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ON SOME CLASSES OF SEMIARTINIAN RINGS

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Abstract: The weakly regularity of all right R-modules with R an arbitrary ring does not imply the same property of all left R-modules. We describe the rings over which every right and left module is weakly regular and also obtain some description of semiartinian CSL-rings.

Keywords: weakly regular module, quasiprojective module, semiartinian ring SV-ring, CSL-ring

This article is a continuation of [1, 2]. The definitions and results from [2–4] are assumed known. The words as "semiartinian ring" imply that the corresponding conditions hold from right and left.

In [5, p. 452] the problem was formulated of existence of a generalized right SV-ring that is not a generalized left SV-ring. There is an appropriate example in [1, p. 584]. In Section 2 we give a series of necessary and sufficient conditions for an arbitrary ring R to be a generalized SV-ring. In Section 3 we study connection between SV-rings, CSL-rings, CC-rings, and mod-retractable rings.

1. Preliminary Results

This section contains several results needed for further, and some of them deal with the quasiprojective modules. Consideration of quasiprojective modules allows us to develop an appropriate theory in Wisbauer's categories and obtain some results connected with the homological classification of modules. This approach was suggested by Wisbauer and expatiated in his book [6].

By [7, p. 76] each primitive right ideal of a semiartinian ring is a primitive left ideal and vice versa. In what follows we do not distinguish between the primitive right and left ideals in semiartinian rings.

Lemma 1.1. *Given a semiartinian right ring R, the following are equivalent*:

(1) *every primitive image of R is artinian*;

(2) *for each quotient ring R/S with $J(R/S) = 0$, every homogeneous component* $\mathrm{Soc}(R/S_R)$ *is of finite length*;

(3) *for each simple right R-module S, the left vector space ${}_{\mathrm{End}(S)}S$ has finite dimension.*

Proof. (1)⇒(2) Assume the contrary. Then for some quotient-ring R/S such that $J(R/S) = 0$ one of the homogeneous components $\mathrm{Soc}(R/S_R)$ is of infinite length. It is easy to see that in this case there are infinitely many mutually orthogonal nonzero idempotents $e_1, e_2, \dots$ in R/S such that e_1R is a simple submodule of R/S_R, and $e_1R \cong e_iR$ for every natural i. Then $T = \mathrm{Ann}(e_1R)$ is a primitive ideal, and $e_i \notin T/S$ for every i. Hence, R/T contains infinitely many mutually orthogonal idempotents, which contradicts the hypothesis of (1).

(2)⇒(1) Let T be a primitive ideal of R. Obviously, $\mathrm{Soc}(R/T_R)$ possesses only one homogeneous component. Hence, the length of $\mathrm{Soc}(R/T_R)$ is finite, and R/T is an artinian ring.

(1)⇒(3) Let S be a simple right R-module, and let P be its annihilator. Then S may be considered as a right R/P-module. Since R/P is a simple artinian ring, $\dim({}_{\mathrm{End}(S)}S) = \lg(R/P_{R/P})$.

(3)⇒(1) Let P be a primitive ideal of R. If R/P is not artinian then $\mathrm{Soc}(R/P)$ is not a module of finite length. Then there are infinitely many $e_1, e_2, \dots$ of primitive mutually orthogonal idempotents in R/P such that $e_1R \cong e_iR$ for every natural i. Since $e_1Re_i \neq 0$; therefore, $\bigoplus_{i=1}^{\infty} e_1Re_i$ is an infinite dimensional subspace of the left vector space ${}_{e_1Re_1}e_1R$ which contradicts (3). □

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Given an arbitrary right R-module M, introduce the condition
$(*)$ for each invariant submodule N of M such that $J(M/N) = 0$, the homogeneous components of $\operatorname{Soc}(M/N)$ are of finite length.

Lemma 1.2. *Let P be a finitely generated quasiprojective R-module. If every primitive image of $\operatorname{End}(P)$ is artinian then P satisfies $(*)$.*

Proof. Assume the contrary. Then there is an invariant submodule P_0 of P such that $J(P/P_0) = 0$ and P/P_0 includes a submodule as $\bigoplus_{i=1}^{\infty} N_i$, where $(N_i)_{i=1}^{\infty}$ are pairwise isomorphic simple submodules of P/P_0. Since P is quasiprojective, the natural homomorphism $\varphi : \operatorname{End}(P)/\operatorname{Hom}(P, P_0) \to \operatorname{End}(P/P_0)$ is an isomorphism. It follows from [6, 22.2] that $J(\operatorname{End}(P/P_0)) = 0$. Since every simple submodule of P/P_0 is a direct summand, $\operatorname{Hom}(P/P_0, N_i) \neq 0$ for each i. By [2, Lemma 1] the right $\operatorname{End}(P/P_0)$-module $\operatorname{Hom}\left(P/P_0, \bigoplus_{i=1}^{\infty} N_i\right) = \bigoplus_{i=1}^{\infty} \operatorname{Hom}(P/P_0, N_i)$ is semisimple. Hence, the socle of $\operatorname{End}(P)/\operatorname{Hom}(P, P_0)$ has a homogeneous component of infinite length. It follows from the proof of (1)⇒(2) of Lemma 1.1 that not every primitive image of $\operatorname{End}(P)$ is artinian. We obtained a contradiction with the hypothesis of the lemma. □

The next lemma is immediate from [8, Corollary 2.2]:

Lemma 1.3. *Given a primitive right ring R, the following are equivalent*:
(1) *R is a classically semisimple ring*;
(2) *there is an idempotent e in R such that eR and Re are simple injective right and left R-modules.*

Given an arbitrary right R-module M, by transfinite induction define the submodule $SI(M)$ for every ordinal α as follows: Put $SI_\alpha(M) = 0$ with $\alpha = 0$. If $\alpha = \beta + 1$ then $SI_{\beta+1}(M)/SI_\beta(M)$ is the sum of all simple $M/SI_\beta(M)$-injective submodules of the right R-module $M/SI_\beta(M)$. Given a limit ordinal α, put $SI_\alpha(M) = \bigcup_{\beta<\alpha} SI_\beta(M)$. Let τ be the least ordinal such that $SI_\tau(M) = SI_{\tau+1}(M)$. Denote the submodule $SI_\tau(M)$ by $SI(M)$. Given an arbitrary ring R and an ordinal α, denote by $SI_\alpha(R)$ the right ideal $SI_\alpha(R_R)$ which is clearly an ideal.

Lemma 1.4. *Let P be an arbitrary right R-module, and let N be a simple $P/SI_\alpha(P)$-injective module. Then N is P-injective.*

Proof. Assume the contrary. Let α be the least ordinal violating the claim. By [6, 16.3] there is an epimorphism $f : \bigoplus_{i\in I} P_i \to E_P(N)$, where $\bigoplus_{i\in I} P_i$ is the external direct sum of modules and $P_i = P$ for every i. Assume that

$$f(SI_\alpha(\bigoplus_{i\in I} P_i)) \neq 0.$$

Let $\beta \leq \alpha$ be the least ordinal such that $f(SI_\beta(\bigoplus_{i\in I} P_i)) \neq 0$. Clearly, β is a nonlimit ordinal. Then f induces the nonzero homomorphism

$$\bar{f} : SI_\beta\big(\bigoplus_{i\in I} P_i\big)/SI_{\beta-1}\big(\bigoplus_{i\in I} P_i\big) \to E_P(N).$$

Hence, $E_P(N)$ contains a simple P-injective submodule, and $E_P(N) = N$. Assume now that

$$f(SI_\alpha(\bigoplus_{i\in I} P_i)) = 0.$$

Then f induces the epimorphism $\bar{f} : \bigoplus_{i\in I} P_i/SI_\alpha(\bigoplus_{i\in I} P_i) \to E_P(N)$. It is easy to see that

$$\bigoplus_{i\in I} P_i/SI_\alpha\big(\bigoplus_{i\in I} P_i\big) \cong \bigoplus_{i\in I}(P_i/SI_\alpha(P_i)).$$

Thus, $E_P(N) \in \sigma(P/SI_\alpha(P))$, and $E_P(N) = N$, since N is a $P/SI_\alpha(P)$-injective module. □

Lemma 1.5. *Let P be a quasiprojective module, let N_0 be a local P-injective submodule in P of length at most 2 such that $N_0/J(N_0)$ is P-injective and $P = N_1 \oplus \cdots \oplus N_k \oplus N$, where $N_i \cong N_0$ for each i, and let N be a submodule of P that has no submodules isomorphic to N_0. If N_0 and $P/I(P)$ do not possess isomorphic simple subfactors then $N_1 \oplus \cdots \oplus N_k = \pi(P)$, where π is a central idempotent in* $\mathrm{End}(P)$.

Proof. Let π be the projection of P on $N_1 \oplus \cdots \oplus N_k$ associated to the decomposition $P = N_1 \oplus \cdots \oplus N_k \oplus N$. Show that π is a central idempotent in $S = \mathrm{End}(P)$. If $(1-\pi)S\pi \neq 0$ then there is a nonzero homomorphism from πP into $(1-\pi)P = N$. It is easy to see that in this case there is a submodule of N isomorphic to N_0, which contradicts the choice of N. Thus, $(1-\pi)S\pi = 0$. Now, we show that $\pi S(1-\pi) = 0$. Assume the contrary. Then there is a nonzero homomorphism from N into πP. We can extend this homomorphism to a homomorphism ϕ from P into πP such that $\phi(\pi P) = 0$. By the choice of N we have $\phi(I_1(P)) = 0$. Then ϕ induces a nonzero homomorphism $\widetilde{\phi} : P/I_1(P) \to \pi P \subset I_1(P)$. By transfinite induction, it is easy to show that $\widetilde{\phi}(I_\alpha(P)/I_1(P)) = 0$ for each ordinal α. Then $\widetilde{\phi}$ induces a nonzero homomorphism $\overline{\phi} : P/I(P) \to \pi P$. Hence, $P/I(P)$ and πP possess nonisomorphic simple subfactors which contradicts the choice of πP. Thus, $(1-\pi)S\pi = \pi S(1-\pi) = 0$, and so π is a central idempotent in $\mathrm{End}(P)$. □

The example of the ring of upper triangular matrices of order 2 over a field shows that the previous lemma fails if we omit the condition of the nonisomorphy of the simple subfactors of N_0 and $P/I(P)$.

The following two claims are immediate from Lemma 1.5.

Corollary 1.6. *Let P be a quasiprojective generalized SV-module and a self-generator, let N_0 be a simple P-injective module, let $P = N_1 \oplus \cdots \oplus N_k \oplus N$, where $N_i \cong N_0$ for each i, and let N be a submodule of P that has no submodules isomorphic to N_0. Then $N_1 \oplus \cdots \oplus N_k = \pi(P)$, where π is a central idempotent in* $\mathrm{End}(P)$.

Corollary 1.7. *Let P be a quasiprojective generalized SV-module. If P is a self-generator satisfying $(*)$ then for every ordinal α there is a set of pairwise orthogonal central idempotents $\{\pi_i\}_{i\in I}$ in $\mathrm{End}(P/SI_\alpha(P))$ such that*

$$SI_{\alpha+1}(P)/SI_\alpha(P) = \bigoplus_{i\in I} \pi_i(P/SI_\alpha(P)).$$

Lemma 1.8. *Let P be a finitely generated quasiprojective generalized right SV-module and a self-generator such that $SI_1(P) = 0$. If P is not a semilocal module then the semisimple module $I_1(P)/J(I_1(P))$ possesses a block of infinite length.*

Proof. Assume that P is not a semilocal module and all blocks of $I_1(P)/J(I_1(P))$ are of finite length. By [2, Theorem 13] $P/I(P)$ is represented as the direct sum of local modules of length at most 2. Then $P/I(P)$ is of finite length, and so $P/I(P)$ possesses only finitely many simple subfactors up to isomorphism. Let $S_1, \ldots, S_n$ be some representatives of isomorphism classes of simple subfactors of $P/I(P)$. Since P is not a semilocal module, $I_1(P)$ is of infinite length. Since $SI(P) = 0$, it is easy to notice that $I_1(P)$ is the direct sum of local modules of length 2. By assumption, there are $n+1$ pairwise nonisomorphic local submodules of length 2 in $I_1(P)$. Then $I_1(P)$ possesses a local submodule N of length 2 that has all subfactors isomorphic to none of $S_1, \ldots, S_n$. By assumption, it is immediate that P can be represented as $P = N_1 \oplus \cdots \oplus N_k \oplus M$, where $N_i \cong N$ for each i, and M has no submodules isomorphic to N. By Lemma 1.5 there is a central idempotent $\pi \in \mathrm{End}(P)$ such that $N_1 \oplus \cdots \oplus N_k = \pi(P)$. Since P is a generating object in the category $\sigma(P)$, there is a nonzero homomorphism from P into $J(N)$. Thus, there is a homomorphism $f \in \mathrm{End}(P)$ such that $Jm(f) = J(N_1)$. Since $f((1-\pi)(P)) = \pi f((1-\pi)(P)) = f\pi((1-\pi)(P)) = 0$; therefore, $f((1-\pi)(P)) = 0$ and $J(N)$ is a homomorphic image of $N_1 \oplus \cdots \oplus N_k$. Then $J(N) \cong N/J(N)$, which contradicts the P-injectivity of $N/J(N)$. The contradiction shows that the semisimple module $I_1(P)/J(I_1(P))$ has a block of infinite length. □

A module M is a *CSL-module* if every module N in $\sigma(M)$ is simple provided that $\mathrm{End}(N)$ is a skew field. A ring R is a *right CSL-ring* if the module R_R is a CSL-module. Following [9], R is a *right CC-ring* provided that $\mathrm{Hom}(M/N, M) \neq 0$ for every nonzero right R-module M and its proper submodule N. A module P is *right* mod*-retractable* if $\mathrm{Hom}(M, N) \neq 0$ for every nonzero right R-module $M \in \sigma(P)$ and its nonzero submodule N. A ring R is *right* mod*-retractable* provided that R_R is a right mod-retractable module. The right mod-retractable rings were introduced in [10]. In what follows, given arbitrary right R-modules M and N, we denote a submodule of N of the shape $\sum_{f\in \mathrm{Hom}_R(M,N)} f(M)$ by $\mathrm{tr}_N(M)$.

Lemma 1.9. *The following hold:*

(1) *If P is a quasiprojective semiartinian CSL-module then every projective simple module in $\sigma(P)$ is P-injective.*

(2) *Let R be a right semiartinian right CSL-ring, let e be an idempotent in R such that the right ideal eR is the direct sum of local right R-modules, and the right ideals eR and $(1-e)R$ have no isomorphic nonzero direct summands. If R is a right* max*-ring then e is a central idempotent.*

(3) *If R is a semiartinian CSL-ring then every primitive image of R is artinian.*

(4) *Each right CC-ring is a right CSL-ring.*

(5) *If R is the full matrix ring over a perfect ring then R is a CC-ring.*

Proof. (1) Assume that $S \neq E_P(S)$ for some simple projective module S in $\sigma(P)$. Then the semiartinianity of P implies the existence of a local module $M \in \sigma(P)$ of length 2 such that $\mathrm{Soc}(M) = S$. The P-projectivity of S implies $S \ncong M/S$. Hence, $\mathrm{End}(M)$ is a skew field which contradicts the hypothesis.

(2) Assume that $eR(1-e) \neq 0$. Then $M = \mathrm{tr}_{eR}(1-e)R \neq 0$, and the hypothesis implies $M \subset J(eR)$. Let M_0 be a maximal submodule of M, and let N be an (intersection) complement of M/M_0 in eR/M_0. Then $L = (eR/M_0)/N$ is a homogeneous module such that $\mathrm{Soc}(L) \cong M/M_0$ and $\mathrm{Soc}(L) \subset J(L)$. Since R is a semiartinian right ring, L contains a local submodule L_0 of length 2. Since $\mathrm{tr}_L(1-e)R = \mathrm{Soc}(L)$ and $\mathrm{tr}_{L/\mathrm{Soc}(L)}(1-e)R = 0$; therefore, $\mathrm{End}(L_0)$ is a skew field which contradicts the hypothesis. The contradiction shows that $eR(1-e) = 0$. Analogously, $(1-e)Re = 0$.

(3) This is immediate from Lemma 1.3 and (1).

(4) Let M be an arbitrary nonzero nonsimple right R-module. By [7, Theorem 3.10] M possesses some maximal submodule N. Since $\mathrm{Hom}(M/N, M) \neq 0$ by hypothesis, $\mathrm{End}(M)$ has a nonzero homomorphism f such that $\mathrm{Ker}(f) = N \neq 0$.

(5) Clearly, R is a semiartinian max-ring. Then the claim is immediate since all simple right (left) modules over R are isomorphic. □

A right R-module M is *regular* provided that its every cyclic submodule is a direct summand of M.

Lemma 1.10. *Given a finitely generated quasiprojective module P, the following are equivalent*:

(1) *P is a regular module*;

(2) *P is a self-generator, and* $\mathrm{End}(P)$ *is a regular ring.*

Proof. (1)⇒(2) Since P is regular, P generates its every submodule. Then [6, 18.5] implies that P is a self-generator.

Take $f \in \mathrm{End}(P)$. Then $Jm(f)$ is finitely generated, and, consequently, it is a direct summand of P. Since P is projective in $\sigma(P)$ by [6, 18.3]; therefore, so is $Jm(f)$. Thus, $\mathrm{Ker}(f)$ is a direct summand of P. Then [11, Theorem 1] implies that f is a regular element in $\mathrm{End}(P)$.

(2)⇒(1) Let P_0 be a finitely generated submodule of P. Since P is a self-generator, there is an epimorphism $f : P_1 \oplus \dots \oplus P_n \to P_0$, where $P_i = P$ for every $1 \leq i \leq n$. Consider the homomorphism $\bar{f} : P_1 \oplus \dots \oplus P_n \to P_1 \oplus \dots \oplus P_n$ acting by the rule $\bar{f}((p_1, p_2, \dots, p_n)) = (f((p_1, p_2, \dots, p_n)), 0, \dots, 0)$. Since $\mathrm{End}(P_1 \oplus \dots \oplus P_n) \cong M_n(\mathrm{End}(P))$; therefore, $\mathrm{End}(P_1 \oplus \dots \oplus P_n)$ is a regular ring. Hence, by [11, Theorem 1] $Jm(f)$ is a direct summand of $P_1 \oplus \dots \oplus P_n$. Then P_0 is a direct summand of P. □

2. Generalized SV-Rings

Theorem 2.1. *Let P be a finitely generated quasiprojective right R-module and a self-generator. Then the following are equivalent*:

(1) *P is a generalized SV-module, and every primitive image of* $\operatorname{End}(P)$ *is artinian*;

(2) *every module in $\sigma(P/SI(P))$ is a lifting module, and for every ordinal α there is a set of mutually orthogonal central idempotents $\{\pi_i\}_{i\in I}$ in $\operatorname{End}(P/SI_\alpha(P))$ such that*

$$SI_{\alpha+1}(P)/SI_\alpha(P) = \bigoplus_{i\in I} \pi_i(P/SI_\alpha(P)).$$

Proof. (1)⇒(2) It follows from Lemmas 1.2 and 1.8 that $P/SI(P)$ is a semilocal module. By [2, Theorem 6] each module in $\sigma(P/SI(P))$ is a lifting module. The second part of claim follows from Corollary 1.7.

(2)⇒(1) The fact that P is a generalized SV-module is proved in a standard manner. Since $\sigma(P/SI(P))$ is equivalent to the category of all right $\operatorname{End}(P/SI(P))$-modules by [6, 46.2], each right $\operatorname{End}(P/SI(P))$-module is a lifting module, and [5, 13.68] implies that $\operatorname{End}(P/SI(P))$ is a serial artinian ring such that $J^2(\operatorname{End}(P/SI(P))) = 0$.

Let T be a primitive ideal of $\operatorname{End}(P)$. If $\operatorname{Hom}(P, SI(P)) \subset T$ then $\operatorname{End}(P)/T$ is a homomorphic image of $\operatorname{End}(P)/\operatorname{Hom}(P, SI(P)) \cong \operatorname{End}(P/SI(P))$, and so $\operatorname{End}(P)/T$ is a simple artinian ring. Let $\operatorname{Hom}(P, SI(P)) \nsubseteq T$, and let α be the least ordinal such that $\operatorname{Hom}(P, SI_\alpha(P)) \nsubseteq T$. It is clear that α is a nonlimit ordinal. Let $\phi : \operatorname{End}(P)/\operatorname{Hom}(P, SI_{\alpha-1}(P)) \to \operatorname{End}(P)/T$ be the natural homomorphism. Then

$$\operatorname{Hom}(P, SI_\alpha(P))/\operatorname{Hom}(P, SI_{\alpha-1}(P)) \neq 0.$$

By assumption $\operatorname{Hom}(P, SI_\alpha(P))/\operatorname{Hom}(P, SI_{\alpha-1}(P))$ is the direct sum of full matrix rings over skew fields. Hence, there is a central idempotent e in $\operatorname{End}(P)/\operatorname{Hom}(P, SI_{\alpha-1}(P))$ such that $\phi(e) \neq 0$, and $e\operatorname{End}(P)/\operatorname{Hom}(P, SI_{\alpha-1}(P))$ is a simple artinian ring. Since $\operatorname{End}(P)/T$ is an indecomposable ring, $\phi(1-e) = 0$, and

$$\operatorname{End}(P)/T \cong e\operatorname{End}(P)/\operatorname{Hom}(P, SI_{\alpha-1}(P)). \quad \square$$

Theorem 2.2. *Given a ring R, the following are equivalent*:

(1) *R is a generalized right SV-ring such that its every primitive image is artinian*;

(2) *$R/SI(R)$ is a serial artinian ring such that $J^2(R/SI(R)) = 0$, and $SI_{\alpha+1}(R)/SI_\alpha(R)$ is the direct sum of full matrix rings of finite order over skew fields for every ordinal α*;

(3) *R is a generalized right SV-ring, and each direct sum of pairwise isomorphic simple injective right R-modules is an injective module*;

(4) *R is a generalized right SV-ring, and each maximal indecomposable factor R/I of R is a serial artinian ring such that $J^2(R/I) = 0$*;

(5) *R is a generalized SV-ring.*

Proof. The equivalence of (1) and (2) follows from Theorem 2.1.

(2)⇒(5) This is obvious.

(5)⇒(1) Let P be a primitive ideal of R. Assume that R/P is not a classically semisimple ring. Then from [1, Theorem 3.4; 2, Theorem 12] it follows that R/P is a primitive semiartinian ring such that $SI_1(R/P_{R/P}) \neq 0$ and $SI_1({}_{R/P}R/P) \neq 0$. Since the semisimple modules $\operatorname{Soc}(R/P_{R/P})$ and $\operatorname{Soc}({}_{R/P}R/P)$ are homogeneous, there is a primitive idempotent $e \in R/P$ such that eR/P and R/Pe are injective right and left modules. Then Lemma 1.3 implies that R/P is a classically semisimple ring, which contradicts the hypothesis.

(2)⇒(3) Let $\{S_i\}_{i\in I}$ be a set of pairwise isomorphic simple injective right R-modules. If

$$E(\bigoplus_{i\in I} S_i)SI(R) \neq 0$$

then for some nonlimit ordinal α we have

$$E(\bigoplus_{i\in I} S_i)SI_{\alpha-1}(R) = 0,$$

and $E(\bigoplus_{i\in I} S_i)SI_\alpha(R) \neq 0$. Hence, $E(\bigoplus_{i\in I} S_i)$ may be considered as an $R/SI_{\alpha-1}(R)$-module, and there is a central idempotent e in $R/SI_{\alpha-1}(R)$ such that $eR/SI_{\alpha-1}(R)$ is a semisimple module and $E(\bigoplus_{i\in I} S_i)e \neq 0$. Then $S_i e \neq 0$ and $S_i(1-e) = 0$ for every $i \in I$. It is easy to see that in this case $E(\bigoplus_{i\in I} S_i)(1-e) = 0$. Thus, $E(\bigoplus_{i\in I} S_i)$ may be considered as a right $eR/SI_{\alpha-1}(R)$-module. Since $eR/SI_{\alpha-1}(R)$ is a classically semisimple ring,

$$E(\bigoplus_{i\in I} S_i) = \bigoplus_{i\in I} S_i.$$

If $E(\bigoplus_{i\in I} S_i)SI(R) = 0$ then $E(\bigoplus_{i\in I} S_i)$ may be considered as a right $R/SI(R)$-module. Since $R/SI(R)$ is a Noetherian ring, $E(\bigoplus_{i\in I} S_i) = \bigoplus_{i\in I} S_i$ by [4, 27.3].

(3)⇒(1) Let P be a primitive ideal of R. Then $\mathrm{Soc}(R/P_R)$ is a homogeneous semisimple module. Assume that R/P is not a classically semisimple ring. By [1, Theorem 3.4; 2, Theorem 12] we infer that R/P_R contains a simple injective submodule. Then $\mathrm{Soc}(R/P_R)$ is injective by assumption. Since $\mathrm{Soc}(R/P_R)$ is essential in R/P_R; therefore, $\mathrm{Soc}(R/P_R) = R/P_R$ which contradicts the hypothesis. Hence, R/P is a classically semisimple ring.

(2)⇒(4) Let R/P be a maximal indecomposable factor of R. If $SI(R) \subset P$ then R/P is a serial artinian ring such that $J^2(R/P) = 0$. Assume that $SI(R) \not\subseteq P$. Then arguing as in the proof of the implication (2)⇒(1) from the proof of Theorem 2.1, we infer that R/P is isomorphic to the full matrix ring of finite order over a skew field.

(4)⇒(1) Let P be a primitive ideal of R. Since R/P is an indecomposable ring, the implication is immediate from [12, Corollary 1.7]. □

3. *SV*- and *CSL*-Rings

Theorem 3.1. *Given a quasiprojective regular semiartinian module P, the following are equivalent*:
(1) *P is a CSL-module*;
(2) *P is a* mod-*retractable module*;
(3) *P is a SV-module.*

Proof. (1)⇒(3) Since the quotient module of a regular module by an invariant submodule is regular; therefore, the implication is immediate from Lemmas 1.4 and 1.9.

(2)⇒(3) Let $S \in \sigma(P)$ be a simple module. Assume that $S \neq E_P(S)$. It follows from the proof of (1) of Lemma 1.9 that $E_P(S)$ contains a local submodule L of length 2 such that $\mathrm{Soc}(L) = S$. It is clear that there is an epimorphism $f : \bigoplus_{i\in I} P_i \to E_P(S)$, where $P_i = P$ for every $i \in I$. For some nonlimit ordinals α and β, we have $f(\mathrm{Soc}_{\alpha-1}(P)) = 0$, $f(\mathrm{Soc}_\alpha(P)) \neq 0$, and $f(\mathrm{Soc}_{\beta-1}(P)) \subset S$, $f(\mathrm{Soc}_\beta(P)) \not\subseteq S$. Since $J(f(\mathrm{Soc}_\beta(P))) \neq 0$, we have $f(\mathrm{Soc}_{\beta-1}(P)) \neq 0$. Hence, $\alpha < \beta$. Since P is a quasiprojective regular module, $\mathrm{Hom}(P/\mathrm{Soc}_{\beta-1}(P), \mathrm{Soc}_\alpha(P)/\mathrm{Soc}_{\alpha-1}(P)) = 0$. Therefore, $L/S \not\cong S$ and $\mathrm{Hom}(L, S) = 0$. We obtained a contradiction with the hypothesis.

The implications (3)⇒(1) and (3)⇒(2) are immediate. □

The equivalence of (1) and (3) of the next corollary was established in [13, Theorem 2.8].

Corollary 3.2. *Given a regular semiartinian ring R, the following are equivalent*:
(1) *R is a right CSL-ring*;
(2) *R is a right* mod-*retractable ring*;
(3) *R is a right SV-ring.*

Theorem 3.3. *Given a semiartinian ring R, the following are equivalent*:

(1) *R is a* mod-*retractable ring*;

(2) *R is a CSL-ring*;

(3) *every maximal indecomposable factor of R is isomorphic to the full matrix ring of finite order over a perfect ring.*

Proof. (1)⇒(2) This is immediate.

(2)⇒(3) It suffices to show that each semiartinian CSL-ring R contains a nonzero central idempotent e such that eR is the full matrix ring of finite order over a perfect ring. Let R be a semiartinian right CSL-ring. It is easy to see that each right ideal of R not lying in $J(R)$ contains a locally direct summand of the module R_R. Then it is immediate from Lemma 1.1 and (2) and (3) of Lemma 1.9 that there is an idempotent e with the property mentioned above.

(3)⇒(1) Let M be a right R-module, and let S be a simple submodule of M. Show that $\mathrm{Hom}(M,S) \neq 0$. If N is an (intersection) complement of S in M then $(S+N)/N$ is an essential submodule of M/N. Without loss of generality, we may assume therefore that S is an essential submodule of M. By assumption and [5, 13.12(2)] $R/\mathrm{Ann}(M)$ is the full matrix ring of finite order over a local ring. Then the implication is immediate from the fact that $R/\mathrm{Ann}(M)$ is a semiartinian max-ring over which all right (left) modules are isomorphic. □

The equivalence of (1) and (5) in the next corollary was established in [14].

Corollary 3.4. *Given a ring R, the following are equivalent*:

(1) *R is a perfect CSL-ring*;

(2) *R is a perfect* mod-*retractable ring*;

(3) *R is a right CC-ring*;

(4) *R is a left CC-ring*;

(5) *R is the finite direct product of some full matrix rings of finite order over perfect rings.*

Proof. By the left and right symmetry of the condition of (5) it suffices to show the equivalence of (1)–(3) and (5). The equivalences (1) ⇔ (2) ⇔ (5) are immediate from the previous theorem. The equivalence (3) ⇔ (5) follows from [9, Theorem 3.10] and (2), (4), and (5) of Lemma 1.9. □

Theorem 3.5. *Given a regular ring with all primitive images artinian, the following are equivalent*:

(1) *R is a right* mod-*retractable ring*;

(2) *R is an SV-ring.*

Proof. (1)⇒(2) Without loss of generality we may assume that R is strictly regular by [15, Theorem 6.6; 10, Theorems 2 and 8]. Let xR be an arbitrary cyclic right R-module. If $E(xR)\,\mathrm{Ann}(x) \neq 0$ then for some nonzero central idempotent $e \in R$ we have $E(xR)e \cap xR \neq 0$ and $xRe = 0$. On the other hand, $E(xR)e \cap xR = xRe = 0$. The contradiction shows that $E(xR)$ may be considered as an injective $R/\mathrm{Ann}(x)$-module. Since $R/\mathrm{Ann}(x)$ is mod-retractable, there is a nonzero $R/\mathrm{Ann}(x)$-homomorphism $f : E(xR) \to xR$. Since $R/\mathrm{Ann}(x)$ is regular, $xR = N \oplus M$ and $N \subset f(E(xR))$, where M and N are some submodules of xR and $N \neq 0$. Let π be the projection to the first summand of the decomposition $xR = N \oplus M$. Since N is obviously a projective $R/\mathrm{Ann}(x)$-module, the epimorphism πf is split. Thus, the right R-module xR contains a nonzero injective submodule. Hence each right module over R contains a nonzero injective submodule. Then R is an SV-ring by Theorem 2.2 and [16].

(2)⇒(1) This is obvious. □

Theorem 3.6. *Let R be a ring whose every right ideal is an ideal. Then the following are equivalent*:

(1) *R is a* mod-*retractable ring*;

(2) *R is a semiartinian ring.*

Proof. (1)⇒(2) If S is a simple right R-module and N is a nonzero submodule of $E(S)$ then $\mathrm{Hom}(N,S) \neq 0$. Hence, N possesses a maximal submodule, and [17, Theorem 1] implies that R is a max-ring. Then from Theorem 3.5 [18, Lemma 3.2; 5, 5.51 and 5.54] it follows that R is a semiartinian ring.

(2)⇒(1) Let S be a simple submodule of a right R-module M. Without loss of generality we may assume that S is an essential submodule of M. Let M_0 be a maximal submodule of M, and $x \in M \backslash M_0$. Clearly, we may consider xR as a right $R/\operatorname{Ann}(x)$-module. Put $\overline{R} = R/\operatorname{Ann}(x)$. Since $\overline{R}_{\overline{R}} \cong x\overline{R}$ is a homogeneous semiartinian module; $\overline{R}$ is a local ring. Hence, $\overline{R}/J(\overline{R})_{\overline{R}} \cong S_{\overline{R}}$. Thus, $S_R \cong xR/J(xR) = xR/(xR \cap M_0) \cong M/M_0$. □

Corollary 3.7. *Given a commutative ring R, the following are equivalent*:
(1) *R is a* mod-*retractable ring*;
(2) *R is a semiartinian ring.*

Theorem 3.8. *Let P be a finitely generated quasiprojective module. If P is an SV-module then P is regular, and* End(P) *is a right SV-ring.*

Proof. It follows from [6, 23.8] that P is a generating object in $\sigma(P)$. Hence, by [6, 46.2] $\sigma(P)$ is equivalent to the category of right End(P)-modules. Then each nonzero right End(P)-module includes a nonzero injective submodule, and [16, Theorem 13] implies that End(P) is a right SV-ring. Since End(P) is a regular ring by [19, Proposition 2.3], P is regular by Lemma 1.10. □

Corollary 3.9 [19, Theorem 2.9]. *Let R be a right SV-ring, and let P be a finitely generated projective right R-module. Then* End(P) *is a right SV-ring.*

Proof. This is immediate from Theorem 3.8 and the fact that each right module over a right SV-ring is an SV-module. □

Theorem 3.10. *Given a finitely generated quasiprojective right R-module P, the following are equivalent*:
(1) *P is an SV-module*;
(2) *P is a regular generalized SV-module*;
(3) *P is a V-module and a generalized SV-module.*

Proof. (1)⇒(2) This follows from Theorem 3.8.
(2)⇒(3) By Lemma 1.10, P is a self-generator, and End(P) is a regular ring. Since $\sigma(P)$ is equivalent to the category of right End(P)-modules by [6, 46.2]; therefore, End(P) is a regular generalized SV-ring. Then by [1, Theorem 3.7] End(P) is a right V-ring. Hence, P is a V-module.
(3)⇒(1) This follows from [2, Theorem 3.5]. □

From the results of Section 3 we immediately have

Theorem 3.11. *Given a ring R, the following are equivalent*:
(1) *R is an SV-ring*;
(2) *$R = SI(R)$, and $SI_{\alpha+1}(R)/SI_{\alpha}(R)$ is the direct sum of full matrix rings of finite order over skew fields for every ordinal α;*
(3) *R is a right SV-ring such that its every primitive image is an artinian ring*;
(4) *R is a right V-ring over every right and left module is weakly regular*;
(5) *R is a regular ring over which every right and left module is weakly regular*;
(6) *R is a regular* mod-*retractable ring such that its every primitive image is artinian*;
(7) *R is a regular semiartinian CSL-ring.*

Remark. Let K be a field, and $R = \prod_{i=1}^{\infty} K_i$, where $K_i = K$ for each natural i. Then by [20, Theorem 18] and Theorem 3.5 R is not a mod-retractable CSL-ring. By the results above the following strict inclusions hold:

$$\{CC\text{-rings}\} \subset \{\text{mod-retractable rings}\} \subset \{CSL\text{-rings}\}.$$

Open problems:. 1. *Describe the regular right CSL-rings.*
2. *Is a regular right* mod-*retractable ring a right SV-ring?*
3. *Is a right* mod-*retractable ring a semiartinian right ring?*

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