assertions hold:*

(a) $\sigma_k(\beta)$ is a nondecreasing function of $\beta \in (\beta_k^0, \infty)$;

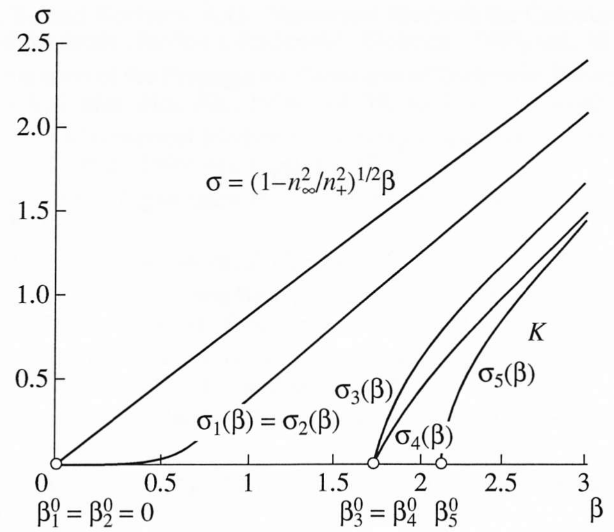

Fig. 3.

(b) $\sigma_k^2(\beta)$ is a locally Lipschitzian function of β on the interval (β_k^0, ∞);

(c) $\lim\limits_{\beta \to \infty} [\sigma_k(\beta)/\beta] = [1 - (n_\infty/n_+)^2]^{1/2}$.

Proof. Let us prove (a). By Lemma 5, for any $k = 1, 2, \ldots$, the function $\gamma_k(\beta, \sigma)$ is a continuous nonincreasing function of $\beta > 0$. Let $\hat{\beta} > \beta$. Note that a solution to Eq. (23) exists only when $\gamma_k < 0$. Taking this fact into account, we derive a sequence of inequalities:

$$-\sigma_k^2(\hat{\beta})/n_\infty^2 = \hat{\beta}\gamma_k(\hat{\beta}, \sigma_k(\hat{\beta})) \le \beta\gamma_k(\hat{\beta}, \sigma_k(\hat{\beta})) \le \beta\gamma_k(\beta, \sigma_k(\hat{\beta})) = \beta[\gamma_k(\beta, \sigma_k(\hat{\beta})) - \gamma_k(\beta, \sigma_k(\beta))] - \sigma_k^2(\beta)/n_\infty^2.$$

Thus,

$$[\sigma_k^2(\hat{\beta}) - \sigma_k^2(\beta)]/n_\infty^2 \ge \beta[\gamma_k(\beta, \sigma_k(\beta)) - \gamma_k(\beta, \sigma_k(\hat{\beta}))].$$

By Lemma 5, for any $k = 1, 2, \ldots$, the function $\gamma_k(\beta, \sigma)$ is a continuous nondecreasing function of $\sigma \ge 0$. From the last inequality, we conclude that $\sigma_k(\hat{\beta}) \ge \sigma_k(\beta)$.

We now prove (b). Without loss of generality, assume that $\hat{\beta} > \beta > 0$. The property (a) and estimate (36) imply that

$$0 \le [\sigma_k^2(\hat{\beta}) - \sigma_k^2(\beta)]/n_\infty^2 = \beta\gamma_k(\beta, \sigma_k(\beta)) - \hat{\beta}\gamma_k(\hat{\beta}, \sigma_k(\hat{\beta}))$$
$$\le \beta\gamma_k(\beta, \sigma_k(\beta)) - \hat{\beta}\gamma_k(\hat{\beta}, \sigma_k(\beta)) \le c\max\{\beta, \hat{\beta}\}|\hat{\beta} - \beta|,$$

because $\gamma_k(\beta, \sigma)$ is a nondecreasing function of σ.

We now prove (c). The equality $-\sigma_k^2(\beta)/\beta^2 = n_\infty^2\gamma_k(\beta, \sigma_k(\beta))/\beta$ and inequality (21) imply the estimate

$$\lim_{\beta \to \infty} \frac{\sigma_k^2(\beta)}{\beta^2} \ge 1 - \frac{n_\infty^2}{n_+^2}.$$

On the other hand, by Theorem 1, if (β, σ, H) is a solution to problem (13), then inequality (14) holds, which implies the validity of (c). The lemma is proved.

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