The Over-determined Boundary Value Problem Method in the Electromagnetic Waves Propagation and Diffraction Theory

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Abstract— The over-determined boundary value problems in the partial domains are proposed to be used as the auxiliary problems to investigate the wave processes in the complex structures. The necessary and sufficient conditions of solvability of the over-determined problem are the dependencies between the boundary functions. These dependencies can be obtained in terms of the Fourier transforms or Fourier coefficients of the boundary functions. The diffraction problems for the electromagnetic waves on the conducting screens in the space and in the waveguides with metallic walls are considered as the examples.

1. INTRODUCTION

The method of partial domains is widely used in the electromagnetic wave propagation and diffraction theory to solve the conjugation problems and boundary value problems with mixed boundary conditions. In the case when the integral or summatorial representations of the field to be found are obtained in some parts of the waveguide structure it is possible to get the integral or summatorial equations equivalent to the initial problem.

It is convenient to consider the over-determined boundary value problems in the partial domains as the auxiliary problems. By this we are to consider more boundary conditions on some pieces of the domains boundaries than it is necessary to choose the unique solution. The review of articles devoted to over-determined boundary value problem method for the partial differential equations is given in the paper [1].

The necessary and sufficient conditions of solvability of the over-determined problems have the form of the supplementary connections between the auxiliary boundary functions. These connections together with the initial boundary conditions and the conjunction conditions form the complete set of equations to determine the electromagnetic field. In many cases the conditions at the infinity for the unbounded domains (the radiation conditions) can be formulated also as the auxiliary conditions for the boundary functions in the over-determined problem.

2. TWO-DIMENSIONAL PROBLEMS IN THE PLANE WAVEGUIDE

Consider the diffraction problem for two-dimensional TE-wave in the plane waveguide on the infinitely thin ideally conducting screen (see Fig. 1).

The eigen TE-wave of the waveguide with potential function $u^0(x, z)$ is falling down on the screen M. It is necessary to find the potential functions $u^{\mp}(x, z)$ of the field being generated by the diffraction of this wave.

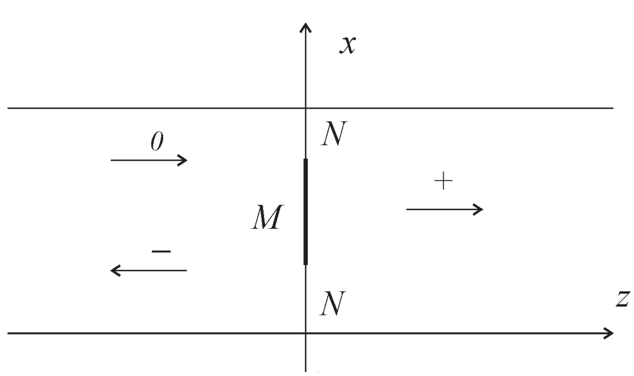

Figure 1: Lateral screen in the plane waveguide.

It is well known that the potential function $u(x,z)$ of any of two possible polarizations of two-dimensional electromagnetic field in the plane waveguide is to satisfy the Helmholtz equation

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial z^2}+k^2 u=0, \qquad (1)$$

here and further k being the real number. The function $u(x,z)$ is to vanish on the walls of the waveguide $x=0$ and $h=h$ for TE-waves.

Consider the auxiliary over-determined boundary value problem in the right-hand part $z>0$ of the waveguide structure. It is necessary to find the solution of the Equation (1) in the half-strip $0<x<h,\ z>0$, satisfying the boundary conditions on the walls of the waveguide $u(0,z)=0,\ u(h,z)=0$ and the conditions $u(x,0+0)=u_0(x),\ \frac{\partial u}{\partial z}(x,0+0)=u_1(x)$ on its cross-section. Besides, the solution to be found is not to contain the elementary harmonics transferring energy from infinity or infinitely increasing by $z\to+\infty$.

It is easy to show that solution of such over-determined boundary value problem for the Helmholtz equation exists and is unique if and only if the Fourier coefficients of the functions $u_0(x)$ and $u_1(x)$ satisfy the conditions

$$u_{1n}+i\gamma_n u_{0n}=0, \quad n=1,2,\ldots, \qquad (2)$$

where

$$\gamma_n=\sqrt{k^2-\left(\frac{\pi n}{h}\right)^2}, \quad n=1,2,\ldots$$

Let us determine these numbers in such way that either $\operatorname{Re}\gamma_n\geq 0$, or $\operatorname{Im}\gamma_n<0$. By this

$$u^-(x,z)=\sum_{n=1}^{+\infty}A_n\varphi_n(x)\,e^{i\gamma_n z}, \quad u^+(x,z)=\sum_{n=1}^{+\infty}B_n\varphi_n(x)\,e^{-i\gamma_n z},$$

where

$$\varphi_m(x)=\sqrt{\frac{2}{h}}\sin\frac{\pi m x}{h}, \quad m=1,2,\ldots$$

In the same time functions $u_0(x)$ and $u_1(x)$ then and only then are traces on the cross $z=0$ of the solution of the Helmholtz equation in the half-strip satisfying the boundary conditions and the condition at the infinity when

$$u_0(x)=\int_0^h u_1(t)\,K_1(t,x)\,dt, \quad x\in(0,h), \quad K_1(t,x)=\sum_{m=1}^{+\infty}\frac{1}{\gamma_m}\varphi_m(t)\varphi_m(x). \qquad (3)$$

The same statements are valid for the left-hand half of the waveguide structure.

As it is shown in [2], by $\varepsilon^{\pm}=\varepsilon$ the diffraction problem for the electromagnetic wave on the lateral screen in the plane waveguide is equivalent to the infinite set of linear algebraic equations

$$A_k-\sum_{n=1}^{+\infty}A_n\gamma_n\sum_{m=1}^{+\infty}\frac{1}{\gamma_m}I_{nm}J_{mk}=-C_l\,I_{lk}, \quad k=1,2,\ldots \qquad (4)$$

or to integral equation

$$u_1^-(x)-\int_M u_1^-(t)\,L(t,x)\,dt=-iC_l\,\psi_l(x), \quad x\in M,$$

where $I_{nm},\ J_{mk}$ are some constants, $L(t,x),\ \psi_l(x)$ are some functions.

3. THE CAUCHY PROBLEM FOR A HALF-SPACE

Consider the Maxwell equations set for complex amplitudes E and H of the electric and magnetic vectors of the electromagnetic field harmonically dependent on time

$$\text{rot}\, H = i\omega\varepsilon_0\varepsilon\, E, \quad \text{rot}\, E = -i\omega\mu_0\mu\, H \tag{5}$$

in the upper half-space R^3_+ and in the lower half-space R^3_-, here $\varepsilon = \varepsilon(z) = \{z > 0 : \varepsilon_+;\ z < 0 : \varepsilon_-\}$ and μ are real numbers. Let the traces of tangent components E and H be given on the plane $z = 0$ and

$$[z_0, E](x, y, 0) = e(x, y), \quad [z_0, H](x, y, 0) = h(x, y). \tag{6}$$

We call the solution E, H of Maxwell equations set in the domain R^3_+ outgoing into a half-space if each of its components is the distribution of slow growth not containing harmonics which transfer the energy on infinity and the traces (6) are defined correctly.

Let k be a wave number, $k^2 = \omega^2\mu_0\mu\varepsilon_0\varepsilon$. Denote by

$$\gamma(\xi, \eta) = \left\{\xi^2 + \eta^2 \geq k^2 \ :\ i\sqrt{\xi^2 + \eta^2 - k^2};\ \ \xi^2 + \eta^2 \leq k^2 \ :\ \sqrt{k^2 - \xi^2 - \eta^2}.\right\}$$

Vector-functions E, H are the solution of the Cauchy problem (5), (6) in the class of solutions outgoing into the half-space if and only if, when [3] the equality for the Fourier transforms of traces of their components on the plane $z = 0$

$$\omega\varepsilon_0\varepsilon\gamma(\xi, \eta)e(\xi, \eta) + P(\xi, \eta)h(\xi, \eta) = 0 \quad \text{or} \quad P(\xi, \eta)e(\xi, \eta) - \omega\mu_0\mu\gamma(\xi, \eta)h(\xi, \eta) = 0 \tag{7}$$

is fulfilled, where

$$P(\xi, \eta) = \begin{pmatrix} \xi\eta & k^2 - \xi^2 \\ \eta^2 - k^2 & -\xi\eta \end{pmatrix}, \quad A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad B(\xi, \eta) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ \eta & -\xi \end{pmatrix}.$$

Formulas (7) follow one from another; they determine the dependence between the traces of the tangent components of the field given on the boundary of the domain in terms of their Fourier transforms. It follows from (7) that

$$\begin{aligned} e(x, y) &= -\frac{1}{\omega\varepsilon_0\varepsilon} \iint K(x_1, y_1; x, y)\, h(x_1, y_1)\, dx_1\, dy_1, \\ h(x, y) &= \frac{1}{\omega\mu_0\mu} \iint K(x_1, y_1; x, y)\, e(x_1, y_1)\, dx_1\, dy_1, \end{aligned} \tag{8}$$

where

$$K(x_1, y_1; x, y) = \frac{1}{(2\pi)^2} \iint \frac{1}{\gamma(\xi, \eta)}\, P(\xi, \eta)\, e^{i(x_1 - x)\xi + i(y_1 - y)\eta}\, d\xi\, d\eta.$$

In the case of analogues statements for the solutions of the Cauchy problem outgoing into the lower half-space in R^3_- one should change a sign of the function $\gamma(\xi, \eta)$ in above given formulas.

4. THE DIFFRACTION PROBLEM ON A PLANE SCREEN

Let M be a metallic screen placed in the plane $z = 0$ and N be its supplement up to the whole plane. Let $\varepsilon_\pm = \varepsilon$, $\mu_\pm = \mu$ and $E_0(x, y, z)$ be the electric field of a wave falling down on the screen. It is necessary to seek the solutions of the Maxwell equations set outgoing from the plane $z = 0$ into half-spaces satisfying the boundary conditions

$$[z_0, E_\pm + E_0](x, y) = 0, \ \ (x, y) \in M,$$

$$[z_0, E_+ - E_-](x, y) = 0, \ \ [z_0, H_+ - H_-](x, y) = 0, \ \ (x, y) \in N.$$

Let us write down the equalities of the form (8) for the traces of tangent components of the vectors $E_\pm$, $H_\pm$. It follows from the boundary conditions that $e_+ = e_- = -e_0$ on M and $e_+ = e_-$ on N. Denote by $e = e_+ = e_-$. Then $h_+ + h_- = 0$ everywhere on the plane $z = 0$, by this $h_+ = h_-$ on N. Thus

$$e(x, y) = \mp\frac{1}{\omega\varepsilon_0\varepsilon} \iint\limits_M K(x_1, y_1; x, y)\, h_\pm(x_1, y_1)\, dx_1\, dy_1,$$

$$h_{\pm}(x,y) = \mp\frac{1}{\omega\mu_0\mu}\iint\limits_M K(x_1,y_1;x,y)\,e_0(x_1,y_1)\,dx_1\,dy_1 \pm \frac{1}{\omega\mu_0\mu}\iint\limits_N K(x_1,y_1;x,y)\,e(x_1,y_1)\,dx_1\,dy_1.$$

It is proved in [3] that the diffraction problem on a plane screen is equivalent to the vector integral equations

$$\begin{aligned}
&\mp\frac{1}{\omega\varepsilon_0\varepsilon}\iint\limits_M K(x_1,y_1;x,y)\;h_{\pm}(x_1,y_1)\;dx_1\;dy_1 = -e_0(x,y);\\
&e(x,y) + \iint\limits_N e(x_1,y_1)L_M(x_1,y_1;x,y)\,dx_1\,dy_1 = \iint\limits_M e_0(x_1,y_1)L_M(x_1,y_1;x,y)\,dx_1\,dy_1,\\
&L_M(x_1,y_1;x,y) = \frac{1}{k^2}\iint\limits_M K(x_1,y_1;x_2,y_2)K(x_2,y_2;x,y)\,dx_2\,dy_2;\\
&h_{\pm}(x,y)+\iint\limits_M h_{\pm}(x_1,y_1)L_N(x_1,y_1;x,y)\,dx_1\,dy_1 = \mp\frac{1}{\omega\mu_0\mu}\iint\limits_M K(x_1,y_1;x,y)e_0(x_1,y_1)dx_1dy_1,\\
&L_N(x_1,y_1;x,y) = \frac{1}{k^2}\iint\limits_N K(x_1,y_1;x_2,y_2)\,K(x_2,y_2;x,y)\,dx_2\,dy_2.
\end{aligned} \tag{9}$$

5. THE CLOSED WAVEGUIDE OF ARBITRARY SECTION

Let C be media interface in the waveguide with metallic walls placed along axis z, and S be a lateral section on the waveguide (see Fig. 2). Let the function $z = f(x,y)$, $(x,y) \in S$ determine the media interface C. We are to obtain the integral identities connecting on C the traces $[n,E] = e(x,y)$, $[n,H] = h(x,y)$, $(x,y) \in S$ of tangent components of the vectors E and H of the field from the right-side on C.

As it is known, any solution of the Maxwell equations set for complex amplitudes in the case of harmonic dependence on time can be represented as a superposition of fields of normal TE- and TM-waves. For TE-waves we have

$$E = i\omega\mu_0\mu\,\mathrm{rot}\,\Pi^m, \quad H = \mathrm{grad}\,\mathrm{div}\,\Pi^m + k^2\Pi^m,$$

for TM-waves we have

$$E = \mathrm{grad}\,\mathrm{div}\,\Pi^e + k^2\Pi^e, \quad H = -i\omega\varepsilon_0\varepsilon\,\mathrm{rot}\,\Pi^e,$$

where

$$\Pi^m = \left(0,0,\sum_{n=1}^{+\infty} B_n^m e^{i\gamma_n^m z}\varphi_n^m(x,y)\right), \quad \Pi^e = \left(0,0,\sum_{n=1}^{+\infty} B_n^e e^{i\gamma_n^e z}\varphi_n^e(x,y)\right),$$

γ^m and $\varphi^m(x,y)$ are the eigen values and eigen functions of the Laplace operator with homogeneous Neumann conditions on the boundary S, γ^e and $\varphi^e(x,y)$ are the eigen values and eigen functions of the Laplace operator with homogeneous Dirichlet conditions. For the waves going in the direction of the axis z the sign of γ is chosen so that $\mathrm{Re}\,\gamma \geq 0$ or $\mathrm{Im}\,\gamma > 0$ by $\mathrm{Re}\,\gamma = 0$. For the waves going in the opposite direction in is suitable to assume that numbers γ have another sign and, more over, the vector E for TE-waves and the vector H for TM-waves have the another direction.

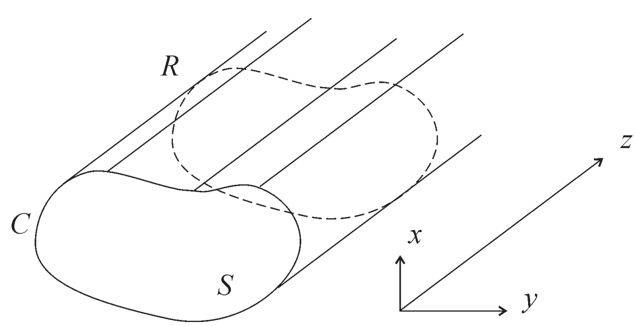

Figure 2: Cross-section of the closed waveguide.

Let us construct 2×2-matrix $K(\xi, \eta, x, y)$ of the elements of the form

$$\sum_{r=0}^{+\infty} \sum_{s=0}^{+\infty} k_{rs} \varphi_r(\xi, \eta) \varphi_s(x, y)$$

so that

$$e(x, y) = \iint_S K(\xi, \eta, x, y) h(\xi, \eta) \, d\xi_S \, d\eta_S, \quad (x, y) \in S. \tag{10}$$

By this the coefficients of all elements of the matrix $K(\xi, \eta, x, y)$ are determined from the infinite set of the linear algebraic equations in the form of the linear equation for the infinite matrices. Formula (10) is the initial integral identity.

Let us transform the conditions of conjugation of fields on C in the diffraction problem

$$[n, E^0] + [n, E^-] = [n, E^+], \quad [n, H^0] + [n, H^-] = [n, H^+]$$

by the following way: we apply the integral operator with the kernel $K(\xi, \eta, x, y)$ to the left-hand side of the second kind equation [4]. Having projected the new vector equation onto the set of functions $\varphi_m^m(x, y)$, $\varphi_m^e(x, y)$, we obtain regular (i.e., with the properties of the operator equation of the first kind) infinite set of linear algebraic equations for propagation constants of the field in the left half of the waveguide. The reduction method can be applied to this set of equation.

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