

NUMERICAL METHOD FOR CALCULATION OF THE GENERALIZED NATURAL MODES OF AN INHOMOGENEOUS OPTICAL FIBER

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Abstract – The eigenvalue problem for generalized natural modes of an inhomogeneous optical fiber without a sharp boundary is formulated as a problem for the set of time-harmonic Maxwell equations with Reichardt condition at infinity in the cross-sectional plane. The generalized eigenvalues of this problem are the complex propagation constants on a logarithmic Riemann surface. The original problem is reduced to a nonlinear spectral problem with Fredholm integral operator. Theorem on spectrum localization is proved, and then it is proved that the set of all eigenvalues of the original problem can only be a set of isolated points on the Riemann surface, and it is also proved that each eigenvalue depends continuously on the frequency and refraction index and can appear and disappear only at the boundary of the Riemann surface. The Galerkin method for numerical calculation of the generalized natural modes is proposed, and the convergence of the method is proved.

I. INTRODUCTION

Optical fibers are dielectric waveguides (DWs), i.e., regular dielectric rods, having various cross sectional shapes, and where generally the refractive index of the dielectric may vary in the waveguide's cross section [1]. Although existing technologies often result in a refractive index that is anisotropic, frequently it is possible to assume that the fiber is isotropic [2], which is the case investigated in this work. The study of the source-free electromagnetic fields, called natural modes, that can propagate on DWs necessitates that longitudinally the rod extend to infinity. Since often DWs are not shielded, the medium surrounding the waveguide transversely forms an unbounded domain, typically taken to be free space. This fact plays an extremely important role in the mathematical analysis of natural waveguide modes, and brings into consideration a variety of possible formulations. Each different formulation can be cast as an eigenvalue problem for the set of time-harmonic Maxwell equations, but they differ in the form of the condition imposed at infinity in the cross-sectional plane, and hence in the functional class of the natural-mode field. This also restricts the localization of the eigenvalues in the complex plane of the eigenparameter [3]. All of the known natural-mode solutions (i.e., guided modes, leaky modes, and complex modes) satisfy the Reichardt condition at infinity [3]. The wavenumbers β may be generally considered on the appropriate logarithmic Riemann surface. The Reichardt condition in this problem is connected with the fact that wavenumbers may be complex. For real wavenumbers on the principal ("proper") sheet of this Riemann surface, one can reduce the Reichardt condition to either the Sommerfeld radiation condition or to the condition of exponential decay. The Reichardt condition may be considered as a generalization of the Sommerfeld radiation condition and can be applied for complex wavenumbers. This condition may also be considered as the continuation of the Sommerfeld radiation condition from a part of the real axis of the complex parameter β to the appropriate logarithmic Riemann surface.

II. STATEMENT OF THE PROBLEM

We consider the generalized natural modes of an inhomogeneous optical fiber without a sharp boundary. Let the three-dimensional space be occupied by an isotropic source-free medium, and let the refractive index be prescribed as a positive real-valued function $n = n(x_1, x_2)$ independent of the longitudinal coordinate x_3 and equal to a constant n_∞ outside a cylinder. The axis of the cylinder is parallel to the x_3 -axis, and its cross-section is a bounded domain Ω with a Lipschitz boundary on the plane $R^2 = \{(x_1, x_2) : -\infty < x_1, x_2 < \infty\}$. Denote by Ω_∞ the unbounded domain $\Omega_\infty = R^2 \setminus \overline{\Omega}$, and denote by n_+ the maximum of the function n in the domain Ω ,

where $n_+ > n_\infty$. Let the function n belong to the space of real-valued continuous in R^2 functions. By $C^2(R^2)$ denote the space of twice continuously differentiable in R^2 complex-valued functions.

The modal problems can be formulated as a vector eigenvalue problem for the set of differential equations (we use notations from [3] for differential operators)

$$\text{Rot}_\beta \mathbf{E} = i\omega\mu_0 \mathbf{H}, \quad \text{Rot}_\beta \mathbf{H} = -i\omega\varepsilon_0 n^2 \mathbf{E}, \quad x \in R^2. \quad (1)$$

Here ε_0 , μ_0 are the free-space dielectric and magnetic constants, respectively. We consider the propagation constant β as an unknown complex parameter and radian frequency $\omega > 0$ as a given parameter. We seek non-zero solutions $[\mathbf{E}, \mathbf{H}]$ of set (1) in the space $(C^2(R^2))^6$. By Λ denote the Riemann surface of the function $\chi(\beta) = \ln \sqrt{k^2 n_\infty^2 - \beta^2}$, where $k^2 = \omega^2 \varepsilon_0 \mu_0$. By $\Lambda_0^{(1)}$ denote the principal (“proper”) sheet of Λ , which is specified by the following conditions: $-\pi/2 < \arg \chi(\beta) < 3\pi/2$, $\text{Im}(\chi(\beta)) \geq 0$, $\beta \in \Lambda_0^{(1)}$.

We say that vector-function $[\mathbf{E}, \mathbf{H}]$ satisfies the Reichardt condition if the vector-function $[\mathbf{E}, \mathbf{H}]$ can be represented for all $|x| > R_0$ as

$$[\mathbf{E}, \mathbf{H}]^T = \sum_{l=-\infty}^{\infty} [\mathbf{A}_l, \mathbf{B}_l]^T H_l^{(1)}(r, \varphi) \exp(il\varphi), \quad (2)$$

where $H_l^{(1)}$ is the Hankel function of the first kind and index l , (r, φ) are the polar coordinates of the point x . The series in (2) should converge uniformly and absolutely.

Definition 1. A nonzero vector $[\mathbf{E}, \mathbf{H}] \in (C^2(R^2))^6$ is referred to as a generalised eigenvector (or eigenmode) of the problem (1), (2) corresponding to an eigenvalue $\beta \in \Lambda$ if the relations of problem (1), (2) are valid. The set of all eigenvalues of the problem (1), (2) is called the spectrum of this problem.

II. GALERKIN METHOD

If $[\mathbf{E}, \mathbf{H}]$ is an eigenvector of problem (1), (2) corresponding to an eigenvalue $\beta \in \Lambda$, then

$$\mathbf{E}(x) = (k^2 n_\infty^2 + \text{Grad}_\beta \text{Div}_\beta) \frac{1}{n_\infty^2} \int_\Omega (n^2(y) - n_\infty^2) \Phi(\beta; x, y) \mathbf{E}(y) dy, \quad (3)$$

$$\mathbf{H}(x) = -i\omega\varepsilon_0 \text{Rot}_\beta \int_\Omega (n^2(y) - n_\infty^2) \Phi(\beta; x, y) \mathbf{E}(y) dy, \quad x \in R^2, \quad y \in \Omega. \quad (4)$$

where $\Phi = i/4H_0^{(1)}(\chi(\beta)|x-y|)$. For any $(x, y) \in \Omega^2$ the function Φ is analytic for $\beta \in \Lambda$. Passing the operator $\text{Grad}_\beta \text{Div}_\beta$ under the integral in relation (3), and using the differentiation rule for weakly singular integrals we obtain a nonlinear spectral problem for a strongly-singular domain integral equation

$$A(\beta)\mathbf{E} = 0, \quad x \in \Omega; \quad A: (L_2(\Omega))^3 \rightarrow (L_2(\Omega))^3. \quad (5)$$

Definition 2. A nonzero vector $\mathbf{E} \in (L_2(\Omega))^3$ is called an eigenvector of operator-valued function $A(\beta)$ corresponding to an eigenvalue $\beta \in \Lambda$ if the relation (5) is valid. The set of all $\beta \in \Lambda$ for which the operator $A(\beta)$ does not have the bounded inverse operator in $(L_2(\Omega))^3$ is called the spectrum of operator-valued function $A(\beta)$. Denote by $\sigma(A) \subset \Lambda$ the spectrum of operator-valued function $A(\beta)$.

Theorem 1. For all $\beta \in \Lambda$ the operator $A(\beta)$ is Fredholm with zero index. The sets $\{\beta \in \Lambda_0^{(1)} : |\beta| < kn_\infty, \text{Im} \beta = 0\}$, $\{\beta \in \Lambda_0^{(1)} : |\beta| \geq kn_+, \text{Im} \beta = 0\}$, and $\{\beta \in \Lambda_0^{(1)} : \text{Re} \beta = 0\}$ are free of the eigenvalues of problem (1), (2). The spectrum of problem (1), (2) is equivalent to the spectrum of the operator-

valued function $A(\beta)$ and can be only a set of isolated points on Λ . Each eigenvalue β of the problem (1), (2) depends continuously on (ω, n_+, n_∞) and can appear and disappear only at the boundary of Λ , i.e., at $\beta = \pm kn_\infty$ and at infinity on Λ .

This theorem was proved in [4]. The eigenvectors of problem (1), (2) are equivalent to the eigenvectors of the operator-valued function $A(\beta)$ corresponding to the same eigenvalues b in the sense of results [4].

Consider the Galerkin method for numerical approximation of integral equation (5). We cover W with small squares D_i and denote by W_n the sub-domain $W_n = \bigcup_{i=1}^n D_i \subseteq W$. We seek the approximate solution \mathbf{E}_n of equation (5) in the form of linear combination $\mathbf{E}_n(x) = \sum_{i=1}^n a_i \mathbf{F}_i(x)$, $x \in W_n$, where \mathbf{F}_i are basis functions, $\mathbf{F}_i(x) = 1$, if $x \in D_i$, $\mathbf{F}_i(x) = 0$, if $x \notin D_i$. We seek the non-zero approximate solution \mathbf{E}_n in the space $H_n = \text{span}\{\mathbf{F}_1, \mathbf{K}, \mathbf{F}_n\}$. The unknown coefficients a_i can be determined from the set of linear algebraic equations:

$$\sum_{i=1}^n a_i (A(b)\mathbf{F}_i, \mathbf{F}_j) = 0, \quad j = 1, \mathbf{K}, n, \quad (6)$$

where (\cdot, \cdot) denotes inner product in $(L_2(\Omega))^3$. The singular Galerkin elements $(A(b)\mathbf{F}_i, \mathbf{F}_j)$ are calculated analytically by formula:

$$\frac{\partial}{\partial x_i} \frac{\partial}{\partial x_i} \frac{1}{2p} \int_{D_i} \ln|x-y| dy = -\frac{1}{2}, \quad (7)$$

that is true if point x is at center of the square D_i .

Therefore, using Galerkin method for solving nonlinear spectral problem for strongly-singular domain integral equation (5), we obtain finite-dimensional nonlinear spectral problem (6), that we can rewrite in the operator form:

$$A_n(\beta)\mathbf{E}_n = 0, \quad x \in \Omega_n; \quad A_n : H_n \rightarrow H_n, \quad (8)$$

where the operator-valued function $A_n(\beta)$ is determined by (6).

Convergence of the presented numerical algorithm is governed by the theorem, which follows from theorem 1 and results of paper [5]. Following [5], we denote by N' the infinite subset of the set of integers N . Denote by $\mathbf{E}_n \rightarrow \mathbf{E}$, $n \in N'$, the convergence $\mathbf{E}_n \rightarrow \mathbf{E}$ for $n \rightarrow \infty$, $n \in N'$.

Theorem 2. If $b_n \in s(A_n)$, $A_n(\beta_n)\mathbf{E}_n = 0$, $\|\mathbf{E}_n\| = 1$, and $b_n \rightarrow b_0 \in L$, $\mathbf{E}_n \rightarrow \mathbf{E}_0$, $n \in N' \subseteq N$, then $b_0 \in s(A)$ and $A(\beta_0)\mathbf{E}_0 = 0$, $\|\mathbf{E}_0\| = 1$.

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