Siberian Mathematical Journal, Vol. 50, *No.* 3, *pp.* 379–384, 2009
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GENERALIZED *SV*-MODULES

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UDC 512.55

Abstract: Given an arbitrary quasiprojective right R-module P, we prove that every module in the category $\sigma(P)$ is weakly regular if and only if every module in $\sigma(M/I(M))$ is lifting, where M is a generating object in $\sigma(P)$. In particular, we describe the rings over which every right module is weakly regular.

Keywords: semiartinian ring, weakly regular module, SV-ring, quasiprojective module

We assume that all rings are associative and unital and all modules are unital. A module M is called *weakly regular* provided that every submodule of M not lying in the Jacobson radical of M has a nonzero direct summand of M. A ring R is called *weakly regular* provided that the module R_R is weakly regular.

Given a ring R and a module M, we denote by $J(R)$ and $J(M)$ the Jacobson radicals of R and M respectively. The injective hull of M is denoted by $E(M)$.

If a module M possesses a composition series then the number of composition factors in this series is called the *length* of the series and denoted by $l(M)$. A module is called M-subgenerated provided that it is isomorphic to a submodule of a homomorphic image of the direct sum of some copies of M. We denote by $\sigma(M)$ the complete subcategory of all right M-subgenerated R-modules.

A module M is called an *SV-module* if M is a semiartinian V-module. A ring R is called a *right SV-ring* provided that R_R is an SV-module. A module M is called a *generalized SV-module* if every module in $\sigma(M)$ is weakly regular. A ring R is called a *generalized right SV-ring* provided that each right R-module is weakly regular.

The weakly regular rings were introduced and studied by Nicholson under the name of I_0-rings [1]. The notion of a weakly regular module was introduced by I. I. Sakhaev at the beginning of the 1990s. The projective weakly regular modules were studied in detail in [2]. The main result of the present work is Theorem 13 in which the quasiprojective generalized SV-modules are characterized. In particular, some description of the generalized right SV-rings is given.

The *Loewy series* of M is the ascending chain

$$0 \subset \operatorname{Soc}_1(M) = \operatorname{Soc}(M) \subset \dots \subset \operatorname{Soc}_\alpha(M) \subset \operatorname{Soc}_{\alpha+1}(M) \subset \dots,$$

where $\operatorname{Soc}_\alpha(M)/\operatorname{Soc}_{\alpha-1}(M) = \operatorname{Soc}(M/\operatorname{Soc}_{\alpha-1}(M))$ for every nonlimit ordinal α and

$$\operatorname{Soc}_\alpha(M) = \bigcup_{\beta<\alpha} \operatorname{Soc}_\beta(M)$$

for every limit ordinal α. Denote by $L(M)$ the submodule $\operatorname{Soc}_\xi(M)$, where ξ is the least ordinal such that $\operatorname{Soc}_\xi(M) = \operatorname{Soc}_{\xi+1}(M)$. A module M is called *semiartinian* provided that $M = L(M)$. A ring R is called *right semiartinian* if the module R_R is semiartinian. Given an arbitrary ring R, we denote the ideals $L(R_R)$ and $\operatorname{Soc}(R_R)$ by $L(R)$ and $\operatorname{Soc}(R)$.

Given an arbitrary right R-module M, by transfinite induction we define the submodule $I_\alpha(M)$ for every ordinal α as follows: If $\alpha = 0$ then $I_\alpha(M) = 0$. If $\alpha = \beta + 1$ then $I_{\beta+1}(M)/I_\beta(M)$ is the sum of all M-injective local right submodules in $M/I_\beta(M)$ of length ≤ 2 whose quotient module by the Jacobson radical is an M-injective module. If α is a limit ordinal then $I_\alpha(M) = \bigcup_{\beta<\alpha} I_\beta(M)$. It is clear that $I_\tau(M) = I_{\tau+1}(M)$ and $I_1(M/I_\tau(M)) = 0$ for some ordinal τ. In what follows, we denote $I_\tau(M)$ by $I(M)$. Given an arbitrary ring R, we denote the right ideal $I(R_R)$ by $I(R)$ which is an ideal as it is easy to see.

Kazan. Translated from *Sibirskiĭ Matematicheskiĭ Zhurnal*, Vol. 50, No. 3, pp. 481–488, May–June, 2009.
Original article submitted April 2, 2008.

Given some right R-modules P and M, the abelian group $\operatorname{Hom}_R(P,M)$ may be considered as a right S-module by putting $(fs)(m) = f(s(m))$, where $S = \operatorname{End}_R(P)$, $f \in \operatorname{Hom}_R(P,M)$, $s \in S$, and $m \in M$.

Lemma 1. *Let P be a finitely generated quasiprojective right R-module and $S = \operatorname{End}_R(P)$. If $M \in \sigma(P)$ then the following are valid*:

(1) *if N is a submodule in M then there is an isomorphism of the right S-modules* $\operatorname{Hom}_R(P, M/N) \cong \operatorname{Hom}_R(P,M)/\operatorname{Hom}_R(P,N)$;

(2) *if M is a simple right R-module then* $\operatorname{Hom}_R(P,M)$ *is either a simple right S-module or zero*;

(3) *if M is a semisimple right R-module then* $\operatorname{Hom}_R(P,M)$ *is a semisimple right S-module*;

(4) *if $M = \sum_{i\in I} N_i$ and* $\operatorname{Hom}_R(P,M) \neq 0$ *then there exists $i_0 \in I$ such that* $\operatorname{Hom}_R(P, N_{i_0}) \neq 0$;

(5) *if $\phi \in \operatorname{Hom}_R(P,M)$ then $\phi S = \operatorname{Hom}_R(P, Jm(\phi))$.*

Proof. (1) Let f be the canonical mapping from M onto M/N. Consider the mapping

$$g : \operatorname{Hom}_R(P,M) \to \operatorname{Hom}_R(P, M/N)$$

acting by the rule $g(\phi) = f\phi$. It is clear that g is an S-homomorphism and $\operatorname{Ker}(g) = \operatorname{Hom}_R(P,N)$. By [3, 18.3] P is a projective object in $\sigma(P)$. Then g is an epimorphism, and this fact proves the claim.

(2) Consider a nonzero homomorphism $\phi \in \operatorname{Hom}_R(P,M)$. Show that $\phi S = \operatorname{Hom}_R(P,M)$. Let $\alpha \in \operatorname{Hom}_R(P,M)$. Since ϕ is nonzero, ϕ is an epimorphism. Then the quasiprojectivity of P implies the existence of a homomorphism $\beta \in S$ such that $\alpha = \phi\beta$.

(3) Let $M = \bigoplus_{i\in I} S_i$, where S_i is simple for every i. Since P is finitely generated, we have $\operatorname{Hom}_R(P, \bigoplus_{i\in I} S_i) \cong \bigoplus_{i\in I} \operatorname{Hom}_R(P, S_i)$. Then the semisimplicity of the S-module $\operatorname{Hom}_R(P,M)$ follows from the previous item.

(4) Let $\phi \in \operatorname{Hom}_R(P,M)$ be a nonzero homomorphism. Consider the natural epimorphism f from $\bigoplus_{i\in I} N_i$ onto $\sum_{i\in I} N_i$. Since P is a projective object in $\sigma(P)$, there exists a homomorphism $g : P \to \bigoplus_{i\in I} N_i$ such that $\phi = fg$. Then g is a nonzero homomorphism. Therefore, there exists $i_0 \in I$ such that $\operatorname{Hom}_R(P, N_{i_0}) \neq 0$.

(5) This assertion follows from the quasiprojectivity of P. □

The next lemma is immediate from [4, 11.35].

Lemma 2. *Let P be a right R-module. Assume that the ring $S = \operatorname{End}_R(P)$ is regular. Then the injectivity of a right R-module M implies the injectivity of the right S-module* $\operatorname{Hom}_R(P,M)$.

Lemma 3. *Let M be a weakly regular right R-module, and let N be a submodule in M such that $(N + J(M))/J(M)$ is a simple submodule in $M/J(M)$. Then N has a local direct summand mR of M such that $(N + J(M))/J(M) = (m + J(M))R$.*

Proof. Take $n \in N$ such that $(N+J(M))/J(M) = (n+J(M))R$. The weak regularity of M implies the existence of a cyclic submodule mR such that $mR \not\subset J(M)$, $mR \subset nR$, and mR is a direct summand in M. Then

$$(N + J(M))/J(M) = (m + J(M))R \cong mR/(J(M) \cap mR) \cong mR/J(mR),$$

which proves the locality of mR. □

Lemma 4. *Let M be a right R-module, and let every module in $\sigma(M)$ be weakly regular. Then every local module in $\sigma(M)$ is of length ≤ 2.*

Proof. Let N be a local module in $\sigma(M)$. If N is not simple then [5, Lemma 3.3] implies that $N \not\subset J(E(N))$, where $E(N)$ is the injective hull of N in $\sigma(M)$. Then the weak regularity of $E(N)$ gives $N = E(N)$. Since every injective indecomposable module is homogeneous and $J(N)$-semisimple by [5, Lemma 3.3], N is a local module of length 2. □

Lemma 5. *If M is a semiartinian module and every module in $\sigma(M)$ is weakly regular then every nonsemisimple module N in $\sigma(M)$ has an injective local submodule of length ≤ 2.*

PROOF. Since N is nonsemisimple, N has a nonzero injective submodule N_0 by [5, Theorem 3.4]. Since M is semiartinian, N_0 is semiartinian by [6, 3.12] as well. Hence, $N_0/J(N_0)$ has a simple submodule. Then Lemmas 3 and 4 imply that N_0 contains a direct summand, which is injective, local, and of length ≤ 2. $\square$

A module M is called *semilocal* provided that $M/J(M)$ is semisimple. Say that a submodule N of M *lies over a direct summand* of M if there exist submodules N_1 and N_2 such that $N_1 \oplus N_2 = M$, $N_1 \subset N$, and $N_2 \cap N$ is coessential in N_2. A right R-module M is called a *lifting module* if its every submodule lies over a direct summand of M. It is easy to see that every lifting module is weakly regular.

Theorem 6. *Let M be a right R-module. Then the following are equivalent*:

(1) *M is semilocal and every module in $\sigma(M)$ is weakly regular*;

(2) *M is locally noetherian and every module in $\sigma(M)$ is weakly regular*;

(3) *every module in $\sigma(M)$ is a lifting module.*

PROOF. (1) $\Rightarrow$ (2) Show that M is locally noetherian. Let N be a finitely generated submodule in M. It is clear that $N/(N \cap J(M))$ is a semisimple module of finite length. Inducting on the length of $N/(N \cap J(M))$, show that N is a module of finite length. If $l(N/(N \cap J(M))) = 1$ then Lemma 3 implies the existence of a local submodule N_0 in N such that $(N_0 + J(M))/J(M) \cong (N + J(M))/J(M)$ and $M = N_0 \oplus L$, where L is a submodule in M. Then $N = N_0 \oplus (N \cap L)$, where $N \cap L \subset J(M)$ and $J(M) \subset \mathrm{Soc}(M)$ by [5, Lemma 3.3]. Since $N \cap L$ is a finitely generated semisimple module and N_0 is a local module of finite length by Lemma 4; therefore N is of finite length. Assume that our assertion is proved for all finitely generated submodules S of M such that $l(S/(S \cap J(M))) < n$ and N is a finitely generated submodule in M with $l(N/(N \cap J(M))) = n$. Choose a submodule N_0 in N such that $N_0/(N_0 \cap J(M))$ is a simple module. Lemma 3 implies the existence of a local submodule mR in N_0 such that $M = mR \oplus L$, where L is a submodule in M. Then $N = mR \oplus (N \cap L)$ and $l(N/(N \cap J(M))) = 1 + l((N \cap L)/((N \cap L) \cap J(M)))$. By the induction hypothesis, mR and $N \cap L$ are of finite length. Therefore, N is of finite length too. Thus, M is locally noetherian.

(2) $\Rightarrow$ (3) Show that every module in $\sigma(M)$ is semiartinian. By [6, 3.12], it suffices to prove that M is semiartinian. Let M/N be a quotient module, and let N_0 be a nonzero finitely generated submodule in M/N. Then N_0 is a noetherian module. Hence, by [7, Proposition 10.14] we have $N_0 = N_1 \oplus \cdots \oplus N_k$, where N_i is indecomposable for every i. Since every indecomposable weakly regular nonradical module is obviously local, Lemma 4 implies $\mathrm{Soc}(N_0) \neq 0$. Thus, every quotient module of M possesses the nonzero socle, which proves that M is semiartinian.

Let N be a nonsemisimple module in $\sigma(M)$. Denote by A the set of all submodules in N, which are local and injective of length ≤ 2. By Zorn's lemma, we may choose a maximal subset A_0 in A with the property $\sum_{U \in A_0} U = \bigoplus_{U \in A_0} U$. Let $N_0 = \bigoplus_{U \in A_0} U$. Since M is locally noetherian by hypothesis, $N = N_0 \oplus L$ for a submodule L in N by [3, 27.3]. If L is nonsemisimple then L has an injective local submodule of length ≤ 2 by Lemma 5. This contradicts the choice of N_0. Thus, every module in $\sigma(M)$ is a direct sum of local modules of length ≤ 2. By [8, Theorem 2.4], each module in $\sigma(M)$ is lifting.

(3) $\Rightarrow$ (1) is immediate from [8, Theorem 2.4]. $\square$

Lemma 7. *If M is a right R-module then every nonzero quotient module of $\bigoplus_{\alpha \in A} I(M_\alpha)$, where $M \cong M_\alpha$ for every α, has a nonzero M-injective local submodule of length ≤ 2.*

PROOF. Let $L = \left(\bigoplus_{\alpha \in A} I(M_\alpha)\right)/N$ be a nonzero quotient module of $\bigoplus_{\alpha \in A} I(M_\alpha)$, and let φ be the canonical homomorphism from $\bigoplus_{\alpha \in A} I(M_\alpha)$ into L. Then there exists an index α_0 such that $\varphi i_{\alpha_0}(I(M_{\alpha_0})) \neq 0$, where i_{α_0} is the canonical embedding of $I(M_{\alpha_0})$ into $\bigoplus_{\alpha \in A} I(M_\alpha)$. Let γ be the least ordinal such that $\varphi i_{\alpha_0}(I_\gamma(M_{\alpha_0})) \neq 0$. It is clear that γ is a nonlimit ordinal. Then L contains a nonzero homomorphic image of $I_\gamma(M_\alpha)/I_{\gamma-1}(M_\alpha)$. Therefore, L possesses an M-injective local submodule of length ≤ 2. $\square$

Theorem 8. *Let M be a semiartinian right R-module, which is a generating object in $\sigma(M)$. Then the following are equivalent*:

(1) *every module in $\sigma(M)$ is weakly regular*;

(2) *every module in $\sigma(M/I(M))$ is lifting*;

(3) *each module in $\sigma(M/I(M))$ is the direct sum of some modules of length ≤ 2.*

PROOF. (1) $\Rightarrow$ (2) If $(M/I(M))/J((M/I(M)))$ has a nonzero M-injective submodule then it obviously has a simple M-injective submodule as well. Then by Lemmas 4 and 5 $M/I(M)$ has an M-injective local submodule of length ≤ 2 whose quotient module by the Jacobson radical is an M-injective module. Since $I_1(M/I(M)) = 0$, we get a contradiction. Thus, $(M/I(M))/J((M/I(M)))$ has no nonzero M-injective submodules. Therefore, it is a semisimple module by [5, Theorem 3.4]. Then $M/I(M)$ is semilocal, and the implication follows from Theorem 6.

The equivalence of (2) and (3) follows from [8, Theorem 2.4].

(3) $\Rightarrow$ (1) Show that every local module N in $\sigma(M/I(M))$ of length 2 is injective in $\sigma(M)$. Let $E(N)$ be the injective hull of N in $\sigma(M)$, and let φ be an epimorphism from $\bigoplus_{\alpha\in A} M_\alpha$ into $E(N)$, where $M \cong M_\alpha$ for every α. If $\varphi\big(\bigoplus_{\alpha\in A} I(M_\alpha)\big) = 0$ then $E(N)$ lies in $\sigma(M/I(M))$. By [8, Theorem 2.4], $E(N)$ is a lifting module, and $N \not\subset J(E(N))$ gives $E(N) = N$. In the case $\varphi\big(\bigoplus_{\alpha\in A} I(M_\alpha)\big) \neq 0$, $E(N)$ has an M-injective local submodule of length ≤ 2 by Lemma 7. The last fact obviously implies that $E(N) = N$.

Consider an arbitrary module N in $\sigma(M)$ which is not semisimple. By hypothesis, there is an epimorphism φ from $\bigoplus_{\alpha\in A} M_\alpha$ into N, where $M \cong M_\alpha$ for every α. If $\bigoplus_{\alpha\in A} I(M_\alpha) \subset \operatorname{Ker}\varphi$ then N lies in $M/I(M)$ and is not semisimple. Therefore, N has a nonzero injective submodule. Let $\bigoplus_{\alpha\in A} I(M_\alpha) \not\subset \operatorname{Ker}\varphi$. Then N has a nonzero M-injective local submodule by Lemma 7. Thus, by the argument above, an arbitrary nonsemisimple module N has a nonzero injective submodule. The implication follows now from [5, Theorem 3.4]. □

Lemma 9. *Let P be a finitely generated quasiprojective generalized right SV-module over R, let $S = \operatorname{End}_R(P)$ be a regular ring, and let $M \in \sigma(P)$. Then the right S-module* $\operatorname{Hom}_R(P, M)$ *is either semisimple or has a nonzero injective submodule.*

PROOF. Assume that the right S-module $\operatorname{Hom}_R(P, M)$ has no nonzero injective submodules. By transfinite induction we define the submodule M_α in M for every ordinal α as follows: If $\alpha = 0$ then $M_\alpha = 0$. If $\alpha = \beta + 1$ then $M_{\beta+1}/M_\beta$ is the sum of all P-injective submodules in M/M_β. If α is a limit ordinal then we put $M_\alpha = \bigcup_{\beta<\alpha} M_\beta$. Denote by M_0 the union of all these modules. By transfinite induction we show that $\operatorname{Hom}_R(P, M_\alpha) = 0$ for every ordinal α. If $\alpha = 0$ then the assertion is trivial. Let α be an ordinal and $\operatorname{Hom}_R(P, M_\beta) = 0$ for every $\beta < \alpha$. If α is a limit ordinal then $\operatorname{Hom}_R(P, M_\alpha) = 0$ is trivial. Assume that α is a nonlimit ordinal and $\alpha = \alpha_0 + 1$. By the induction hypothesis, $\operatorname{Hom}_R(P, M_{\alpha_0}) = 0$. Then by Lemma 1

$$\operatorname{Hom}_R(P, M_\alpha/M_{\alpha_0}) \cong \operatorname{Hom}_R(P, M_\alpha)/\operatorname{Hom}_R(P, M_{\alpha_0}) \cong \operatorname{Hom}_R(P, M_\alpha).$$

If $\operatorname{Hom}_R(P, M_\alpha) \neq 0$ then by Lemma 1 there exists a P-injective submodule L in M_α/M_{α_0} such that $\operatorname{Hom}_R(P, L) \neq 0$. Since $\operatorname{Hom}_R(P, L)$ is an injective S-module by Lemma 2; therefore $\operatorname{Hom}_R(P, M_\alpha)$ and, consequently, $\operatorname{Hom}_R(P, M)$ have nonzero P-injective submodules, which contradicts the initial hypothesis. Thus, $\operatorname{Hom}_R(P, M_\alpha) = 0$ for every ordinal α. Therefore, $\operatorname{Hom}_R(P, M_0) = 0$. Since M/M_0 has no P-injective submodules, M/M_0 is semisimple by [5, Theorem 3.4]. Then by Lemma 1 $\operatorname{Hom}_R(P, M/M_0)$ is a semisimple module. Hence, by $\operatorname{Hom}_R(P, M/M_0) \cong \operatorname{Hom}_R(P, M)$, $\operatorname{Hom}_R(P, M)$ is also semisimple. □

Lemma 10. *Let P be a finitely generated quasiprojective right R-module, let $S = \operatorname{End}_R(P)$ be a regular ring, and let N be a cyclic right S-module. Then there exists a right R-module M such that $M \in \sigma(P)$ and $N \cong \operatorname{Hom}_R(P, M)$.*

PROOF. By [3, 25.5], the homomorphism of the right S-modules $\phi : N \bigotimes_S S \to \operatorname{Hom}_R(P, N \bigotimes_S P)$ such that $n \otimes s \to [p \to n \otimes s(p)]$ is a monomorphism. It is clear that $N \bigotimes_S P \in \sigma(P)$. Thus, without loss of generality we may assume that N is a submodule in $\operatorname{Hom}_R(P, N \bigotimes_S P)$. Since N is a cyclic module, $N = \phi S$ for some $\phi \in \operatorname{Hom}_R(P, N \bigotimes_S P)$. Then $N = \operatorname{Hom}_R(P, Jm(\phi))$ by Lemma 1. □

Lemma 11. *Let P be a finitely generated quasiprojective generalized right SV-module over R, and let $S = \operatorname{End}_R(P)$ be a regular ring. Then S is a generalized right SV-ring.*

Proof. By [5, Theorem 3.4], it suffices to show that each cyclic nonsemisimple right S-module has a nonzero injective submodule. Let N be a cyclic nonsemisimple right S-module. Lemma 10 implies the isomorphism $N \cong \operatorname{Hom}_R(P, M)$ for some $M \in \sigma(P)$. By Lemma 9, N has a nonzero injective submodule. □

Theorem 12. *If P is a quasiprojective generalized SV-module then P is a semiartinian module.*

Proof. Assume that $P \neq L(P)$. Denote by M the quotient module $P/L(P)$ which is quasiprojective by [3, 18.2]. By [5, Lemma 3.3], $J(M) = 0$. Since M is nonsemisimple, M has a nonzero cyclic quasiprojective P-injective submodule mR with $m \in M$ by [5, Theorem 3.4]. By [3, 22.1 and 22.2], $\operatorname{End}_R(mR)$ is a regular ring. By Lemma 11 and [5, Theorem 3.7], $\operatorname{End}_R(mR)$ is a right SV-ring. Then $\operatorname{End}_R(mR)$ has a primitive idempotent e and emR is a simple module, which contradicts $\operatorname{Soc}(M) = 0$. □

Theorem 13. *Given a quasiprojective right R-module P, the following are equivalent*:
(1) *P is a generalized SV-module*;
(2) *if M is a generating object in $\sigma(P)$ then every module in $\sigma(M/I(M))$ is lifting.*

Proof. The equivalence of (1) and (2) is immediate from Theorems 8 and 12. □

The following assertion is straightforward from the previous theorem and [8, Corollary 2.5].

Corollary 14. *Given a ring R, the following are equivalent*:
(1) *R is a generalized right SV-ring*;
(2) *$R/I(R)$ is artinian uniserial ring and $J^2(R/I(R)) = 0$*;
(3) *every right module over $R/I(R)$ is lifting.*

Corollary 15 [9]. *Given a quasiprojective right R-module P, the following are equivalent*:
(1) *P is an SV-module*;
(2) *each nonzero module in $\sigma(P)$ has a nonzero P-injective submodule.*

Proof. (1) $\Rightarrow$ (2) By [6, 3.12], every module in $\sigma(P)$ is semiartinian. Then the implication follows from the fact that every nonzero module in $\sigma(P)$ has a simple P-injective submodule.

(2) $\Rightarrow$ (1) It is easy to see that P is a generalized SV-module. Therefore, by Theorem 12, P is semiartinian. By hypothesis, we deduce immediately that every simple module in $\sigma(P)$ is P-injective, i.e., P is a V-module. □

The right SV-rings and the artinian uniserial rings with the zero square of the Jacobson radical are some examples of generalized SV-rings. We give an example of a generalized SV-ring which is distinct from the examples above.

Example 16. Let R be a classically semisimple ring, and let R_0 be an artinian uniserial subring in R such that $J^2(R_0) = 0$. Consider the ring $S = \prod_{i \geq 1} R_i$, where $R_i = R$ for every i, and the subring $T = \{a \in S \mid \exists N \forall i, j > N : a_i = a_j \& a_i \in R_0\}$ in S. By [5, Lemma 1.2], we immediately deduce that $\operatorname{Soc}(T) = I_1(T) = \bigoplus_{i \geq 1} R_i$.

Let N be an injective right $T/\operatorname{Soc}(T)$-module which may be naturally considered as a right T-module. Consider the embedding $N \subset E(N)$, where $E(N)$ is the injective hull of the right T-module N. If $E(N)\operatorname{Soc}(T) \neq 0$ then $E(N)e \neq 0$ for some primitive idempotent e in $\operatorname{Soc}(T)$. Since N is essential in $E(N)$ and e is a central idempotent in T; therefore $Ne \neq 0$, which contradicts $N\operatorname{Soc}(T) = 0$. The so-obtained contradiction shows that $E(N)\operatorname{Soc}(T) = 0$. Therefore, we may consider $E(N)$ as a module over $T/\operatorname{Soc}(T)$. Since N is an injective right $T/\operatorname{Soc}(T)$-module, we have $N = E(N)$.

Thus, every module injective over $T/\operatorname{Soc}(T)$ is injective over T. In particular, $I_2(T)/I_1(T) = I_1(T/\operatorname{Soc}(T)_{T/\operatorname{Soc}(T)})$. Therefore, $I(T) = I_2(T) = \{a \in S \mid \exists N \forall i, j > N : a_i = a_j \& a_i \in I_1(R_0)\}$ and $T/I(T) \cong R_0/I(R_0)$. By Corollary 14, S is a generalized SV-ring. Notice that the case when $R = M_2(P)$ and R_0 is the ring of uppertriangular matrices of order 2 over a field P was considered in [10] as an example of a semiartinian nonregular ring with the zero square of the Jacobson radical.

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