ISSN 1995-0802, Lobachevskii Journal of Mathematics, 2011, Vol. 32, No. 3, pp. 175–179. © Pleiades Publishing, Ltd., 2011.

The Peierls–Bogoliubov Inequality in C^*-Algebras and Characterization of Tracial Functionals

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(Submitted by D.H. Mushtari)

Received February 21, 2011

Abstract—We prove that either of the Peierls-Bogoliubov and Araki-Lieb-Thirring inequalities characterizes the tracial functionals among all positive functionals on a C^*-algebra. We also give the affirmative answer to the question of J. Zemánek.

DOI: 10.1134/S1995080211030061

Keywords and phrases: *Algebra of matrices, functional calculus, positive functional, tracial functional, the Peierls-Bogoliubov inequality, von Neumann algebra, C^*-algebra.*

In what follows, M_n stands for the algebra of $n \times n$ complex matrices, M_n^+ stands for the cone of positive semi-definite matrices.

It is an important issue in statistical mechanics to calculate the value of the so-called *partition function* $\mathrm{tr}(e^{\widetilde{H}})$, where the Hermitian matrix $\widetilde{H}$ is the Hamiltonian of a physical system. Since that computation is often difficult, it is simpler to compute the related quantity $\mathrm{tr}(e^H)$, where H is a convenient approximation of the Hamiltonian $\widetilde{H}$. Indeed, let $\widetilde{H} = H + K$. The *Peierls–Bogoliubov inequality* provides useful information on $\mathrm{tr}(e^{H+K})$ from $\mathrm{tr}(e^H)$. This inequlity states that, for two Hermitian operators H and K

$$\mathrm{tr}(e^H)\exp\frac{\mathrm{tr}(e^H K)}{\mathrm{tr}(e^H)} \le \mathrm{tr}(e^{H+K}). \tag{1}$$

The equality occurs in the Peierls-Bogoliubov inequality if and only if K is a scalar matrix. The elegant proof of the inequality (1) for Hermitian $H, K \in M_n$ is given in § 7 of [1].

Recall that (for basic references, see, e.g., [10, 14]) a positive linear functional φ on a von Neumann algebra $\mathcal{M}$ is said to be *normal* if $\varphi(\sup A_i) = \sup \varphi(A_i)$ for every bounded increasing net $\{A_i\}$ of positive operators in $\mathcal{M}$. A linear functional φ on a C^*-algebra $\mathcal{A}$ is said to be *tracial* if $\varphi(AB) = \varphi(BA)$ for all A, B in $\mathcal{A}$.

Lemma 1. *A positive normal functional φ on von Neumann algebra $\mathcal{M}$ is tracial if and only if*

$$\varphi(e^H)\exp\frac{\varphi(e^{H/2}Ke^{H/2})}{\varphi(e^H)} \le \varphi(e^{H+K}) \tag{2}$$

for all positive operators H, K in $\mathcal{M}$.

Proof. It was proved in Theorem 7 of [12] that

$$\tau(e^H)\exp\frac{\tau(e^H K)}{\tau(e^H)} \le \tau(e^{H+K})$$

for every semifinite normal trace τ on $\mathcal{M}$ and any Hermitian operators H, K in $\mathcal{M}$ with $\tau(e^H) < \infty$. We have

$$\tau(e^H K) = \tau(e^{H/2}e^{H/2}K) = \tau(e^{H/2}Ke^{H/2}) < \infty$$

Supported by the Russian Federal Agency of Science and Innovations (contract No 02.740.0193).