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SHORT COMMUNICATIONS

Conditions for the Solvability of a System of Integral Equations by Quadratures

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Abstract—By reduction to Goursat problems, we obtain various versions of conditions ensuring that the solution of the system in question can be constructed in closed form.

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A number of partial integral equations (for the term, e.g., see [2, p. 4]) were reduced in [1] to Goursat problems admitting closed-form solutions. In the present paper, we apply the same approach to the system of equations

$$\varphi_k(x,y) = a_{k1}(x,y)\int_{x_0}^{x}[\lambda(t,y)\varphi_1(t,y)+\mu(t,y)\varphi_2(t,y)]\,dt$$
$$+\,a_{k2}(x,y)\int_{y_0}^{y}[\nu(x,\tau)\varphi_1(x,\tau)+\sigma(x,\tau)\varphi_2(x,\tau)]\,d\tau + f_k(x,y), \qquad k=1,2, \tag{1}$$

in the domain $D=\{x_0<x<x_1,\ y_0<y<y_1\}$. We assume that the coefficients of the system are continuous in $\overline{D}$ and

$$\Delta(x,y) = \det\|a_{ik}(x,y)\| \neq 0. \tag{2}$$

We denote the integrals with respect to x and y occurring in system (1) by M and N, respectively. Then, by condition (2), we obtain

$$M = (a_{22}\varphi_1 - a_{12}\varphi_2 - a_{22}f_1 + a_{12}f_2)\Delta^{-1}, \qquad N = (a_{11}\varphi_2 - a_{21}\varphi_1 - a_{11}f_2 + a_{21}f_1)\Delta^{-1}. \tag{3}$$

Let us introduce the new unknown functions

$$a_{22}\varphi_1 - a_{12}\varphi_2 = u, \qquad -a_{21}\varphi_1 + a_{11}\varphi_2 = v \tag{4}$$

and consider relations (4) as a system of equations for φ_1 and φ_2. Since its determinant coincides with Δ, we have

$$\varphi_1 = (a_{11}u + a_{12}v)\Delta^{-1}, \qquad \varphi_2 = (a_{21}u + a_{22}v)\Delta^{-1}. \tag{5}$$

It follows from relations (4) and (5) that the problems of finding the pairs (φ_1,φ_2) and (u,v) are equivalent.

On the other hand, by substituting the expressions (4) into (3), we obtain

$$M = (u - a_{22}f_1 + a_{12}f_2)\Delta^{-1}, \qquad N = (v + a_{21}f_1 - a_{11}f_2)\Delta^{-1}. \tag{6}$$

We differentiate the first relation with respect to x and the second with respect to y, compute u_x and u_y, and take into account relations (5) to obtain

$$u_x = \alpha u + \beta v + F_1, \qquad v_y = \gamma u + \delta v + F_2, \tag{7}$$

where

$$\begin{gathered}\alpha = \lambda a_{11} + \mu a_{21} + (\ln\Delta)_x, \qquad \beta = \lambda a_{12} + \mu a_{22},\\ \gamma = \nu a_{11} + \sigma a_{21}, \qquad \delta = \nu a_{12} + \sigma a_{22} + (\ln\Delta)_y,\\ F_1 = (a_{22}f_1 - a_{12}f_2)_x - (a_{22}f_1 - a_{12}f_2)(\ln\Delta)_x,\\ F_2 = -(a_{21}f_1 - a_{11}f_2)_y + (a_{21}f_1 - a_{11}f_2)(\ln\Delta)_y.\end{gathered} \tag{8}$$

Since the definitions of M and N imply the identities $M(x_0, y) \equiv N(x, y_0) \equiv 0$, from relations (3) and (4), we obtain the boundary values

$$\begin{gathered}u(x_0, y) = a_{22}(x_0, y)f_1(x_0, y) - a_{12}(x_0, y)f_2(x_0, y),\\ v(x, y_0) = a_{11}(x, y_0)f_2(x, y_0) - a_{21}(x, y_0)f_1(x, y_0).\end{gathered} \tag{9}$$

Thus, we have reduced system (1) to the Goursat problem (7)–(9), which was studied, e.g., in [3, 4], where, in particular, its unique solvability was proved. To obtain conditions for its solvability in closed form, we reduce Eqs. (7) to two equations of the form

$$\Theta_{xy} + a\Theta_x + b\Theta_y + c\Theta = f, \tag{10}$$

which are obtained by eliminating one of the unknown functions from the system: if the first or second of the inequalities

$$\beta \neq 0, \qquad \gamma \neq 0 \tag{11}$$

holds, then we arrive at Eq. (10) for $\Theta = u$ or $\Theta = v$, respectively. One can readily see that coefficients of the fist equation have the form

$$\begin{gathered}-a = \delta + (\ln\beta)_y, \qquad -b = \alpha, \qquad -c = \alpha_y + \beta\gamma - \alpha(\delta + (\ln\beta)_y),\\ f = (a_{22}f_1 - a_{12}f_2)_{xy} - [(a_{22}f_1 - a_{12}f_2)(\ln\Delta)_x]_y - [\delta + (\ln\beta)_y][(a_{22}f_1 - a_{12}f_2)_x\\ - (a_{22}f_1 - a_{12}f_2)(\ln\Delta)_x] + \beta[-(a_{21}f_1 - a_{11}f_2)_y + (a_{21}f_1 - a_{11}f_2)(\ln\Delta)_y],\end{gathered} \tag{12}$$

while the coefficients of the second equation have the form

$$\begin{gathered}-a = \delta, \qquad -b = \alpha + (\ln\gamma)_x, \qquad -c = \delta_x + \beta\gamma - \delta(\alpha + (\ln\gamma)_x),\\ f = -(a_{21}f_1 - a_{11}f_2)_{xy} + [(a_{21}f_1 - a_{11}f_2)(\ln\Delta)_y]_x - [\alpha + (\ln\gamma)_x][-(a_{21}f_1 - a_{11}f_2)_y\\ + (a_{21}f_1 - a_{11}f_2)(\ln\Delta)_y] + \gamma[(a_{22}f_1 - a_{12}f_2)_x - (a_{22}f_1 - a_{12}f_2)(\ln\Delta)_x].\end{gathered} \tag{13}$$

Given a solution $u(x, y)$ of the first equation obtained for $\beta \neq 0$, one can compute the function $v(x, y)$ from the first equation in (7). Likewise, if $\gamma \neq 0$, then, for a given solution v of the second equation, the function u can be found from the second relation in (7). However, condition (9) is insufficient to find the function $\Theta = u$ or $\Theta = v$ from Eq. (10), because one also needs the values

$$u(x, y_0) = r(x), \qquad v(x_0, y) = g(y). \tag{14}$$

They can be found from system (1). Indeed, by setting $y = y_0$ and $x = x_0$ in system (1) and by the subsequent differentiation of the resulting relations with respect to x and y, respectively, we obtain linear equations of the form

$$r' + p_1 r = \Phi_1(x), \qquad g' + p_2 g = \Phi_2(y),$$

where

$$p_1 = (\ln\Delta)_x - a_{11}\lambda - a_{21}\mu, \qquad p_2 = (\ln\Delta)_y - a_{12}\nu - a_{22}\sigma.$$

By virtue of conditions (9), we know the functions $\Phi_1(x)$ and $\Phi_2(y)$ and the initial data $r(x_0)$ and $g(y_0)$. Therefore, it is easy to compute the functions $r(x)$ and $g(y)$. Obviously, the first (respectively, second) relations in (9) and (14) are the boundary conditions of the Goursat problem for the first (respectively, second) equation of the form (10). To find a solution of the original system (1), it suffices to construct a solution of at least one of these Goursat problems.

It is well known [5, p. 172; 6, p. 14] that the solutions of these Goursat problems can be written out via the corresponding Riemann functions; moreover, for the latter we have various cases of their construction in closed form [6, pp. 15–16; 7; 8]. In the cited papers, these cases are determined by the following relations.

1. $a_x + ab - c \equiv 0$.
2. $b_y + ab - c \equiv 0$.
3. $a_x \equiv b_y$ and $c - a_x - ab \equiv \xi_0(x)\eta_0(y) \neq 0$.
4. $b_y - a_x \equiv a_x + ab - c \equiv \xi_1(x)\eta_1(y) \neq 0$.
5. $a_x - b_y \equiv b_y + ab - c \equiv \xi_2(x)\eta_2(y) \neq 0$.
6. $ma_x - b_y \equiv mb_y - a_x \equiv (m-1)(ab - c)$.
7. $w = \dfrac{2s'(x)t'(y)}{(2-m)[s(x)+t(y)]^2}$ and $[s(x)+t(y)]s'(x)t'(y) \neq 0$.

Here $\xi_0, \eta_0 \in C$ and $\xi_k, \eta_k \in C^1$ $(k = 1, 2)$, $s, t, m \in C^2$, and m depends only on one of the variables x and y and does not take the value 2. In other aspects, the above-mentioned functions are arbitrary: i.e., the corresponding class should contain functions for which the above-listed relations hold. The coefficients a, b, and c have smoothness sufficient for the represented formulas to be satisfiable. The smoothness classes are defined on the closed domains of the corresponding functions. Each of identities 1 and 2 and systems 3–5 is sufficient to obtain the Riemann functions in closed form. Relations 6 and 7 should be used together: if relations 6 are satisfied, then the Riemann function can be constructed provided that the left-hand side of at least one of relations 1 and 2 has the form w indicated in relations 7. In other words, we have seven cases of conditions for solvability by quadratures for each of the two Goursat problems. The form of the Riemann functions for all of the cases can be found in [6–8]. Obviously, the total number of solvability cases discussed here is 14.

By using the formulas in (12) and (13), let us rewrite relations 1–7 via the coefficients of system (7). For the first Goursat problem related to the first inequality in (11), we have

$$\alpha_y + \beta\gamma - \delta_x - (\ln\beta)_{xy} \equiv 0, \qquad \beta\gamma \equiv 0, \tag{15}$$
$$\alpha_y - \delta_x - (\ln\beta)_{xy} \equiv 0, \qquad \delta_x + (\ln\beta)_{xy} - \alpha_y - \beta\gamma \equiv \xi_1(x)\eta_1(y) \neq 0, \tag{16}$$
$$2[\delta_x + (\ln\beta)_{xy} - \alpha_y] \equiv \beta\gamma, \qquad \delta_x + (\ln\beta)_{xy} - \alpha_y \equiv \xi_2(x)\eta_2(y) \neq 0, \tag{17}$$
$$\alpha_y - \delta_x - (\ln\beta)_{xy} \equiv \beta\gamma \equiv \xi_3(x)\eta_3(y) \neq 0, \tag{18}$$
$$m[-\delta_x - (\ln\beta)_{xy}] + \alpha_y \equiv -m\alpha_y + \delta_x + (\ln\beta)_{xy} \equiv (m-1)(\alpha_y + \beta\gamma), \tag{19}$$
$$w_k = \frac{2s_k'(x)t_k'(y)}{(2-m)[s_k(x)+t_k(y)]^2}, \qquad [s_k(x)+t_k(y)]s_k'(x)t_k'(y) \neq 0, \qquad k = 1, 2. \tag{20}$$

In the last row, one should assume that w_1 and w_2 are equal to the left-hand sides of the first and second identities in (15), respectively.

From the preceding argument, we obtain the following assertion.

Theorem 1. *Let $\beta \neq 0$, and let relations* (2) *be satisfied*; *in addition*, *suppose that either at least one of the identities in* (15) *holds or there exist functions m, ξ_k, η_k $(k = 1, 2, 3)$, s_k, and t_k $(k = 1, 2)$ of the above-mentioned classes such that either at least one of three groups of relations* (16)–(18) *holds or*, *in addition to identity* (19), *the representation* (20) *holds at least for one of the functions w_1 and w_2. Then system* (1) *is solvable by quadratures.*

Note that in (16)–(18) the indices of the functions $\xi(x)$ and $\eta(y)$ are changed to distinguish them from the indices in formulas 3–5.

The following relations are analogs of formulas (15)–(19) for the second Goursat problem [corresponding to the condition $\gamma \neq 0$ in (11)]:

$$\beta\gamma \equiv 0, \qquad \delta_x + \beta\gamma - \alpha_y - (\ln\gamma)_{xy} \equiv 0, \tag{21}$$

$$\alpha_y + (\ln\gamma)_{xy} - \delta_x \equiv 0, \qquad -\beta\gamma \equiv \xi_4(x)\eta_4(y) \neq 0, \tag{22}$$

$$\delta_x - \alpha_y - (\ln\gamma)_{xy} \equiv \beta\gamma \equiv \xi_5(x)\eta_5(y) \neq 0, \tag{23}$$

$$2[\alpha_y + (\ln\gamma)_{xy} - \delta_x] \equiv \beta\gamma, \qquad \alpha_y + (\ln\gamma)_{xy} - \delta_x \equiv \xi_6(x)\eta_6(y) \neq 0, \tag{24}$$

$$-m\delta_x + \alpha_y + (\ln\gamma)_{xy} \equiv m[-\alpha_y - (\ln\gamma)_{xy}] + \delta_x \equiv (m-1)(\delta_x + \beta\gamma). \tag{25}$$

We can (and will) assume that these relations are supplemented with relations (20) numbered like (26), where $k = 3, 4$ and w_3 and w_4 are the left-hand sides of the first and second identities in (21), respectively.

We have thereby obtained the following assertion.

Theorem 2. *If, in addition to the inequality $\gamma \neq 0$ and relation* (20), *either at least one of identities* (21) *holds or there exist functions m, ξ_k, η_k ($k = 4, 5, 6$), s_k, and t_k ($k = 3, 4$) of the above-mentioned classes such that either at least one group of relations* (22)–(24) *holds or, in addition to identity* (25), *the representation* (26) *is true for at least one of the functions w_3 and w_4, then system* (1) *is solvable by quadratures.*

Obviously, relations (15)–(26) can be written out directly in terms of the coefficients of system (1); to this end, it suffices to substitute the values in (8) into them.

Note that, in each of the above-listed 14 cases, the number of functions related by the above-formulated relations exceeds the number of constraints themselves. Therefore, each version describes a whole set of systems of equations of the considered form solvable by quadratures.

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