Coring stabilizers for a Hopf algebra coaction

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Let H be a Hopf algebra and A a right H-comodule algebra. Suppose that P is a prime ideal of A such that the factor ring A/P is either right or left Goldie, so that A/P has a simple artinian classical right or left quotient ring Q_P. We will introduce two Q_P-corings $\mathrm{Stab}(P)$ and $\mathrm{Inert}(P)$ called the *stabilizer* and the *inertializer* of P. The latter is a factor coring of the former. When A is commutative and H is the function algebra on a finite group, these corings are described explicitly in terms of the set-theoretic stabilizer and the inertia group of P with respect to the group action corresponding to the coaction of H on A. When A is commutative and H is an arbitrary commutative Hopf algebra, $\mathrm{Inert}(P)$ coincides with the Hopf algebra over the field Q_P representing the scheme-theoretic stabilizer (inertia group) of P in the group scheme represented by H. It seems that in the noncommutative case too much of information about the original coaction of H may get lost in $\mathrm{Inert}(P)$, and we will not use the inertializers in any way.

Our interest in stabilizers stems from the desire to generalize basic facts about homogeneous spaces. Consider a right action $X \times G \to G$ of a group scheme G of finite type over a field k on a k-scheme X. If $x \in X$ is a rational point such that the corresponding orbit morphism $G \to X$ is surjective and flat, then X is isomorphic with the quotient $G_x\backslash G$ where G_x stands for the stabilizer of x in G. In this case the category of G-linearized quasicoherent sheaves on X is equivalent to the category of G_x-modules [7]. However, if x is a point with residue field strictly larger than k, then the former category cannot be recovered from the latter alone since a kind of descent datum is additionally needed.

The coring stabilizers $\mathrm{Stab}(P)$ remedy this defect of the inertializers. One could anticipate the appearance of corings in this question, knowing a description of descent data for a ring extension in terms of comodule structures over the Sweedler coring (see [3, 25.4] or [4]). Theorems 3.1 and 4.4 establish, under suitable hypotheses, two category equivalences

$$\mathcal{M}_A^H/\,\mathcal{T}_A^H \approx \mathcal{M}^{\mathrm{Stab}(P)}, \qquad {}_A\mathcal{M}/{}_A\mathcal{T} \approx {}^{\mathrm{Stab}(P)}_{\;\;H}\mathcal{M}\,.$$

In general we denote by $\mathcal{M}_R$ and ${}_R\mathcal{M}$ the categories of right and left modules over a ring R, by $\mathcal{M}^C$ and ${}^C\mathcal{M}$ the categories of right and left comodules over a coring C, by $\mathcal{M}_A^H$ and ${}^C_H\mathcal{M}$ the categories of Hopf modules. The localizing subcategories $\mathcal{T}_A^H$ and ${}_A\mathcal{T}$ are defined by the filters of H-costable right and left ideals of A containing a regular element. From the viewpoint of noncommutative geometry the quotient categories $\mathcal{M}_A^H/\,\mathcal{T}_A^H$ and ${}_A\mathcal{M}/{}_A\mathcal{T}$ represent quasicoherent sheaves on a “noncommutative scheme”. The assumption that P contains no nonzero H-costable ideals of A means that this scheme is sufficiently close to a homogeneous space.

If P is a maximal ideal of codimension 1 in A, then the aforementioned results reduce to the ungraded versions of Theorems 0.1, 0.4 in [23] since in this case A is isomorphic to a right coideal subalgebra of H. Certainly, the previous condition on P imposes serious limitations, especially for noncommutative algebras. Unlike [23] we do not consider the graded versions of results in this paper.

The proof of the second equivalence is based on the first equivalence in one crucial argument concerning the flatness of a certain ring extension of A. There are also several other interesting consequences of the first equivalence. It is proved in section 3 that $\operatorname{Tor}_i^A(M, Q_P)$ and $\operatorname{Ext}_A^i(M, Q_P)$ vanish for all $M \in \mathcal{M}_A^H$ and $i > 0$. If P' is a second prime ideal of A satisfying the same hypotheses as P in Theorem 3.1, then $\operatorname{Stab}(P)$ and $\operatorname{Stab}(P')$ are Morita-Takeuchi equivalent.

In section 5 we look at birational H-coequivariant extensions by which we mean embeddings of right H-comodule algebras $A \hookrightarrow B$ with the same classical quotient ring $Q(A) \cong Q(B)$. We are interested to find objects associated with H-comodule algebras which are preserved in such extensions. So it is shown in Proposition 5.4 that the quotient categories $\mathcal{M}_A^H / \mathcal{T}_A^H$ and $\mathcal{M}_A / \mathcal{T}_A$ are birational invariants of A. Still deeper is the correspondence between the prime ideals in a birational extension which makes the content of Theorem 5.6. There is a subset $\operatorname{Spec}' A$ of the prime spectrum $\operatorname{Spec} A$ which exhibits desired behaviour. These results support the intuitive idea that comodule algebras are relevant only up to birational equivalence when we view them as models of noncommutative homogeneous spaces.

The notion of relative Hopf modules has proved to be of fundamental importance since the work of Takeuchi [26] and Doi [10]. A special attention was given to the categories of Hopf modules in the case of Hopf Galois extensions and their generalizations [1], [5], [12], [19], [20]. If the H-comodule algebra A is an H-Galois extension, then $\operatorname{Stab}(P)$ is generated as a Q_P-bimodule by its distinguished grouplike element, i.e. $\operatorname{Stab}(P)$ is a homomorphic image of the Q_P-coring $Q_P \otimes Q_P$. For general comodule algebras the stabilizers may be quite complicated.

The main results of sections 3 and 4 assume the base ring k to be a field. In all other results k is an arbitrary commutative ring. The tensor product $\otimes_k$ is abbreviated to $\otimes$.

1. Localizing filters and classical quotient rings

We recall some terminology which will be used in this paper. Let R be a ring. A nonempty set $\mathcal{I}$ of right ideals of R is a *localizing filter* (also called an *idempotent topologizing filter* or a *Gabriel topology*) if the following four conditions are satisfied:

(T1) *if* $J \in \mathcal{I}$ *then* $\mathcal{I}$ *contains each right ideal* I *with* $J \subset I$,

(T2) *if* $I, J \in \mathcal{I}$ *then* $I \cap J \in \mathcal{I}$,

(T3) *if* $I \in \mathcal{I}$ *then for each* $a \in R$ *there exists* $I_a \in \mathcal{I}$ *such that* $aI_a \subset I$,

(T4) *if* $J \in \mathcal{I}$ *and* I *is a right ideal such that for each* $a \in J$ *there exists* $I_a \in \mathcal{I}$ *satisfying* $aI_a \subset I$, *then* $I \in \mathcal{I}$.

In any right R-module V the subset $\tau_{\mathcal{I}}(V)$ consisting of those elements $v \in V$ whose annihilator in R belongs to $\mathcal{I}$ is a submodule called the *$\mathcal{I}$-torsion submodule.* The R-module V is said to be *$\mathcal{I}$-torsion* when $\tau_{\mathcal{I}}(V) = V$, and V is *$\mathcal{I}$-torsion-free* when $\tau_{\mathcal{I}}(V) = 0$. Axiom (T4) ensures that the class of $\mathcal{I}$-torsion modules is closed under extensions. When a set of right ideals satisfies (T1)–(T3), but not necessarily (T4), it is called a *topologizing filter*.

A full subcategory of a Grothendieck category is *localizing* if and only if it is closed under subobjects, factor objects, extensions and small direct limits [13, Ch. III, Prop. 8]. A full subcategory of $\mathcal{M}_R$ is localizing precisely when it consists of the torsion modules for some localizing filter of right ideals of R [13, Ch. V, p. 412].

Denote by $\Sigma(R)$ the set of regular elements, i.e. nonzerodivisors, of R. If $\Sigma(R)$ satisfies the right Ore condition, then the set $\mathcal{G}(R)$ of those right ideals of R which intersect $\Sigma(R)$ is a localizing filter. The ring of fractions $Q(R) = R\Sigma(R)^{-1}$ is called the classical right quotient ring of R (occasionally $Q(R)$ will stand for the classical left quotient ring of R). More generally, any overring of R isomorphic to $Q(R)$ is a *classical right quotient ring* of R. If $\Sigma(R)$ satisfies both the right and the left Ore conditions, then $Q(R)$ is the *classical two-sided quotient ring*.

Let H be a k-flat Hopf algebra with a bijective antipode $S : H \to H$ over the base ring k, and let A be a right H-comodule algebra. The comodule structure map $\rho : A \to A \otimes H$ is a homomorphism of algebras. All algebras are assumed to be associative and unital. Denote by $\mathcal{G}_H(A)$ the set of right ideals of A characterized by the property that a right ideal I belongs to $\mathcal{G}_H(A)$ if and only if there exists an H-costable right ideal $I' \in \mathcal{G}(A)$ such that $I' \subset I$. Since the largest H-subcomodule I_H of A contained in I is a right ideal, we have $I \in \mathcal{G}_H(A)$ if and only if $I_H \in \mathcal{G}(A)$, which can also be rewritten as $I_H \cap \Sigma(A) \neq \varnothing$. Note that

$$I_H = \rho^{-1}(I \otimes H).$$

By definition $\mathcal{G}_H(A) \subset \mathcal{G}(A)$ and an H-costable right ideal of A belongs to $\mathcal{G}_H(A)$ if and only if it belongs to $\mathcal{G}(A)$.

Assuming that $\Sigma(A)$ satisfies the right Ore condition, we denote by $\mathcal{T}_A$ the class of $\mathcal{G}_H(A)$-torsion right A-modules (thus $\mathcal{T}_A$ actually depends on H). Let $\mathcal{T}_A^H$ be the class of right (H, A)-Hopf modules which are $\mathcal{G}_H(A)$-torsion in $\mathcal{M}_A$.

Lemma 1.1. *If $\Sigma(A)$ is right Ore, then $\mathcal{G}_H(A)$ is a localizing filter. In this case $\mathcal{T}_A$ and $\mathcal{T}_A^H$ are localizing subcategories, respectively, of $\mathcal{M}_A$ and of $\mathcal{M}_A^H$.*

Proof. We know that $\mathcal{G}(A)$ is a localizing filter. Properties (T1), (T2) of $\mathcal{G}_H(A)$ are immediate from the respective properties of $\mathcal{G}(A)$ since the set of H-subcomodules of A is closed under finite intersections. For each right ideal I of A and an element $a \in A$ put $I_a = \{x \in A \mid ax \in I\}$, and let $\lambda_a : A \to A$ denote the left multiplication by a. Then I_a is the unique right ideal of A such that λ_a induces an injective map $A/I_a \to A/I$. Since $\lambda_a \otimes H$ coincides with the left multiplication by $a \otimes 1$ in the algebra $A \otimes H$ and induces an injective map $A/I_a \otimes H \to A/I \otimes H$, we get

$$I_a \otimes H = \{y \in A \otimes H \mid (a \otimes 1)y \in I \otimes H\}.$$

Suppose that $I \in \mathcal{G}(A)$ is an H-costable right ideal. Given $a \in A$, we can find a finitely generated k-submodule $U \subset A$ such that $\rho(a) \in U \otimes H$. Put $I_U = \bigcap_{u \in U} I_u$, so that $UI_U \subset I$. If $u_1, \ldots, u_n$ generate U, then $I_U = I_{u_1} \cap \cdots \cap I_{u_n}$. Since $\mathcal{G}(A)$ satisfies (T3), we have $I_u \in \mathcal{G}(A)$ for all $u \in A$. Therefore $I_U \in \mathcal{G}(A)$ by (T2). An easy calculation in the algebra $A \otimes H$ shows that

$$
\begin{aligned}
(a \otimes 1) \cdot \rho(x) &= \sum \big(a_{(0)} \otimes S^{-1}(a_{(2)})\, a_{(1)}\big) \cdot \rho(x) \\
&= \sum (1 \otimes S^{-1} a_{(1)}) \cdot \rho(a_{(0)} x) \in (1 \otimes H) \cdot \rho(U I_U) \subset I \otimes H
\end{aligned}
$$

for all $x \in I_U$. Hence $\rho(I_U) \subset I_a \otimes H$, and therefore $I_U \subset (I_a)_H$. The last inclusion entails $(I_a)_H \in \mathcal{G}(A)$, whence $I_a \in \mathcal{G}_H(A)$. This proves (T3).

Now let $J \in \mathcal{G}_H(A)$, and let I be any right ideal of A such that $I_a \in \mathcal{G}_H(A)$ for each $a \in J$. Pick any element s in the nonempty set $J_H \cap \Sigma(A)$. Since $\rho(s) \in J \otimes H$, we can find a finitely generated k-submodule $U \subset J$ such that $\rho(s) \in U \otimes H$. Note that $s \in U' \subset U$ where $U' = \rho^{-1}(U \otimes H)$ is an H-subcomodule of A. The right ideal $I_U = \bigcap_{u \in U} I_u$ belongs to $\mathcal{G}_H(A)$ by (T2) and satisfies $U I_U \subset I$. Hence there exists an H-costable $K \in \mathcal{G}(A)$ such that $U'K \subset UK \subset I$. Since the multiplicatively closed set $\Sigma(A)$ intersects both U' and K, we get $U'K \cap \Sigma(A) \neq \varnothing$. But $U'K$ is an H-costable right ideal of A, so that $I \in \mathcal{G}_H(A)$. Thus (T4) also holds. □

Lemma 1.2. *We have* $\mathcal{T}_A^H = \{M \in \mathcal{M}_A^H \mid M \otimes_A Q(A) = 0\}$.

Proof. A right A-module V satisfies $V \otimes_A Q(A) = 0$ if and only if each element of V is annihilated by an element of $\Sigma(A)$. In particular, $V \otimes_A Q(A) = 0$ whenever $V \in \mathcal{T}_A$. Conversely, suppose that $M \otimes_A Q(A) = 0$ where $M \in \mathcal{M}_A^H$, and denote by I the annihilator in A of some element $v \in M$. Let $U \subset M$ be any finitely generated k-submodule such that $\delta(v) \in U \otimes H$ where $\delta : M \to M \otimes H$ denotes the comodule structure map. There exists $J \in \mathcal{G}(A)$ such that $UJ = 0$. We may regard $M \otimes H$ as an $(H, A \otimes H)$-bimodule. Since $\delta(ma) = \delta(m)\rho(a)$ for all $m \in M$ and $a \in A$, we get

$$
(v \otimes 1) \cdot \rho(x) = \sum \big(v_{(0)} \otimes S^{-1}(v_{(2)})\, v_{(1)}\big) \cdot \rho(x) = \sum S^{-1}(v_{(1)}) \cdot \delta(v_{(0)} x) = 0
$$

for all $x \in J$. By the k-flatness of H the annihilator in $A \otimes H$ of $v \otimes 1 \in M \otimes H$ coincides with $I \otimes H$. Hence $\rho(J) \subset I \otimes H$, i.e. $J \subset I_H$. It follows that $I_H \in \mathcal{G}(A)$, and therefore $I \in \mathcal{G}_H(A)$. Thus $M \in \mathcal{T}_A^H$. □

When $\Sigma(A)$ satisfies the left Ore condition, there is a localizing filter $\mathcal{G}_H^l(A)$ of left ideals of A defined similarly to $\mathcal{G}_H(A)$. In this case the class ${}_A\mathcal{T}$ of $\mathcal{G}_H^l(A)$-torsion left A-modules is a localizing subcategory of ${}_A\mathcal{M}$.

Included below are two subsidiary results on the existence of classical quotient rings. Proposition 1.4 will be used in the proof of Theorem 5.6. A special case of this result was given in [24, Prop. 7.1]. Proposition 1.5 will allow us to shorten slightly the hypotheses in the results of sections 3 and 4. We will denote by $\operatorname{rann}_Q S$ and $\operatorname{lann}_Q S$, respectively, the right and left annihilators of a subset S in a ring Q.

Lemma 1.3. *Suppose that R is a subring of a right artinian ring Q. Then every right ideal I of R contains an element x such that $Q = xQ \oplus \operatorname{rann}_Q x$ and the right ideal $\operatorname{rann}_I x = I \cap \operatorname{rann}_Q x$ of R is nilpotent. If Q is a classical right quotient ring of R and $I \in \mathcal{G}(R)$ then $x \in \Sigma(R)$ for any such x.*

Proof. Put $X = \{x \in Q \mid \operatorname{rann}_Q x = \operatorname{rann}_Q x^2\}$. Each $u \in Q$ satisfies $u^n \in X$ for sufficiently large $n > 0$ since in Q the ascending chain of right ideals $\operatorname{rann}_Q u^i$ with $i = 1, 2, \ldots$ has to stabilize. The condition on x in the definition of X means precisely that $xQ \cap \operatorname{rann}_Q x = 0$. Since $xQ \cong Q/\operatorname{rann}_Q x$ in $\mathcal{M}_Q$, we have the equality between the composition series lengths

$$\operatorname{length} xQ + \operatorname{length} \operatorname{rann}_Q x = \operatorname{length} Q.$$

It follows that

$$X = \{x \in Q \mid Q = xQ \oplus \operatorname{rann}_Q x\}.$$

Now pick $x \in I \cap X$ for which the right ideal $\operatorname{rann}_Q x$ of Q is minimal possible. Let $u \in \operatorname{rann}_I x$. There is an integer $n > 0$ such that $v = u^n$ lies in X. Since $xv = 0$, we have $xQ \cap vQ = 0$. Then $y = x + v$ also lies in X (an easy check is given in the proof of [24, Lemma 7.5]), actually $y \in I \cap X$, with

$$\operatorname{rann}_Q y = \operatorname{rann}_Q x \bigcap \operatorname{rann}_Q v.$$

In particular, $\operatorname{rann}_Q y \subset \operatorname{rann}_Q x$. If $v \neq 0$ then the previous inclusion is proper since $yv = v^2 \neq 0$. But this is impossible by the choice of x. Thus $u^n = 0$. We conclude that $\operatorname{rann}_I x$ is nil. By [15] every nil multiplicatively closed subset of a right artinian ring is nilpotent. In particular, this applies to $\operatorname{rann}_I x$.

Suppose that Q is a classical right quotient ring of R and $I \in \mathcal{G}(R)$. The first of these two assumptions implies that the nil radical N of R is contained in the Jacobson radical J of Q. Since all elements of $\Sigma(R)$ are invertible in Q, the second assumption entails $IQ = Q$. Given any $q \in Q$, it is possible to find $s \in \Sigma(R)$ such that $qs \in I$. If $q \in \operatorname{rann}_Q x$, then $qs \in \operatorname{rann}_I x$. Hence $\operatorname{rann}_Q x = (\operatorname{rann}_I x)\, Q$. Since $\operatorname{rann}_I x$ is a nil right ideal of R, we have $\operatorname{rann}_I x \subset N \subset J$. Then $\operatorname{rann}_Q x \subset J$ as well. Since J is nilpotent, so too is $\operatorname{rann}_Q x$. On the other hand, $\operatorname{rann}_Q x = eQ$ for a suitable idempotent $e \in Q$ since $\operatorname{rann}_Q x$ is an $\mathcal{M}_Q$-direct summand of Q. It follows that $e = 0$, i.e. $\operatorname{rann}_Q x = 0$. Thus x is right regular in Q. By [17, Prop. 3.1.1] right regular elements of a right artinian ring are invertible. So x is invertible in Q, and therefore $x \in \Sigma(R)$. □

Proposition 1.4. *Let B be a ring with a quasi-Frobenius classical right quotient ring Q. Suppose that R is a subring of B and $\mathcal{I}$ is a topologizing filter of right ideals of R with the following properties*:

(a) *each $I \in \mathcal{I}$ has zero left and right annihilators in Q,*

(b) *for each $b \in B$ there exists $I \in \mathcal{I}$ such that $bI \subset R$.*

Then $I \cap \Sigma(R) \neq \varnothing$ for all $I \in \mathcal{I}$ and Q is a classical right quotient ring of R.

Proof. Since Q is quasi-Frobenius, each left (right) ideal of Q is the left (right) annihilator of a uniquely determined right (left) ideal. Denote by N the nil radical of R. It is a nilpotent two-sided ideal of R containing every nil right ideal of R (such an ideal exists because R is a subring of an artinian ring). There are several intermediate steps in the proof:

Step 1. *$QI = IQ = Q$ for all $I \in \mathcal{I}$.*

Step 2. *If $I \in \mathcal{I}$ and J is any right ideal of Q then $J = (J \cap I) \cdot Q$.*

Step 3. *$QT = TQ$ for any two-sided ideal T of R.*

Step 4. $\operatorname{rann}_Q N = \operatorname{lann}_Q N$.

By (a) $\operatorname{rann}_Q QI = 0$ and $\operatorname{lann}_Q IQ = 0$ for $I \in \mathcal{I}$. Therefore Step 1 is immediate from the bijective correspondence between the left and the right ideals of Q.

In Step 2 let $q \in J$. Since $Q \cong B\Sigma(B)^{-1}$, there exists $s \in \Sigma(B)$ for which $qs \in B$. According to (b) we can find $I' \in \mathcal{I}$ with the property $qsI' \subset R$. Since s is invertible in Q, we have $sI'Q = sQ = Q$ by Step 1, whence $q \in qsI'Q \subset (J \cap R)\,Q$. Hence $J = (J \cap R)\,Q$. Furthermore, if $q \in J \cap R$, then (T3) allows us to find $I'' \in \mathcal{I}$ such that $qI'' \subset J \cap I$. In this case $q \in qI''Q \subset (J \cap I)\,Q$ since $1 \in I''Q$ by Step 1. So $J \cap R \subset (J \cap I)\,Q$, and the desired conclusion follows.

Step 3 generalizes [24, Lemma 7.3]. First we note that the inclusion $BT \subset TQ$ implies that $QT \subset TQ$, in which case TQ is a two-sided ideal of Q. Assuming that $BT \subset TQ$, we have $sTQ \subset TQ$ for any $s \in \Sigma(B)$. If sTQ were properly contained in TQ, then we would get an infinite strictly descending chain of right ideals s^iTQ of Q with $i = 0, 1, \ldots$, but this is impossible since Q is artinian. Hence $sTQ = TQ$, and therefore $s^{-1}T \subset TQ$. Then $qT \subset TQ$ for any $q \in Q$ since q can be written as bs^{-1} for some $b \in B$ and $s \in \Sigma(B)$.

Suppose now that there exist two-sided ideals T of R such that $QT \not\subset TQ$. Let us choose such a T with the additional property that the right ideal TQ of Q is minimal possible. If we had $ITQ = TQ$ for all $I \in \mathcal{I}$ then, taking $I_b \in \mathcal{I}$ such that $bI_b \subset R$, we would get $bTQ = bI_bTQ \subset RTQ = TQ$ for each $b \in B$, but this yields the inclusion $BT \subset TQ$ contradicting our previous observation. Hence there exist $I \in \mathcal{I}$ for which $ITQ \neq TQ$. We pick such an I with the additional property that the right ideal ITQ of Q is minimal possible.

If $J \in \mathcal{I}$ is arbitrary then $(I \cap J)TQ = ITQ$ by the minimality assumption since $I \cap J \in \mathcal{I}$ by (T2). It follows that $ITQ \subset JTQ$. Given any $a \in R$, we can use (T3) to find $J \in \mathcal{I}$ such that $aJ \subset I$ and get $aITQ \subset aJTQ \subset ITQ$. Thus $RITQ = ITQ$. So the two-sided ideal $V = RIT$ of R satisfies $VQ = ITQ \neq TQ$. On the other hand, $VQ \subset TQ$ because $V \subset T$. The choice of T ensures that VQ has to be a two-sided ideal of Q. Since $QI = Q$ by Step 1, we get $QT = QIT \subset QV \subset VQ \subset TQ$, a contradiction.

We conclude that $QT \subset TQ$ for any two-sided ideal T of R. Now $QT = \operatorname{lann}_Q K$ where we put $K = \operatorname{rann}_Q QT$. As $TRK \subset TK = 0$, we have $RK \subset K$; hence $K \cap R$ is a two-sided ideal of R. By Step 2 $K = (K \cap R)Q$. As we have proved already, this implies that K is a two-sided ideal of Q. Then so is its left annihilator QT. Hence $TQ \subset QT$ as well.

In Step 4 denote by J the Jacobson radical of Q. Since $J \cap R$ is a nilpotent two-sided ideal of R, we have $J \cap R \subset N$. Step 2 yields $J = (J \cap R)Q \subset NQ$. By Step 3 $NQ = QN$. Hence NQ is a nilpotent ideal of Q, which implies that $NQ \subset J$. Thus $J = NQ = QN$. It follows that the right and the left annihilators of N in Q are equal to the respective annihilators of J. However, $\operatorname{lann}_Q J = \operatorname{rann}_Q J$ since the right and the left socles of a quasi-Frobenius ring coincide.

Now we show that each $I \in \mathcal{I}$ contains an element invertible in Q, and therefore $I \cap \Sigma(R) \neq \varnothing$. By Lemma 1.3 there exists $x \in I$ such that $Q = xQ \oplus \operatorname{rann}_Q x$ and $\operatorname{rann}_I x$ is nil. In particular, $\operatorname{rann}_I x \subset N$. Suppose that the left ideal $L = \operatorname{lann}_Q x$ of Q is nonzero. Since N is nilpotent, there exists $0 \neq t \in L$ such that $Nt = 0$. Step 4 allows us to deduce that $tN = 0$ as well. Then $t\,(\operatorname{rann}_I x) = 0$. By Step 2 we have $\operatorname{rann}_Q x = (\operatorname{rann}_I x)\,Q$. Hence $tq = 0$ for all $q \in \operatorname{rann}_Q x$. Since $tx = 0$, it follows that $0 = t(xQ + \operatorname{rann}_Q x) = tQ$. This entails $t = 0$, a contradiction. Thus $L = 0$, i.e. x is left regular in Q. But left regular elements of a left artinian ring are invertible [17, 3.1.1]. Hence x is invertible in Q, as desired.

To conclude that Q is a classical right quotient ring of R it suffices to check that R is a right order in Q, i.e. every $q \in Q$ can be written as $q = au^{-1}$ for some $a, u \in R$ such that u is invertible in Q [17, 3.1.4]. We have $qs \in B$ for a suitable $s \in \Sigma(B)$. By (b) there exist $I', I'' \in \mathcal{I}$ such that $sI' \subset R$ and $qsI'' \subset R$. By (T2) $I' \cap I'' \in \mathcal{I}$. Let $v \in I' \cap I''$ be any element invertible in Q. Then we may take $u = sv$ and $a = qsv$, completing the proof. □

Proposition 1.5. *Let A, B be two k-algebras where B is faithfully k-flat. If $A \otimes B$ has a right artinian classical right quotient ring then so does A as well. Moreover, the canonical map $i : A \to A \otimes B$, $a \mapsto a \otimes 1$, extends to an injective ring homomorphism $Q(A) \to Q(A \otimes B)$.*

Proof. Put $Q = Q(A \otimes B)$ for short and denote by $\mathcal{I}$ the set of those right ideals I of A for which $I \otimes B \in \mathcal{G}(A \otimes B)$. Then $\mathcal{I}$ is a localizing filter. In the subsequent argument we will use only property (T3) of $\mathcal{I}$ which is checked as follows. For each right ideal I of A and an element $a \in A$ put $I_a = \{x \in A \mid ax \in I\}$. As in the proof of Lemma 1.1 we have

$$I_a \otimes B = \{y \in A \otimes B \mid (a \otimes 1)y \in I \otimes B\}.$$

If $I \in \mathcal{I}$, then property (T3) of $\mathcal{G}(A \otimes B)$ yields $I_a \otimes B \in \mathcal{G}(A \otimes B)$, so that $I_a \in \mathcal{I}$.

The canonical ring homomorphism $i : A \to A \otimes B$ is injective. Indeed, it suffices to check that $i \otimes B : A \otimes B \to A \otimes B \otimes B$ is injective. But the latter map admits a retraction arising from the multiplication in B. Thus A may be identified with a subring in $A \otimes B$ and in Q. The left multiplication $\lambda_a : A \to A$ by $a \in A$ is injective if and only if $\lambda_a \otimes B$ is injective, and a similar observation is valid for the right multiplications. This shows that

$$\Sigma(A) = \{a \in A \mid a \otimes 1 \in \Sigma(A \otimes B)\}.$$

Each $I \in \mathcal{I}$ intersects $\Sigma(A)$. To prove this claim we first apply Lemma 1.3 to get an element $x \in I$ such that $Q = (x \otimes 1)Q \oplus \operatorname{rann}_Q(x \otimes 1)$ and the right ideal $\operatorname{rann}_I x$ of A is nilpotent. The k-flatness of B ensures that $\operatorname{rann}_{I \otimes B}(x \otimes 1) = (\operatorname{rann}_I x) \otimes B$ is a nilpotent right ideal of $A \otimes B$. Lemma 1.3 applied this time to the subring $A \otimes B$ of Q yields $x \otimes 1 \in \Sigma(A \otimes B)$, i.e. $x \in \Sigma(A)$.

If $s \in \Sigma(A)$, then $sA \in \mathcal{I}$ since $s \otimes 1 \in \Sigma(A \otimes B)$. Taking $I = sA$, we get $I_a \in \mathcal{I}$ for any $a \in A$. As we have proved already, this implies that $I_a \cap \Sigma(A) \neq \varnothing$. In other words, for each pair s, a there exists $t \in \Sigma(A)$ such that $at \in sA$, i.e. $\Sigma(A)$ satisfies the right Ore condition. So A has a classical right quotient ring $Q_A = Q(A)$. Since i is injective and takes regular elements to regular ones, Q_A embeds in Q.

Let I be any right ideal of A. The right ideal $IQ \cap A$ consists of those $a \in A$ for which there exists $u \in \Sigma(A \otimes B)$ such that $(a \otimes 1)u \in I \otimes B$. This condition on a can be rewritten, in the earlier notation, as $(I_a \otimes B) \cap \Sigma(A \otimes B) \neq \varnothing$, and therefore as $I_a \in \mathcal{I}$. As we have seen, the last inclusion implies that $I_a \cap \Sigma(A) \neq \varnothing$. Thus for each $a \in IQ \cap A$ there exists $s \in \Sigma(A)$ such that $as \in I$. Since all right ideals of Q_A are extensions of right ideals of A, we get $JQ \cap Q_A = J$ for each right ideal J of Q_A. Hence the assignment $J \mapsto JQ$ embeds the lattice of right ideals of Q_A into that of Q. Since Q is right artinian, so too is Q_A. □

2. Definition of stabilizers and some special cases

A thorough treatment of corings and comodules is given by Brzeziński and Wisbauer [3]. We refer to this book for all details. Some aspects of the coring theory are also covered in a book of Caenepeel, Militaru and Zhu [6].

Let R be a ring. An R-coring is an (R,R)-bimodule C together with two bimodule homomorphisms $\Delta : C \to C \otimes_R C$ and $\varepsilon : C \to R$ satisfying the coassociativity and the counit conditions. A right C-comodule is a right R-module V together with a right R-linear map $\delta : V \to V \otimes_R C$ satisfying the coassociativity and the counit conditions. If C is left R-flat, then the category $\mathcal{M}^C$ of right C-comodules is Grothendieck and the forgetful functor $\mathcal{M}^C \to \mathcal{M}_R$ is exact [3, 18.6]. All corings considered in this paper will satisfy the flatness condition. For each right R-module X we will view $X \otimes_R C$ as a right C-comodule with respect to the map $\mathrm{id} \otimes_R \Delta$.

Given an R-coring C and a ring homomorphism $R \to R'$, the (R',R')-bimodule $C' = R' \otimes_R C \otimes_R R'$ has a natural R'-coring structure obtained by extending Δ and ε to C' [3, 17.2]. There is a functor

$$\Phi : \mathcal{M}^C \to \mathcal{M}^{C'}, \qquad \Phi = ? \otimes_R R'.$$

If C is left R-flat, then Φ has a right adjoint

$$\Psi : \mathcal{M}^{C'} \to \mathcal{M}^C, \qquad \Psi = ? \,\Box_{C'} (R' \otimes_R C)$$

(see [3, 23.9 and 24.11]). Here $\Box_{C'}$ stands for the cotensor product and $R' \otimes_R C$ is viewed as a (C',C)-bicomodule.

In the next lemma we encounter the assumption about the flatness of injective modules. Rings with this property were studied by Colby [8]. The class of such rings contains all von Neumann regular rings and all quasi-Frobenius rings.

Lemma 2.1. *Assume C to be left R-flat and C' to be left R'-flat. Suppose also that all injectives in $\mathcal{M}_{R'}$ are flat. Then the counit of adjunction $\eta_E : \Phi\Psi(E) \to E$ is an isomorphism whenever E is an injective in $\mathcal{M}^C$. If Φ is exact, then Ψ is fully faithful.*

Proof. First of all, $\Psi(C') \cong R' \otimes_R C$ and $\Phi(R' \otimes_R C) \cong C'$. The map $\eta_{C'}$ coincides with the resulting isomorphism $\Phi\big(\Psi(C')\big) \cong C'$. If X is any flat right R'-module, then $\Psi(X \otimes_{R'} C') \cong X \otimes_{R'} \Psi(C')$ by properties of cotensor products [3, 21.4] and $\Phi\big(X \otimes_{R'} \Psi(C')\big) \cong X \otimes_{R'} \Phi\big(\Psi(C')\big)$ by the associativity of tensor products. Hence $\eta_{X \otimes_{R'} C'}$ is an isomorphism as well. By [3, 18.10] there are bijections

$$\mathrm{Hom}^{C'}(V, X \otimes_{R'} C') \cong \mathrm{Hom}_{R'}(V, X) \quad \text{for } V \in \mathcal{M}^{C'}.$$

Taking X to be an injective hull of V in $\mathcal{M}_{R'}$, we get a right C'-colinear embedding $V \to X \otimes_{R'} C'$. If V is injective in $\mathcal{M}^{C'}$, then V has to be a direct summand of $X \otimes_{R'} C'$, whence η_V is an isomorphism. In general we can find an exact sequence of right C'-comodules $0 \to V \to X \otimes_{R'} C' \to Y \otimes_{R'} C'$ for some injective modules $X, Y \in \mathcal{M}_{R'}$. By a general property of right adjoint functors Ψ is left exact. If Φ is exact, then $\Phi\Psi$ takes the above exact sequence to an exact sequence

$$0 \to \Phi\Psi(V) \to \Phi\Psi(X \otimes_{R'} C') \to \Phi\Psi(Y \otimes_{R'} C').$$

Since $\eta_{X\otimes_{R'}C'}$ and $\eta_{Y\otimes_{R'}C'}$ are isomorphisms, so too is $\eta_V : \Phi\Psi(V) \to V$ for any V. Thus we have checked a necessary and sufficient condition for Ψ to be fully faithful [16, p. 88, Th. 1]. $\square$

Let A be a right H-comodule algebra with the comodule structure map ρ. If R is an arbitrary algebra, then $H_R = R \otimes H$ will be viewed as a right H-comodule algebra with respect to the comodule structure map $\mathrm{id} \otimes \Delta$. Each algebra homomorphism $\alpha : A \to R$ gives rise to a homomorphism of H-comodule algebras

$$\varphi_\alpha : A \xrightarrow{\rho} A \otimes H \xrightarrow{\alpha\otimes\mathrm{id}} R \otimes H.$$

Denoting by ε_R the algebra homomorphism $\mathrm{id} \otimes \varepsilon : R \otimes H \to R$ where $\varepsilon : H \to k$ is the counit, we have $\alpha = \varepsilon_R \circ \varphi_\alpha$. Conversely, any homomorphism of H-comodule algebras $A \to H_R$ is obtained in this way. For an arbitrary homomorphism of right H-comodule algebras $A \to B$ there is a functor $? \otimes_A B : \mathcal{M}_A^H \to \mathcal{M}_B^H$. In particular, this functor is defined for $B = H_R$.

Lemma 2.2. *Let $\varphi : A \to H_R$ be a homomorphism of right H-comodule algebras. Suppose that H is k-flat. Then for each $M \in \mathcal{M}_A^H$ the map*

$$\xi : M \otimes_A H_R \to (M \otimes_A R) \otimes H, \qquad m \otimes b \mapsto \sum \big((m_{(0)} \otimes 1) \otimes m_{(1)}\big) \cdot b,$$

is an isomorphism in $\mathcal{M}_{H_R}^H$.

Proof. Since k is an $\mathcal{M}_k$-direct summand of H, the flatness of H in $\mathcal{M}_k$ implies faithful flatness. Hence H_R is a faithfully flat H-Galois extension of R, and therefore the functor $? \otimes_R H_R$ gives a category equivalence $\mathcal{M}_R \to \mathcal{M}_{H_R}^H$ by [11, Th. 9] or [20, Th. I]. Thus each object of $\mathcal{M}_{H_R}^H$ can be written as $V \otimes_R H_R \cong V \otimes H$ for some $V \in \mathcal{M}_R$. The retraction $\varepsilon_R : H_R \to R$ of the canonical embedding $R \to H_R$ gives rise to a quasi-inverse equivalence $? \otimes_{H_R} R : \mathcal{M}_{H_R}^H \to \mathcal{M}_R$. It is easy to see that ξ is a morphism in $\mathcal{M}_{H_R}^H$. The final conclusion follows from the fact that $\xi \otimes_{H_R} R$ is an isomorphism in $\mathcal{M}_R$. $\square$

By a general construction in [3, 32.6] $C(A, H) = A \otimes H$ is an A-coring with the bimodule structure

$$x(a \otimes h) = xa \otimes h, \qquad (a \otimes h)x = \sum ax_{(0)} \otimes hx_{(1)}$$

where $a, x \in A,\ h \in H$, the comultiplication and the counit

$$\Delta(a \otimes h) = (a \otimes h_{(1)}) \otimes_A (1 \otimes h_{(2)}), \qquad \varepsilon(a \otimes h) = a\varepsilon(h).$$

Note that $C(A, H)$ is left A-flat provided H is k-flat.

Suppose that P is a prime ideal of A such that the factor ring A/P is either right or left Goldie. So A/P has a simple artinian classical right or left quotient ring $Q_P = Q(A/P)$. We define $\mathrm{Stab}(P)$ as the Q_P-coring obtained from $C(A, H)$ by the base ring extension $\alpha : A \to A/P \to Q_P$ where α is the composite of two canonical maps. Thus

$$\mathrm{Stab}(P) = Q_P \otimes_A C(A, H) \otimes_A Q_P \cong \big(Q_P \otimes H\big) \otimes_A Q_P.$$

In the previous line $H_P = Q_P \otimes H$ is viewed as a ring extension of A via the homomorphism of H-comodule algebras $\varphi : A \to H_P$ corresponding to $\alpha : A \to Q_P$. Note that $\operatorname{Stab}(P)$ has a left H_P-module structure and, in particular, a left H-module structure. If A/P is simple artinian, then $Q_P \cong A/P$ and $\operatorname{Stab}(P) \cong H_P/H_P\varphi(P)$ with $H_P \cong A/P \otimes H$.

The coring $\operatorname{Stab}(P)$ has a distinguished grouplike e, the image of $1 \otimes 1 \in A \otimes H$. Moreover, $\operatorname{Stab}(P)$ is generated by e as an $\big(H_P, Q_P\big)$-bimodule. The comultiplication in $\operatorname{Stab}(P)$ is expressed as

$$\Delta(beq) = \sum b_{(0)}e \otimes b_{(1)}eq \quad \text{for } b \in H_P,\ q \in Q_P.$$

Also, $\varepsilon(beq) = \varepsilon_P(b)q$ where $\varepsilon_P : H_P \to Q_P$ is the map $\mathrm{id} \otimes \varepsilon$.

Next we turn to the inertializer of P. Let us view the algebra $H_P = Q_P \otimes H$ itself as a Q_P-coring with respect to the natural Q_P-bimodule structure, the counit ε_P and the comultiplication $\Delta : H_P \to H_P \otimes_{Q_P} H_P$ which extends by left and right Q_P-linearity the comultiplication of H. The ring unity $e = 1 \otimes 1 \in H_P$ is a distinguished grouplike of H_P. We have

$$\Delta(b) = \sum b_{(0)} \otimes b_{(1)}e \quad \text{for } b \in H_P.$$

Lemma 2.3. *The (H_P, Q_P)-subbimodule I of H_P generated by $\{\varphi(a) - \alpha(a)e \mid a \in A\}$ is a coideal of H_P.*

Proof. For $a \in A$ we have $\varphi(a) = \sum \alpha(a_{(0)}) \otimes a_{(1)}$, whence

$$\varepsilon_P\big(\varphi(a) - \alpha(a)e\big) = \alpha(a) - \alpha(a) = 0.$$

Since ε_P is a ring homomorphism, we get $\varepsilon_P(I) = 0$. Also,

$$\begin{aligned}\Delta\big(\varphi(a) - \alpha(a)e\big) &= \sum \varphi(a_{(0)}) \otimes a_{(1)}e - \alpha(a)e \otimes e \\ &= \sum \big(\varphi(a_{(0)}) - \alpha(a_{(0)})e\big) \otimes a_{(1)}e + e \otimes \big(\varphi(a) - \alpha(a)e\big).\end{aligned}$$

Since Δ is right Q_P-linear and left H_P-linear if we let H_P operate on $H_P \otimes_{Q_P} H_P$ via the comodule structure map $H_P \to H_P \otimes H$, we deduce that

$$\Delta(I) \subset I \otimes_{Q_P} H_P + H_P \otimes_{Q_P} I.$$

□

We define $\operatorname{Inert}(P)$ as the factor coring H_P/I of H_P where I is given in Lemma 2.3. The coset $\overline{e} = e + I$ is taken to be the distinguished grouplike of $\operatorname{Inert}(P)$.

Lemma 2.4. *There is a surjective left H_P-linear homomorphism of Q_P-corings* $\operatorname{Stab}(P) \to \operatorname{Inert}(P)$ *compatible with the distinguished grouplikes. Its kernel coincides with the left H_P-submodule L of* $\operatorname{Stab}(P)$ *generated by $\{eq - qe \mid q \in Q_P\}$.*

Proof. Since $b\varphi(a)\overline{e}q = b\overline{e}\alpha(a)q$ for all $a \in A$, $b \in H_P$ and $q \in Q_P$, there is a well-defined left H_P-linear and right Q_P-linear map $\pi : \operatorname{Stab}(P) \to \operatorname{Inert}(P)$ which sends e to $\overline{e}$. Since $\pi(eq - qe) = \overline{e}q - q\overline{e} = 0$, we have $L \subset \operatorname{Ker}\pi$. On the other hand, $\operatorname{Stab}(P) = L + H_Pe$. Furthermore, $be \in \operatorname{Ker}\pi$ for $b \in H_P$ if and only if $b \in I$

where I is as in Lemma 2.3. It is easy to check that L is a right Q_P-submodule of $\operatorname{Stab}(P)$. Hence $I' = \{b \in H_P \mid be \in L\}$ is an (H_P, Q_P)-subbimodule of H_P. Since

$$\varphi(a)e - \alpha(a)e = e\alpha(a) - \alpha(a)e \in L$$

for all $a \in A$, we deduce that $I \subset I'$. It follows that $Ie \subset L$, whence $\operatorname{Ker} \pi = L$. It is also clear that π is compatible with the comultiplications and the counits. □

The notions of the stabilizer and the inertializer are explained in several examples below. Propositions 2.5–2.7 will not be used later in this paper.

Proposition 2.5. *Suppose that the base ring k is a field and P is a maximal ideal of codimension 1 in A. Then $\operatorname{Stab}(P) = \operatorname{Inert}(P)$ and $\operatorname{Stab}(P)$ coincides with the largest left H-module factor coalgebra C of H such that P is stable under the induced C-comodule structure $A \to A \otimes C$.*

Proof. By the assumptions $Q_P \cong A/P \cong k$. In this case α is the algebra homomorphism $A \to k$ with $\operatorname{Ker} \alpha = P$ and φ is the corresponding homomorphism of right H-comodule algebras $A \to H$. Thus $\operatorname{Stab}(P) = H/H\varphi(P)$. The map $A \to H$ given by the rule $a \mapsto \varphi(a) - \alpha(a)1$ vanishes on the image of k in A and coincides with φ on P. Hence $\{\varphi(a) - \alpha(a)1 \mid a \in A\} = \varphi(P)$, so that $\operatorname{Inert}(P) = H/H\varphi(P)$ as well. For $a \in A$ we have

$$\rho(a) = \sum a_{(0)} \otimes a_{(1)} \equiv \sum \alpha(a_{(0)})1 \otimes a_{(1)} \equiv 1 \otimes \varphi(a) \pmod{P \otimes H}.$$

If I is a left ideal and a coideal of H, then P is stable under the induced comodule structure $A \to A \otimes H/I$ if and only if $\varphi(P) \subset I$, if and only if $H\varphi(P) \subset I$. □

Suppose now that Γ is a finite group which acts on an algebra A via automorphisms. The decomposition group of a prime ideal P of A is the set-theoretic stabilizer $D_P = \{x \in \Gamma \mid xP = P\} = \{x \in \Gamma \mid xP \subset P\}$, while the inertia group T_P is its subgroup consisting of those $x \in D_P$ which induce the identity transformation of A/P. Although this terminology is ordinarily used only in the commutative algebra, commutativity of A is not needed in the next result.

We may view A as a right H-comodule algebra where $H = k[\Gamma]$ is the function algebra on Γ consisting of all maps $\Gamma \to k$. Algebraic operations on H are pointwise, and the comultiplication in H is dual to the multiplication in the group algebra of Γ. The group D_P operates on Q_P via automorphisms, so that Q_P may be regarded as a right $k[D_P]$-comodule algebra. The action of T_P and the corresponding coaction of $k[T_P]$ on Q_P are trivial.

Proposition 2.6. *Let Γ, D_P, T_P, H be as above. If A/P is a right Ore domain, then*

$$\operatorname{Stab}(P) \cong C(Q_P, k[D_P]) = Q_P \otimes k[D_P]\,,$$
$$\operatorname{Inert}(P) \cong C(Q_P, k[T_P]) = Q_P \otimes k[T_P]\,.$$

Proof. The ring $H_P = Q_P \otimes H$ may be identified with the ring of functions $\Gamma \to Q_P$, and so H_P is isomorphic to a direct product of finitely many copies of Q_P. With this identification $\varphi : A \to H_P$ is expressed as $\varphi(a)(x) = \alpha(xa)$ for $a \in A$ and $x \in \Gamma$ where $\alpha : A \to Q_P$ is the canonical map with kernel P. Since Q_P is a skew field, any one-sided ideal I of H_P is a two-sided ideal, and moreover

$$I = \{f \in H_P \mid f(X) = 0\}$$

for some subset $X \subset \Gamma$. The factor ring H_P/I may be identified with the ring of functions $X \to Q_P$. In particular, this applies to the left ideal $I = H_P\,\varphi(P)$. Given $x \in \Gamma$, we have $f(x) = 0$ for all $f \in I$ if and only if $\varphi(a)(x) = 0$ for all $a \in P$, if and only if $xP \subset P$, i.e. $x \in D_P$. Thus I corresponds to the subset $X = D_P$ of Γ. If $a \in A$, $a \notin P$, then $\varphi(a)(x) \neq 0$ for all $x \in D_P$, whence the coset $\varphi(a) + I$ is an invertible element of H_P/I. It follows that

$$\mathrm{Stab}(P) \cong H_P \otimes_A Q_P \cong H_P/I \otimes_{A/P} Q_P \cong H_P/I \cong Q_P \otimes k[D_P].$$

It is straightforward to check that the corresponding coring structure on $Q_P \otimes k[D_P]$ is the one defined in $C(Q_P, k[D_P])$.

In the case of $\mathrm{Inert}(P)$ we replace I with the left ideal of H_P defined in Lemma 2.3. Then $f(x) = 0$ for all $f \in I$ if and only if $\big(\varphi(a) - \alpha(a)e\big)(x) = 0$ for all $a \in A$. The last equality can be rewritten as $\alpha(xa - a) = 0$, which amounts to the condition $xa \equiv a \pmod{P}$. Thus I corresponds to the subset $X = T_P$ of Γ, and therefore

$$\mathrm{Inert}(P) = H_P/I \cong Q_P \otimes k[T_P].$$

□

Proposition 2.7. *Suppose that A and H are commutative. Then* $\mathrm{Inert}(P)$ *is a factor Hopf algebra of the Hopf algebra $H_P = Q_P \otimes H$ over the field Q_P. Let G and $G^T(P)$ denote the group schemes represented by H and* $\mathrm{Inert}(P)$*, respectively. Then $G^T(P)$ coincides with the scheme-theoretic stabilizer of P in G.*

Proof. We may view the commutative algebra $A_P = Q_P \otimes A$ over Q_P as a right H_P-comodule algebra. Let $\alpha_P : A_P \to Q_P$ be the homomorphism of Q_P-algebras extending the canonical map $\alpha : A \to Q_P$ and $\varphi_P : A_P \to H_P$ the homomorphism of H_P-comodule algebras such that $\varepsilon_P \circ \varphi_P = \alpha_P$. Denote by J the ideal of A_P generated by $\{1 \otimes a - \alpha(a) \otimes 1 \mid a \in A\}$. Clearly $J \subset \mathrm{Ker}\,\alpha_P$. Since $A_P = J + Q_P$, it follows that $J = \ker \alpha_P$. Since

$$\varphi_P\big(1 \otimes a - \alpha(a) \otimes 1\big) = \varphi(a) - \alpha(a) \otimes 1,$$

the ideal I of H_P defined in Lemma 2.3 is generated by the image of J in H_P. Hence

$$\mathrm{Inert}(P) = H_P/I \cong H_P \otimes_{A_P} A_P/J \cong H_P \otimes_{A_P} Q_P$$

and $G^T(P) = \mathrm{Spec}\,\mathrm{Inert}(P)$ is described as the product $\mathrm{Spec}\,H_P \times_{\mathrm{Spec}\,A_P} \mathrm{Spec}\,Q_P$ which defines the scheme-theoretic inertia group of P [9, III.2.2.3]. □

3. The first equivalence

Let A be a right H-comodule algebra and $P \in \mathrm{Spec}\,A$. When A/P is right or left Goldie, we put

$$Q_P = Q(A/P), \qquad H_P = Q(A/P) \otimes H.$$

Let $\varphi : A \to H_P$ be the homomorphism of H-comodule algebras such that $\varepsilon_P \circ \varphi$ coincides with the canonical map $A \to Q_P$ where $\varepsilon_P : H_P \to Q_P$ stands for $\mathrm{id} \otimes \varepsilon$. Since $\mathrm{Ker}(\varepsilon_P \circ \varphi) = P$, the kernel of φ is an H-costable ideal of A contained in P. Therefore φ is injective whenever $P_H = 0$.

Recalling the A-coring $C(A, H) = A \otimes H$, we have $\mathcal{M}^{C(A,H)} \approx \mathcal{M}_A^H$ by [3, 32.6]. Thus a special case of considerations in section 2 yields a pair of adjoint functors

$$\Phi : \mathcal{M}_A^H \to \mathcal{M}^{\mathrm{Stab}(P)}, \qquad \Psi : \mathcal{M}^{\mathrm{Stab}(P)} \to \mathcal{M}_A^H.$$

In particular, $\Phi(M) = M \otimes_A Q_P$. A Hopf algebra over a field is said to be *residually finite dimensional* if its ideals of finite codimension have zero intersection [18].

Theorem 3.1. *Let H be a residually finite dimensional Hopf algebra over a field, A a right H-comodule algebra and P a prime ideal of A with the property $P_H = 0$. If both A and $A/P \otimes H$ have right artinian classical right quotient rings, then Φ induces a category equivalence $\mathcal{M}_A^H / \mathcal{T}_A^H \approx \mathcal{M}^{\mathrm{Stab}(P)}$.*

Proof. We follow the proof of [23, Th. 1.8] very closely. By Proposition 1.5 A/P and H have right artinian classical right quotient rings. In particular, A/P is right Goldie. The existence of a right artinian ring $Q(H)$ implies that the antipode of H is bijective [21, Th. A]. By Proposition 1.5 the ring Q_P embeds in the quotient ring $Q(A/P \otimes H)$. The latter is then also a classical right quotient ring of H_P.

We may view A and H_P as left module algebras over the finite dual H° of H. Since H is residually finite dimensional, the H°-submodules in A and H_P are precisely the H-subcomodules. By [24, Th. 2.2] the action of H° extends to $Q(A)$ and $Q(H_P)$. Since $P_H = 0$, products of nonzero H°-stable two-sided ideals of A are nonzero, i.e. the H°-module algebra A is H°-prime. Then so too is $Q(A)$, whence $Q(A)$ is H°-simple by [24, Lemma 4.2].

Since H embeds in H_P as a right H-comodule algebra, each H-costable right ideal K of H_P may be regarded as an object of $\mathcal{M}_H^H$. By Sweedler's structure theorem for Hopf modules [25, Th. 4.1.1] we have $K \cong K_0 \otimes H$ where

$$K_0 = \{x \in K \mid (\mathrm{id} \otimes \Delta)(x) = x \otimes 1\} \subset Q_P \otimes 1 \cong Q_P.$$

Applying ε_P, we deduce that $K_0 = \varepsilon_P(K_0 \otimes H) = \varepsilon_P(K)$.

Suppose that $K = \varphi(I)H_P$ where I is an H-costable two-sided ideal of A. Then K_0 coincides with the extension IQ_P of I. It follows from standard properties of classical quotient rings [17, 2.1.16] that K_0 is a two-sided ideal of Q_P. If $I \neq 0$, then $I \not\subset P$ since $P_H = 0$. In this case $K_0 \neq 0$, and therefore $K_0 = Q_P$ since Q_P is simple artinian. We conclude that $\varphi(I)H_P = H_P$ for each nonzero H°-stable (i.e. H-costable) ideal I of A. Thus the assumptions of [23, Lemma 1.7] are satisfied. By that lemma φ extends to a homomorphism of H°-module algebras $Q(A) \to Q(H_P)$. This extension is injective since so is φ.

For $M \in \mathcal{M}_A^H$ we have

$$M \otimes_A Q(H_P) \cong \big(M \otimes_A Q(A)\big) \otimes_{Q(A)} Q(H_P).$$

Here $M \otimes_A Q(A)$ is an H°-equivariant $Q(A)$-module. It coincides with the union of H°-stable finitely generated $Q(A)$-submodules $VA \otimes_A Q(A)$ where V runs over the finite dimensional H-subcomodules $V \subset M$. This means that $M \otimes_A Q(A)$ is locally $Q(A)$-finite. By [23, Lemma 1.6] the functor $? \otimes_{Q(A)} Q(H_P)$ is faithfully exact on the category of locally $Q(A)$-finite H°-equivariant $Q(A)$-modules. Since $Q(A)$ is left A-flat, the functor $? \otimes_A Q(A)$ is exact on $\mathcal{M}_A$. It follows that $? \otimes_A Q(H_P)$ is exact on $\mathcal{M}_A^H$. On the other hand,

$$\begin{aligned} M \otimes_A Q(H_P) &\cong (M \otimes_A H_P) \otimes_{H_P} Q(H_P) \cong (M \otimes_A Q_P) \otimes_{Q_P} Q(H_P) \\ &\cong \Phi(M) \otimes_{Q_P} Q(H_P) \end{aligned}$$

since $M \otimes_A H_P \cong (M \otimes_A Q_P) \otimes_{Q_P} H_P$ according to Lemma 2.2 (in the left hand side H_P is viewed as a ring extension of A via φ). As Q_P is simple artinian, the functor $? \otimes_{Q_P} Q(H_P)$ is faithfully exact on $\mathcal{M}_{Q_P}$. It follows that Φ has to be exact. Now Lemma 2.1 shows that Ψ is fully faithful. By [13, Ch. III, Prop. 5] $\operatorname{Ker} \Phi$ is a localizing subcategory of $\mathcal{M}_A^H$, and Φ induces an equivalence $\mathcal{M}_A^H / \operatorname{Ker} \Phi \cong \mathcal{M}^{\operatorname{Stab}(P)}$. Moreover, $\Phi(M) = 0$ if and only if $M \otimes_A Q(H_P) = 0$, which is equivalent to the equality $M \otimes_A Q(A) = 0$. Hence $\operatorname{Ker} \Phi = \mathcal{T}_A^H$ by Lemma 1.2. □

Remarks. (1) If $\dim A/P = 1$, then φ is an isomorphism of A onto a right coideal subalgebra of H. In this case Theorem 3.1 reduces to [23, Th. 1.8].

(2) The right quotient rings in Theorem 3.1 cannot be replaced with the left quotient rings because it is essential in the proof that $Q(A)$ is left A-flat.

(3) There is a version of Theorem 3.1 for an arbitrary base ring k which reduces, however, to the case of a field. Denote by $\mathfrak{p} \in \operatorname{Spec} k$ the preimage of $P \in \operatorname{Spec} A$ in k. The assumption $P_H = 0$ forces $\mathfrak{p} A = 0$, so that A is an algebra over the domain $k/\mathfrak{p}$. Passing to the field of fractions $Q_\mathfrak{p} = Q(k/\mathfrak{p})$, we may view $A' = Q_\mathfrak{p} \otimes A$ as a right comodule algebra over the Hopf algebra $H' = Q_\mathfrak{p} \otimes H$. There is a unique prime $P' \in \operatorname{Spec} A'$ whose preimage in A coincides with P. Since $Q_\mathfrak{p}$ embeds in Q_P, we have $Q_P \cong Q_{P'}$ and $\operatorname{Stab}(P) \cong \operatorname{Stab}(P')$. There is a functor $Q_\mathfrak{p} \otimes ? : \mathcal{M}_A^H \to \mathcal{M}_{A'}^{H'}$. Viewing right (H', A')-Hopf modules as right (H, A)-Hopf modules gives another functor $\mathcal{M}_{A'}^{H'} \to \mathcal{M}_A^H$. It is easy to see that these two functors induce quasi-inverse equivalences between $\mathcal{M}_A^H / \mathcal{T}_A^H$ and $\mathcal{M}_{A'}^{H'} / \mathcal{T}_{A'}^{H'}$. Thus we may replace the pair A, H with A', H'.

For $W \in {}_A\mathcal{M}$ and $U \in {}_H\mathcal{M}$ we regard $W \otimes U$ as a left A-module, letting A operate on $W \otimes U$ via $\rho : A \to A \otimes H$. In a similar way $V \otimes U' \in \mathcal{M}_A$ for $V \in \mathcal{M}_A$ and $U' \in \mathcal{M}_H$. Let U_S denote U with the right H-module structure $uh = S(h)u$ ($u \in U$, $h \in H$) where S is the antipode of H.

Lemma 3.2. *There are natural k-linear bijections*

$$(V \otimes U_S) \otimes_A W \cong V \otimes_A (W \otimes U) \quad \textit{for } V \in \mathcal{M}_A,\ W \in {}_A\mathcal{M},\ U \in {}_H\mathcal{M}.$$

Assuming $S : H \to H$ to be bijective, we have $(V \otimes H) \otimes_A W \cong V \otimes_A (W \otimes H)$.

Proof. Clearly $(V \otimes U_S) \otimes_A W \cong (V \otimes U \otimes W)/K$ where K is the k-linear span of elements

$$\sum va_{(0)} \otimes S(a_{(1)})u \otimes w - v \otimes u \otimes aw$$

with $u \in U$, $v \in V$, $w \in W$ and $a \in A$. Similarly $V \otimes_A (W \otimes U) \cong (V \otimes W \otimes U)/L$ where L is the k-linear span of elements

$$va \otimes w \otimes u - \sum v \otimes a_{(0)} w \otimes a_{(1)} u.$$

If $\zeta : V \otimes U \otimes W \cong V \otimes W \otimes U$ is the canonical k-linear bijection, then

$$\begin{aligned}\zeta\big(\sum va_{(0)} \otimes S(a_{(1)})u \otimes w\big) &= \sum va_{(0)} \otimes w \otimes S(a_{(1)})u \\ &\equiv \sum v \otimes a_{(0)} w \otimes a_{(1)} S(a_{(2)})u \\ &= v \otimes aw \otimes u = \zeta(v \otimes u \otimes aw) \pmod{L}\end{aligned}$$

Hence $\zeta(K) \subset L$. A similar calculation shows that $\zeta^{-1}(L) \subset K$. In other words, $\zeta(K) = L$. Now take $U = H$, $U' = H$ with the H-module structures given, respectively, by left and right multiplications. The map $S : H \to H$ is a homomorphism of right H-modules $U' \to U_S$. Thus $U_S \cong U'$ in $\mathcal{M}_H$ when S is bijective. □

For each $V \in \mathcal{M}_A$ we may regard $V \otimes H$ as an object of $\mathcal{M}_A^H$ with the action of A obtained via $\rho : A \to A \otimes H$ and the coaction of H given by the map $\mathrm{id} \otimes \Delta$.

Lemma 3.3. *If $S : H \to H$ is bijective then $\Phi(V \otimes H) \cong V \otimes_A H_P$ in $\mathcal{M}_{Q_P}$.*

Proof. The desired isomorphism can be rewritten as

$$(V \otimes H) \otimes_A Q_P \cong V \otimes_A (Q_P \otimes H).$$

It is obtained by taking $W = Q_P$ in Lemma 3.2. By naturality the isomorphisms of Lemma 3.2 are $\mathrm{End}_A W$-linear. In particular, they are right Q_P-linear for $W = Q_P$. □

Proposition 3.4. *Under the hypotheses of Theorem* 3.1 *H_P is left A-flat with respect to φ and $V \otimes_A H_P = 0$ for $V \in \mathcal{M}_A$ if and only if $V \in \mathcal{T}_A$. Moreover, we have $\mathrm{Tor}_i^A\big(M, Q_P\big) = 0$ and $\mathrm{Ext}^i_A\big(M, Q_P\big) = 0$ for all $M \in \mathcal{M}_A^H$ and $i > 0$.*

Proof. The functor $\Phi(\,?\otimes H)$ is exact on $\mathcal{M}_A$ by Theorem 3.1, whence so is $?\otimes_A H_P$ in view of Lemma 3.3. This verifies the flatness of H_P. By Lemma 3.3 $V \otimes_A H_P = 0$ if and only if $\Phi(V \otimes H) = 0$, and Theorem 3.1 allows us to rewrite the last condition as $V \otimes H \in \mathcal{T}_A^H$. Note that $\mathrm{id} \otimes \varepsilon : V \otimes H \to V$ is an epimorphism in $\mathcal{M}_A$. Therefore $V \in \mathcal{T}_A$ whenever $V \otimes H \in \mathcal{T}_A$. Conversely, suppose that $V \in \mathcal{T}_A$. Then for each $v \in V$ there exists an H-costable right ideal I of A such that $I \in \mathcal{G}(A)$ and $vI = 0$. Since $(v \otimes H) \cdot I = 0$, we conclude that $V \otimes H \in \mathcal{T}_A$ as well.

As mentioned in the proof of Lemma 2.2, each object $N \in \mathcal{M}_{H_P}^H$ can be written as $X \otimes_{Q_P} H_P$ for some right Q_P-module X. Since the ring Q_P is simple artinian, all Q_P-modules are projective. Hence N is projective in $\mathcal{M}_{H_P}$, so that $\mathrm{Tor}_i^{H_P}(N, ?) = 0$ for all $i > 0$. The ring homomorphism $A \to Q_P$ factors through $\varphi : A \to H_P$, and $? \otimes_A H_P$ is defined as a functor $\mathcal{M}_A^H \to \mathcal{M}_{H_P}^H$. Therefore

$$\mathrm{Tor}_i^A\big(M, Q_P\big) \cong \mathrm{Tor}_i^{H_P}\big(M \otimes_A H_P, Q_P\big) = 0$$

by standard homological algebra [27, Prop. 3.2.9]. A similar argument applies to the functors Ext^i. □

For a finitely generated right A-module M we define the normalized rank at P and another quantity which does not depend on P. The lengths of right Q_P-modules are used in the former case and the lengths of right $Q(A)$-modules in the latter:

$$r_P(M) = \frac{\operatorname{length}_{Q_P} M \otimes_A Q_P}{\operatorname{length} Q_P}, \qquad r(M) = \frac{\operatorname{length}_{Q(A)} M \otimes_A Q(A)}{\operatorname{length} Q(A)}.$$

Proposition 3.5. *Under the hypotheses of Theorem* 3.1 *we have* $r_P(M) = r(M)$ *for each A-finite object* $M \in \mathcal{M}_A^H$.

Proof. As was mentioned in the proof of Theorem 3.1, $Q(A)$ is an H°-simple right artinian H°-module algebra and $M \otimes_A Q(A)$ is an H°-equivariant right module over $Q(A)$. By [22, Th. 7.6] a suitable finite direct sum of copies of $M \otimes_A Q(A)$ is a free module. Thus there exist integers $n \geq 0$ and $l \geq 0$ such that

$$M^l \otimes_A Q(A) \cong Q(A)^n$$

in $\mathcal{M}_{Q(A)}$. Comparing the lengths of these two modules, we find $r(M) = n/l$. Now applying the functor $? \otimes_{Q(A)} Q(H_P)$ and comparing similarly the lengths of the resulting right $Q(H_P)$-modules, we get

$$\frac{\operatorname{length}_{Q(H_P)} M \otimes_A Q(H_P)}{\operatorname{length} Q(H_P)} = \frac{n}{l} = r(M).$$

On the other hand, $M \otimes_A Q(H_P) \cong (M \otimes_A Q_P) \otimes_{Q_P} Q(H_P)$ (see again the proof of Theorem 3.1). Since the ring Q_P is simple artinian, a suitable finite direct sum of copies of $M \otimes_A Q_P$ is a free Q_P-module. Repeating the previous argument, but now with respect to the ring extension $Q_P \to Q(H_P)$, we deduce that

$$\frac{\operatorname{length}_{Q(H_P)} M \otimes_A Q(H_P)}{\operatorname{length} Q(H_P)} = r_P(M).$$

□

Corollary 3.6. *Let* P, P' *be two prime ideals of* A, *both satisfying the hypotheses of Theorem* 3.1. *Then* $\mathcal{M}^{\mathrm{Stab}(P)} \approx \mathcal{M}^{\mathrm{Stab}(P')}$. *If* $V \in \mathcal{M}^{\mathrm{Stab}(P)}$ *and* $V' \in \mathcal{M}^{\mathrm{Stab}(P')}$ *correspond to each other under this equivalence, then*

$$\frac{\operatorname{length}_{Q_P} V}{\operatorname{length} Q_P} = \frac{\operatorname{length}_{Q_{P'}} V'}{\operatorname{length} Q_{P'}}.$$

Proof. By Theorem 3.1 $\mathcal{M}^{\mathrm{Stab}(P)} \approx \mathcal{M}_A^H \approx \mathcal{M}^{\mathrm{Stab}(P')}$. The second assertion follows from Proposition 3.5. □

Equivalences of comodule categories over corings are described in [3, 23.3, 23.12].

4. The second equivalence

Let H be a Hopf algebra over the base ring k, and R a k-algebra. A *left H-module R-coring* is an R-coring C equipped with a left H-module structure such that the three module structures on C restrict to the same k-module structure, the action of H on C commutes both with the left and the right actions of R and

$$\Delta(hc) = \Delta(h)\Delta(c) = \sum h_{(1)}c_{(1)} \otimes h_{(2)}c_{(2)}, \qquad \varepsilon(hc) = \varepsilon(h)\varepsilon(c)$$

for all $h \in H$, $c \in C$ (the second of the previous two identities is actually a consequence of the first one).

A *left* (C,H)*-Hopf module* M is a left C-comodule and a left H-module such that the two module structures on M restrict to the same k-module structure, the action of H commutes with the action of R, and

$$\delta(hm) = \Delta(h)\delta(m) = \sum h_{(1)}m_{(-1)} \otimes h_{(2)}m_{(0)} \quad \text{for all } h \in H,\ m \in M$$

where $\delta : M \to C \otimes_R M$ is the comodule structure map and $C \otimes_R M$ is viewed as an $H \otimes H$-module in a natural way. Denote by ${}^C_H\mathcal{M}$ the category of left (C,H)-Hopf modules. We will view $C \otimes H$ as an $R \otimes H$-coring with the bimodule structure

$$(a \otimes g)(c \otimes h) = \sum ag_{(1)}c \otimes g_{(2)}h, \qquad (c \otimes h)(a \otimes g) = ca \otimes hg$$

where $a \in R,\ g, h \in H,\ c \in C$, the comultiplication and the counit

$$\Delta(c \otimes h) = \sum (c_{(1)} \otimes 1) \otimes_{(R \otimes H)} (c_{(2)} \otimes h), \qquad \varepsilon(c \otimes h) = \varepsilon(c) \otimes h.$$

When C is considered with a distinguished grouplike e, then $e \otimes 1$ is taken to be the distinguished grouplike of $C \otimes H$. For $M \in {}^C\mathcal{M}$ one defines

$$M^{\mathrm{co}C} = \{m \in M \mid \delta(m) = e \otimes m\}.$$

Similarly, $M^{\mathrm{co}C \otimes H}$ is defined for each $M \in {}^{C \otimes H}\mathcal{M}$. Given a ring homomorphism $R \to R'$, the H-module structure on C passes to $C' = R' \otimes_R C \otimes_R R'$. It makes C' into a left H-module R'-coring. The distinguished grouplike of C' is $1 \otimes e \otimes 1$.

When $R = k$, a left H-module R-coring is just a left H-module coalgebra, and the next lemma reduces to a well-known fact [3, 32.6].

Lemma 4.1. *Structures of a left* (C,H)*-Hopf module may be identified with structures of a left comodule over the* $R \otimes H$*-coring* $C \otimes H$*. Thus* ${}^C_H\mathcal{M} \approx {}^{C \otimes H}\mathcal{M}$*. Furthermore,* $M^{\mathrm{co}C} = M^{\mathrm{co}C \otimes H}$ *for each* $M \in {}^C_H\mathcal{M}$*.*

Proof. A pair of commuting left R-module and H-module structures on M may be interpreted as a single left $R \otimes H$-module structure. Next,

$$(C \otimes H) \otimes_{(R \otimes H)} M \cong C \otimes_R (R \otimes H) \otimes_{(R \otimes H)} M \cong C \otimes_R M.$$

This bijection is left $R \otimes H$-linear if we let H operate on $C \otimes_R M$ via the comultiplication $H \to H \otimes H$. Thus the left R-linear maps $\delta : M \to C \otimes_R M$ satisfying the required compatibility condition with the action of H correspond to the left $R \otimes H$-linear maps $\delta' : M \to (C \otimes H) \otimes_{(R \otimes H)} M$. It is straightforward to check that the coassociativity and the counit conditions for δ are equivalent to similar conditions for δ'. For $m \in M$ one has $\delta(m) = e \otimes m$ if and only if $\delta'(m) = (e \otimes 1) \otimes m$. □

Let A be a right H-comodule algebra, and P a prime ideal of A such that A/P is right or left Goldie. The A-coring $A \otimes H$ is a left H-module coring in the obvious way. Hence $\mathrm{Stab}(P)$ is a left H-module coring too. We continue to use the notations Q_P, H_P, φ of section 3. With $\varphi : A \to H_P$ one associates the Sweedler H_P-coring $H_P \otimes_A H_P$ whose distinguished grouplike is taken to be $1 \otimes 1$ [3, 25.1]. Taking C to be the Q_P-coring $\mathrm{Stab}(P)$ in the previous discussion, we derive an H_P-coring structure on $\mathrm{Stab}(P) \otimes H$.

Lemma 4.2. *Assume H to be k-flat. Then there is an isomorphism of H_P-corings $\mathrm{Stab}(P)\otimes H\cong H_P\otimes_A H_P$ compatible with the distinguished grouplikes.*

Proof. Recall that $\mathrm{Stab}(P)\cong H_P\otimes_A Q_P$. By Lemma 2.2 there is an isomorphism

$$H_P\otimes_A H_P\cong\big(H_P\otimes_A Q_P\big)\otimes H\cong \mathrm{Stab}(P)\otimes H$$

in $\mathcal{M}^H_{H_P}$. Explicit formula for this isomorphism shows that the action of H_P by left multiplications on the first tensorand of $H_P\otimes_A H_P$ corresponds to the left action of H_P on $\mathrm{Stab}(P)\otimes H$ defined earlier in this section. Thus $b\otimes b'\in H_P\otimes_A H_P$ is sent to $b(e\otimes 1)b'\in\mathrm{Stab}(P)\otimes H$ where e stands for the distinguished grouplike of $\mathrm{Stab}(P)$. This map respects the H_P-coring structures. □

By Lemma 4.2 the H_P-coring $\mathrm{Stab}(P)\otimes H$ is a Galois coring, in the terminology of [2]. There are functors

$$\Phi: {}_A\mathcal{M}\to {}^{\mathrm{Stab}(P)}_{\quad H}\mathcal{M}\quad\text{and}\quad \Psi: {}^{\mathrm{Stab}(P)}_{\quad H}\mathcal{M}\to {}_A\mathcal{M}$$

defined as $\Phi=H_P\otimes_A ?,\quad \Psi=?^{\mathrm{co}\,\mathrm{Stab}(P)}$.

Lemma 4.3. *Assume H to be k-flat. Then Ψ is right adjoint of Φ. If H_P is right A-flat, then Φ is exact, while Ψ is fully faithful.*

Proof. By Lemmas 4.1 and 4.2 ${}^{\mathrm{Stab}(P)}_{\quad H}\mathcal{M}\approx {}^{H_P\otimes_A H_P}\mathcal{M}$. Under this equivalence Φ and Ψ correspond to the canonical pair of functors

$${}_A\mathcal{M}\to {}^{H_P\otimes_A H_P}\mathcal{M}\quad\text{and}\quad {}^{H_P\otimes_A H_P}\mathcal{M}\to {}_A\mathcal{M}.$$

Their adjointness is verified in [6, p. 206, Prop. 107] (compared with [6] we switched the left and right sides). Exactness of Φ is immediate from the right A-flatness of H_P. In this case Ψ is fully faithful by [6, p. 207, Prop. 108]. □

Theorem 4.4. *Let H be a residually finite dimensional Hopf algebra over a field, A a right H-comodule algebra and P a prime ideal of A with the property $P_H=0$. If both A and $A/P\otimes H$ have left artinian classical left quotient rings, then Φ induces a category equivalence ${}_A\mathcal{M}/{}_A\mathcal{T}\approx {}^{\mathrm{Stab}(P)}_{\quad H}\mathcal{M}$.*

Proof. By the assumptions A^{op} and $(A/P)^{\mathrm{op}}\otimes H^{\mathrm{op}}$ have right artinian classical right quotient rings. Proposition 1.5 then shows that so does H^{op} too. This implies, in view of [21, Th. A], that the antipode of the Hopf algebra $H^{\mathrm{op,cop}}$ is bijective. Then H^{op} is itself a Hopf algebra. The right H^{op}-comodule algebra A^{op} and its prime ideal P satisfy the hypotheses of Theorem 3.1. Applying Proposition 3.4 to A^{op}, we deduce that $(H_P)^{\mathrm{op}}$ is left A^{op}-flat with respect to φ. Hence H_P is right A-flat with respect to φ. This allows us to apply Lemma 4.3 which, in conjunction with [13, Ch. III, Prop. 5], entails a category equivalence

$${}_A\mathcal{M}/\operatorname{Ker}\Phi\approx {}^{\mathrm{Stab}(P)}_{\quad H}\mathcal{M}.$$

By Proposition 3.4 applied again to A^{op} we have $H_P\otimes_A V=0$ for $V\in{}_A\mathcal{M}$ if and only if $V\in{}_A\mathcal{T}$. Thus $\operatorname{Ker}\Phi={}_A\mathcal{T}$. □

We denote by $\operatorname{Stab}^r(P)$ the stabilizer of P considered as a prime ideal of the right $H^{\rm op}$-comodule algebra $A^{\rm op}$. Then $\operatorname{Stab}^r(P)$ is a left $H^{\rm op}$-module $Q_P^{\rm op}$-coring, hence a right H-module $Q_P^{\rm op}$-coring. For example, if P is a maximal ideal of codimension 1 in A and $\varphi : A \to H$ is the corresponding homomorphism of right H-comodule algebras, then $\operatorname{Stab}^r(P) \cong H/\varphi(P)H$. For each right H-module coring C the category ${}^C\mathcal{M}_H$ of left-right Hopf (C, H)-modules can be defined. Replacing A with $A^{\rm op}$ in Theorem 4.4, we get an equivalent formulation:

Theorem 4.5. *Under the hypotheses of Theorem* 3.1 *there is a category equivalence* $\mathcal{M}_A/\mathcal{T}_A \approx {}^{\operatorname{Stab}^r(P)}\mathcal{M}_H$.

5. Birational extensions

In this section we assume that H is a k-flat Hopf algebra with a bijective antipode over an arbitrary commutative ring k. Suppose that $\psi : A \to B$ is a homomorphism of right H-comodule algebras which extends to an isomorphism of classical right quotient rings $Q(A) \to Q(B)$ (thus $Q(A)$ and $Q(B)$ exist but are not necessarily artinian). In this case we say that B is a *birational H-coequivariant extension* of A. Identifying $Q(A)$ with $Q(B)$ and the algebras A, B with their images in the quotient ring, we may assume that $A \subset B \subset Q(A)$. Put

$$\operatorname{Spec}' A = \{P \in \operatorname{Spec} A \mid \text{the ring } A/P \text{ is right Goldie and } P_H \cap \Sigma(A) = \varnothing\}.$$

The condition $P_H \cap \Sigma(A) = \varnothing$ means precisely that $P \notin \mathcal{G}_H(A)$.

Lemma 5.1. *The right A-module B/A is $\mathcal{G}_H(A)$-torsion.*

Proof. By the initial assumptions $B \otimes_A Q(A) \cong Q(A)$, whence $B/A \otimes_A Q(A) = 0$. Since B/A is an object of $\mathcal{M}_A^H$, the conclusion follows from Lemma 1.2. □

Lemma 5.2. *Let I and J be right ideals of A and B, respectively. If $I \in \mathcal{G}_H(A)$ then $IB \in \mathcal{G}_H(B)$. If $J \in \mathcal{G}_H(B)$ then $J \cap A \in \mathcal{G}_H(A)$.*

Proof. We may assume I and J to be H-costable. Then IB and $J \cap A$ are H-costable right ideals of B and A, respectively. Since $I \cap \Sigma(A) \neq \varnothing$ and $\Sigma(A) \subset \Sigma(B)$, we get $IB \cap \Sigma(B) \neq \varnothing$, which proves the first conclusion.

Take any $s \in J \cap \Sigma(B)$. By the finiteness theorem (see [3, 3.16]) there exists an H-subcomodule $U \subset J$ such that $s \in U$ and U is contained in a finitely generated k-submodule of B. In view of Lemma 5.1 $UK \subset A$ for a suitable $K \in \mathcal{G}_H(A)$. We may assume K to be H-costable. Then UK is an H-costable right ideal of A, and $UK \subset J \cap A$. Since $U \cap \Sigma(B) \neq \varnothing$ and $K \cap \Sigma(B) \neq \varnothing$, we have $UK \cap \Sigma(B) \neq \varnothing$ as well. But $UK \cap \Sigma(B) \subset \Sigma(A)$ because $UK \subset A$. It follows that $UK \in \mathcal{G}_H(A)$, whence the second conclusion. □

Lemma 5.3. *A right B-module W is $\mathcal{G}_H(B)$-torsion* (*resp. $\mathcal{G}_H(B)$-torsion-free*) *if and only if W is $\mathcal{G}_H(A)$-torsion* (*resp. $\mathcal{G}_H(A)$-torsion-free*).

Proof. Let $w \in W$ and denote by I and J the annihilators of w in A and B, respectively. Then $I = J \cap A$ and $IB \subset J$. Hence $I \in \mathcal{G}_H(A)$ if and only if $J \in \mathcal{G}_H(B)$ by Lemma 5.2. This shows that the $\mathcal{G}_H(B)$-torsion submodule of W in $\mathcal{M}_B$ coincides with the $\mathcal{G}_H(A)$-torsion submodule in $\mathcal{M}_A$. □

The canonical functor $\mathcal{M}_A \to \mathcal{M}_A/\mathcal{T}_A$ sends a morphism f in $\mathcal{M}_A$ to an isomorphism in $\mathcal{M}_A/\mathcal{T}_A$ if and only if the kernel and the cokernel of f are $\mathcal{G}_H(A)$-torsion [13, Ch. III, Lemme 4]. Moreover, the category $\mathcal{M}_A/\mathcal{T}_A$ is universal with respect to inverting such morphisms [14, I.2.5.4]. In other words, for an arbitrary category $\mathcal{C}$ a functor $F : \mathcal{M}_A \to \mathcal{C}$ factors through $\mathcal{M}_A/\mathcal{T}_A$ if and only if $F(f)$ is an isomorphism for each $\mathcal{M}_A$-morphism f with $\operatorname{Ker} f \in \mathcal{T}_A$ and $\operatorname{Coker} f \in \mathcal{T}_A$. When $\mathcal{C}$ is abelian and F is exact, the previous condition means that F vanishes on $\mathcal{T}_A$.

Proposition 5.4. *Assuming that B is a birational H-coequivariant extension of A, we have $\mathcal{M}_A/\mathcal{T}_A \approx \mathcal{M}_B/\mathcal{T}_B$ and $\mathcal{M}_A^H/\mathcal{T}_A^H \approx \mathcal{M}_B^H/\mathcal{T}_B^H$.*

Proof. Consider the adjoint functors $\mathcal{M}_A \to \mathcal{M}_B$ and $\mathcal{M}_B \to \mathcal{M}_A$ given, respectively, by extension and restriction of scalars. Let

$$\xi_V : V \to V \otimes_A B, \qquad v \mapsto v \otimes 1, \qquad V \in \mathcal{M}_A,$$
$$\eta_W : W \otimes_A B \to W, \quad w \otimes b \mapsto wb, \quad W \in \mathcal{M}_B,$$

be the unit and the counit of adjunction.

Since tensor products commute with inductive direct limits, we have $V \otimes_A B \cong \varinjlim V \otimes_A F$ where F runs over the finitely generated left A-submodules of B, and we may use only those F for which $A \subset F$. Since B/A is $\mathcal{G}_H(A)$-torsion by Lemma 5.1, for such an F there exists $I_F \in \mathcal{G}_H(A)$ with the property that $FI_F \subset A$. Suppose that $v \in \operatorname{Ker} \xi_V$. Then $v \otimes 1 = 0$ in $V \otimes_A F$ for some F as above. The right multiplication by an element $a \in I_F$ gives a left A-linear map $\mu_a : F \to A$. The map $\mathrm{id} \otimes \mu_a : V \otimes_A F \to V$ sends $v \otimes 1$ to va. Hence $vI_F = 0$. This shows that $\operatorname{Ker} \xi_V$ is $\mathcal{G}_H(A)$-torsion. Any element $x \in V \otimes_A B$ lies in the image of some $V \otimes_A F$, and then xI_F is contained in the image of $V \otimes_A A \cong V$. In other words, $\operatorname{Coker} \xi_V$ is also $\mathcal{G}_H(A)$-torsion. Thus ξ_V is an isomorphism in $\mathcal{M}_A/\mathcal{T}_A$.

Since $\eta_W \circ \xi_W = \mathrm{id}$ and ξ_W is an isomorphism in $\mathcal{M}_A/\mathcal{T}_A$, so too is η_W. It follows that $\operatorname{Ker} \eta_W$ is $\mathcal{G}_H(A)$-torsion, hence $\mathcal{G}_H(B)$-torsion by Lemma 5.3. At the same time $\operatorname{Coker} \eta_W = 0$. Therefore η_W is an isomorphism in $\mathcal{M}_B/\mathcal{T}_B$.

Suppose that $f : V \to V'$ is a morphism in $\mathcal{M}_A$ with $\mathcal{G}_H(A)$-torsion kernel and cokernel. Since $(f \otimes \mathrm{id}) \circ \xi_V = \xi_{V'} \circ f$ and f, ξ_V, $\xi_{V'}$ are isomorphisms in $\mathcal{M}_A/\mathcal{T}_A$, so is $f \otimes \mathrm{id} : V \otimes_A B \to V' \otimes_A B$. In other words, $\operatorname{Ker}(f \otimes \mathrm{id})$ and $\operatorname{Coker}(f \otimes \mathrm{id})$ are $\mathcal{G}_H(A)$-torsion. Since these two B-modules have to be $\mathcal{G}_H(B)$-torsion by Lemma 5.3, $f \otimes \mathrm{id}$ is an isomorphism also in $\mathcal{M}_B/\mathcal{T}_B$. It follows that $? \otimes_A B : \mathcal{M}_A \to \mathcal{M}_B$ gives rise to a functor $\mathcal{M}_A/\mathcal{T}_A \to \mathcal{M}_B/\mathcal{T}_B$. Since all $\mathcal{G}_H(B)$-torsion right B-modules are $\mathcal{G}_H(A)$-torsion in $\mathcal{M}_A$ by Lemma 5.3, the exact functor $\mathcal{M}_B \to \mathcal{M}_A$ induces a functor $\mathcal{M}_B/\mathcal{T}_B \to \mathcal{M}_A/\mathcal{T}_A$. The fact that ξ_V and η_W are isomorphisms in the quotient categories for all V and W means that the two induced functors between the quotient categories are quasi-inverse to each other.

For each object $M \in \mathcal{M}_A^H$ the map $\xi_M : M \to M \otimes_A B$ is a morphism in $\mathcal{M}_A^H$. Since we have proved already that $\operatorname{Ker} \xi_M$ and $\operatorname{Coker} \xi_M$ are $\mathcal{G}_H(A)$-torsion, ξ_M is an isomorphism in $\mathcal{M}_A^H/\mathcal{T}_A^H$. Similarly, η_N is an isomorphism in $\mathcal{M}_B^H/\mathcal{T}_B^H$ for each object $N \in \mathcal{M}_B^H$. Then the second category equivalence also follows. □

Lemma 5.5. *Let $P \in \operatorname{Spec} A \setminus \mathcal{G}_H(A)$. If A/P is either right or left Goldie, then the quotient ring $Q_P = Q(A/P)$ is $\mathcal{G}_H(A)$-torsion-free in $\mathcal{M}_A$ and $IQ_P = Q_P$ for each $I \in \mathcal{G}_H(A)$.*

Proof. Denote by T the $\mathcal{G}_H(A)$-torsion right A-submodule of Q_P. Since T is stable under all $\mathcal{M}_A$-endomorphisms of Q_P, it is a left ideal of Q_P. If s is any regular element of A/P then the left ideal Ts of Q_P has the same length as T; since $Ts \subset T$, we deduce that $Ts = T$, and therefore $Ts^{-1} = T$. It follows that T is a two-sided ideal of Q_P. Since Q_P is a simple artinian ring, either $T = 0$ or $T = Q_P$. Note that P is the annihilator in A of $1 \in Q_P$. Since $P \notin \mathcal{G}_H(A)$, we have $1 \notin T$. Therefore $T = 0$, i.e. Q_P is $\mathcal{G}_H(A)$-torsion-free.

We can conclude that each right Q_P-module is $\mathcal{G}_H(A)$-torsion-free in $\mathcal{M}_A$ since it embeds in a free Q_P-module. If $I \in \mathcal{G}_H(A)$, then the $\mathcal{G}_H(A)$-torsion submodule of Q_P/IQ_P contains the coset $1 + IQ_P$. It follows that $1 \in IQ_P$ since Q_P/IQ_P is $\mathcal{G}_H(A)$-torsion-free, but then $IQ_P = Q_P$. □

Theorem 5.6. *Let B be a birational H-coequivariant extension of A. Suppose that $Q(A)$ is a classical two-sided quotient ring of A. Then the assignment $\mathfrak{P} \mapsto \mathfrak{P} \cap A$ gives a bijection $\operatorname{Spec}'B \to \operatorname{Spec}'A$.*

If $P \in \operatorname{Spec}'A$ corresponds to $\mathfrak{P} \in \operatorname{Spec}'B$, then the canonical map $A/P \to B/\mathfrak{P}$ extends to an isomorphism of quotient rings $Q_P \to Q_{\mathfrak{P}}$ and to coring isomorphisms $\operatorname{Stab}(P) \to \operatorname{Stab}(\mathfrak{P})$, $\operatorname{Inert}(P) \to \operatorname{Inert}(\mathfrak{P})$.

Proof. By the assumptions the opposite ring $Q(A)^{\mathrm{op}}$ is a classical right quotient ring of A^{op} and B^{op}. Thus B^{op} is a birational H^{op}-coequivariant extension of A^{op}. This observation allows us to use freely the left hand versions of the previous results in this section.

Given some $\mathfrak{P} \in \operatorname{Spec}' B$, put $\overline{B} = B/\mathfrak{P}$, $P = A \cap \mathfrak{P}$, and $\overline{A} = A/P$. We will apply Proposition 1.4 to the chain of rings $\overline{A} \subset \overline{B} \subset Q_{\mathfrak{P}} = Q(\overline{B})$. The ring $Q_{\mathfrak{P}}$ is simple artinian, hence quasi-Frobenius. Denote by $\mathcal{I}$ the set of right ideals of $\overline{A}$ consisting of the images of right ideals in $\mathcal{G}_H(A)$. Conditions (T1)–(T3) for $\mathcal{I}$ are immediate from the respective conditions for $\mathcal{G}_H(A)$ (condition (T4) also holds, but we do not need it). Thus $\mathcal{I}$ is a topologizing filter.

Applying Lemma 5.5 with B in place of A, we see that $Q_{\mathfrak{P}}$ is $\mathcal{G}_H(B)$-torsion-free in $\mathcal{M}_B$. By Lemma 5.3 $Q_{\mathfrak{P}}$ is $\mathcal{G}_H(A)$-torsion-free in $\mathcal{M}_A$. This means that each right ideal $I \in \mathcal{G}_H(A)$ has zero left annihilator in $Q_{\mathfrak{P}}$. By the left hand version of Lemma 5.5 $Q_{\mathfrak{P}}$ is $\mathcal{G}^l_H(B)$-torsion-free in ${}_B\mathcal{M}$. Note that BI_H is an H-costable left ideal of B which intersects $\Sigma(B)$. Since $BI_H \subset BI$, we get $BI \in \mathcal{G}^l_H(B)$. Therefore the right annihilator of I in $Q_{\mathfrak{P}}$, equal to the right annihilator of BI, is 0. This verifies condition (a) of Proposition 1.4. Condition (b) holds because $\overline{B}/\overline{A}$ is $\mathcal{G}_H(A)$-torsion in $\mathcal{M}_A$ by Lemma 5.1.

Thus Proposition 1.4 shows that $Q_{\mathfrak{P}}$ is a classical right quotient ring of $\overline{A}$. Since $Q_{\mathfrak{P}}$ is simple artinian, $\overline{A}$ is prime right Goldie. In particular, $P \in \operatorname{Spec} A$. Since $Q_{\mathfrak{P}}$ is $\mathcal{G}_H(A)$-torsion-free, we get $P \notin \mathcal{G}_H(A)$, i.e. $P \in \operatorname{Spec}'A$. Note that $\mathfrak{P}/P \in \mathcal{T}_A$ in view of Lemma 5.1 since $\mathfrak{P}/P$ embeds in B/A. Since $\overline{B} \subset Q_{\mathfrak{P}}$ is $\mathcal{G}_H(A)$-torsion-free, $\mathfrak{P}/P$ must coincide with the $\mathcal{G}_H(A)$-torsion submodule of B/P in $\mathcal{M}_A$. Thus $\mathfrak{P}$ is recovered from P in a unique way.

Next we will work out the correspondence between the primes of A and B in the opposite direction. Starting with an arbitrary $P \in \operatorname{Spec}'A$, we first prove that the map

$$\xi : Q_P \to B \otimes_A Q_P, \qquad q \mapsto 1 \otimes q,$$

is bijective. Given any $b \in B$, Lemma 5.1 shows that $bI \subset A$ for some $I \in \mathcal{G}_H(A)$.

Then the image of ξ contains all elements $u \otimes q$ with $u \in bI$ and $q \in Q_P$. It follows that $b \otimes Q_P \subset \operatorname{Im} \xi$ since $IQ_P = Q_P$ by Lemma 5.5. Hence ξ is surjective. On the other hand, $\operatorname{Ker} \xi$ is $\mathcal{G}^l_H(A)$-torsion by (the left hand version of) Proposition 5.4. Since Q_P is $\mathcal{G}^l_H(A)$-torsion-free in ${}_A\mathcal{M}$ by Lemma 5.5, ξ is injective.

Thus the (B, Q_P)-bimodule $B \otimes_A Q_P$ is freely generated by $e = 1 \otimes 1$ as a right Q_P-module. Hence there is a ring homomorphism $f : B \to Q_P$ defined by the formula $be = ef(b)$ for $b \in B$. The restriction of f to A coincides with the canonical map $A \to Q_P$. Therefore the subring $f(B)$ of Q_P contains the image of A/P. In particular, $f(B)$ is a right order in Q_P. Letting $\mathfrak{P} = \operatorname{Ker} f$, we see that $B/\mathfrak{P} \cong f(B)$ is prime right Goldie with the right quotient ring $Q_\mathfrak{P} \cong Q_P$. Hence $\mathfrak{P}$ is a prime ideal of B and $\mathfrak{P} \cap A = \operatorname{Ker} f|_A = P$. Since $P \notin \mathcal{G}_H(A)$, Lemma 5.2 yields $\mathfrak{P} \notin \mathcal{G}_H(B)$. This shows that $\mathfrak{P} \in \operatorname{Spec}' B$.

Recall that $\operatorname{Stab}(P) \cong (Q_P \otimes H) \otimes_A Q_P$ and $\operatorname{Stab}(\mathfrak{P}) \cong (Q_\mathfrak{P} \otimes H) \otimes_B Q_\mathfrak{P}$. The obvious map $\operatorname{Stab}(P) \to \operatorname{Stab}(\mathfrak{P})$ is a homomorphism of corings, and we need only to check its bijectivity. Since $B \otimes_A Q_P \cong Q_P$ and $Q_\mathfrak{P} \cong Q_P$, we have

$$\operatorname{Stab}(P) \cong (Q_P \otimes H) \otimes_B (B \otimes_A Q_P) \cong (Q_P \otimes H) \otimes_B Q_P \cong \operatorname{Stab}(\mathfrak{P}).$$

Lemma 2.4 allows us to pass easily to the inertializers. □

It is not clear whether Theorem 5.6 remains true when $Q(A)$ is not a two-sided quotient ring. By the left-right symmetry of the hypotheses in Theorem 5.6 the conclusion holds also when the right Goldie condition on the factor rings corresponding to prime ideals in $\operatorname{Spec}' A$ and $\operatorname{Spec}' B$ is replaced with the left Goldie condition.

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