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ON THE τ-COMPACTNESS OF PRODUCTS OF τ-MEASURABLE OPERATORS ADJOINT TO SEMI-FINITE VON NEUMANN ALGEBRAS

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Abstract. Let $\mathcal{M}$ be the von Neumann algebra of operators in a Hilbert space $\mathcal{H}$ and τ be an exact normal semi-finite trace on $\mathcal{M}$. We obtain inequalities for permutations of products of τ-measurable operators. We apply these inequalities to obtain new submajorizations (in the sense of Hardy, Littlewood, and Pólya) of products of τ-measurable operators and a sufficient condition of orthogonality of certain nonnegative τ-measurable operators. We state sufficient conditions of the τ-compactness of products of self-adjoint τ-measurable operators and obtain a criterion of the τ-compactness of the product of a nonnegative τ-measurable operator and an arbitrary τ-measurable operator. We present an example that shows that the nonnegativity of one of the factors is substantial. We also state a criterion of the elementary nature of the product of nonnegative operators from $\mathcal{M}$. All results are new for the *-algebra $\mathcal{B}(\mathcal{H})$ of all bounded linear operators in $\mathcal{H}$ endowed with the canonical trace $\tau = \mathrm{tr}$.

***Keywords and phrases*:** Hilbert space, linear operator, von Neumann algebra, normal semi-finite trace, τ-measurable operator, τ-compact operator, elementary operator, nilpotent, permutation, submajorization.

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Introduction. Let $\mathcal{M}$ be the von Neumann algebra of operators in a Hilbert space $\mathcal{H}$ and τ be an exact normal semi-finite trace on $\mathcal{M}$. Products of τ-measurable operators appear in various problems of the theory of noncommutative integration (e.g., in [20] in the definition of dual spaces in the sense of Köthe, the Golden–Thompson inequality [7], the Peierls–Bogolyubov inequality [6], etc.). Sufficient conditions for the integrability of products of τ-measurable operators were found in [14]. This paper is a continuation of the papers [4, 10], in which criteria of the τ-compactness of products of nonnegative τ-measurable operators were obtained. Similar problems were examined in [3, 8, 30, 31]. Compact products of operators were studied in [16, 17, 19, 23, 25, 27, 32]. Applications of compact (respectively, τ-compact) products of operators are discussed in [22] (respectively, in [5]).

In Sec. 3 we obtain inequalities for permutations of products of τ-measurable operators. We apply these inequalities to obtain new submajorizations (in the sense of Hardy, Littlewood, and Pólya) of products of τ-measurable operators and a sufficient condition of orthogonality of certain nonnegative τ-measurable operators. In Sec. 4, we state sufficient conditions of the τ-compactness of products of self-adjoint τ-measurable operators and obtain a criterion of the τ-compactness of the product of a nonnegative τ-measurable operator and an arbitrary τ-measurable operator. We present an example that shows that the nonnegativity of one of factors is substantial. From a well-known property of permutations (see item (6) of Lemma 2.1) we deduce that a nonnegative operator $A \in \mathcal{M}$ is an elementary operator if and only if A^p is elementary for all $p > 0$. Theorem 4.2 shows that a similar situation also occurs for products of nonnegative operators $A, B \in \mathcal{M}$: the operator AB is elementary if and only if the operators A^pB^r are elementary for all $p, r > 0$. We describe some applications of results obtained to symmetric spaces on $(\mathcal{M}, \tau)$. All results are new for the *-algebra $\mathcal{B}(\mathcal{H})$ of all bounded linear operators in $\mathcal{H}$ endowed with the canonical trace $\tau = \mathrm{tr}$.

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1. Basic definitions, preliminaries, and notation. Let $\mathcal{M}$ be the von Neumann algebra of operators in a Hilbert space $\mathcal{H}$, $\mathcal{M}^{\mathrm{pr}}$ be a lattice of projectors in $\mathcal{M}$, and $\mathcal{M}^+$ be the cone of positive elements from $\mathcal{M}$. Let I be the unit of the algebra $\mathcal{M}$ and $\mathcal{M}_1 = \{X \in \mathcal{M} : \|X\| \leq 1\}$.

A mapping $\varphi : \mathcal{M}^+ \to [0, +\infty]$ is called a *trace* if

$$\varphi(X+Y) = \varphi(X) + \varphi(Y), \quad \varphi(\lambda X) = \lambda\varphi(X) \quad \forall X, Y \in \mathcal{M}^+,\ \lambda \geq 0$$

(in this case $0 \cdot (+\infty) \equiv 0$) and

$$\varphi(Z^*Z) = \varphi(ZZ^*) \quad \forall Z \in \mathcal{M}.$$

A trace φ is said to be *exact* if $\varphi(X) > 0$ for all $X \in \mathcal{M}^+$, $X \neq 0$; *semi-finite* if

$$\varphi(X) = \sup\Big\{\varphi(Y) :\ Y \in \mathcal{M}^+,\ Y \leq X,\ \varphi(Y) < +\infty\Big\} \quad \forall X \in \mathcal{M}^+;$$

normal if

$$X_i \nearrow X, \quad \text{i.e.,} \quad (X_i, X \in \mathcal{M}^+) \ \Rightarrow\ \varphi(X) = \sup \varphi(X_i).$$

An operator in $\mathcal{H}$ (not necessarily bounded or densely defined) is said to be *adjoint to the von Neumann algebra* $\mathcal{M}$ if it commutes with any unitary operator from the commutator subalgebra $\mathcal{M}'$ of the algebra $\mathcal{M}$. A self-adjoint operator is adjoint to $\mathcal{M}$ if and only if all projectors from its spectral decomposition of unity belong to $\mathcal{M}$.

Let τ be an exact, normal, semi-finite trace on $\mathcal{M}$. A closed operator X adjoint to $\mathcal{M}$ with everywhere dense in $\mathcal{H}$ domain $\mathcal{D}(X)$ is said to be τ-*measurable*, if for arbitrary $\varepsilon > 0$ there exists $P \in \mathcal{M}^{\mathrm{pr}}$ such that $P\mathcal{H} \subset \mathcal{D}(X)$ and $\tau(I - P) < \varepsilon$. The set $\widetilde{\mathcal{M}}$ of all τ-measurable operators is a *-algebra with respect to the transition to conjugate operator, multiplication by scalars, and the operations of strong addition and multiplication obtained by the closure of the ordinary operations (see [28, 29]). For a family $\mathcal{L} \subset \widetilde{\mathcal{M}}$, we denote by $\mathcal{L}^+$ and $\mathcal{L}^{\mathrm{sa}}$ its positive and Hermitian parts, respectively. We denote the partial order in $\widetilde{\mathcal{M}}^{\mathrm{sa}}$ generated by the proper cone $\widetilde{\mathcal{M}}^+$ by $\leq$.

If X is a closed, densely defined linear operator adjoint to $\mathcal{M}$ and $|X| = \sqrt{X^*X}$, then the spectral decomposition of $P^{|X|}(\cdot)$ is contained in $\mathcal{M}$ and $X \in \widetilde{\mathcal{M}}$ if and only if there exists $\lambda \in \mathbb{R}$ such that

$$\tau(P^{|X|}((\lambda, +\infty))) < +\infty.$$

If $X \in \widetilde{\mathcal{M}}$ and $X = U|X|$ is the polar decomposition of X, then $U \in \mathcal{M}$ and $|X| \in \widetilde{\mathcal{M}}^+$. Moreover, if

$$|X| = \int_0^\infty \lambda P^{|X|}(d\lambda)$$

is the spectral decomposition, then $\tau(P^{|X|}((\lambda, +\infty))) \to 0$ as $\lambda \to +\infty$.

We denote by $\mu_t(X)$ a *permutation* of the operator $X \in \widetilde{\mathcal{M}}$, i.e., a nonincreasing, right-continuous function $\mu(X) : (0, \infty) \to [0, \infty)$ defined by the formula

$$\mu_t(X) = \inf\Big\{\|XP\| :\ P \in \mathcal{M}^{\mathrm{pr}},\ \tau(I - P) \leq t\Big\}, \quad t > 0.$$

The set of τ-compact operators

$$\widetilde{\mathcal{M}}_0 = \Big\{X \in \widetilde{\mathcal{M}} :\ \mu_\infty(X) \equiv \lim_{t \to \infty} \mu_t(X) = 0\Big\}$$

is an ideal in $\widetilde{\mathcal{M}}$ (see [33]). The set of elementary operators

$$\mathcal{F}(\mathcal{M}) = \Big\{X \in \mathcal{M} :\ \mu_t(X) = 0 \text{ for some } t > 0\Big\}$$

is an ideal in $\mathcal{M}$. If $\tau(I) < +\infty$, then $\widetilde{\mathcal{M}}_0 = \widetilde{\mathcal{M}}$.

Let m be a linear Lebesgue measure on $\mathbb{R}$. The noncommutative Lebesgue L_p-space associated with $(\mathcal{M}, \tau)$ $(0 < p < \infty)$ can be defined as follows:

$$L_p(\mathcal{M}, \tau) = \left\{ X \in \widetilde{\mathcal{M}} : \ \mu(X) \in L_p(\mathbb{R}^+, m) \right\}$$

with the F- (norm for $1 \le p < \infty$)

$$\|X\|_p = \|\mu(X)\|_p, \quad X \in L_p(\mathcal{M}, \tau).$$

We have $\mathcal{F}(\mathcal{M}) \subset L_p(\mathcal{M}, \tau) \subset \widetilde{\mathcal{M}}_0$ for all $0 < p < \infty$.

For operators $X, Y \in (L_1 + L_\infty)(\mathcal{M}, \tau)$, the submajorization (or Hardy–Littlewood–Polýa weak spectral order), $X \prec\prec Y$, means that

$$\int\limits_0^t \mu_s(X) ds \le \int\limits_0^t \mu_s(Y) ds \quad \text{for all } t > 0.$$

For operators $X, Y \in \widetilde{\mathcal{M}}$ we also consider their Jordan product $X \circ Y = \frac{1}{2}(XY + YX)$ and Lie product (commutator) $[X, Y] = XY - YX$. An operator $X \in \widetilde{\mathcal{M}}$ is said to be *normal* if $X^*X = XX^*$, *hyponormal* if $X^*X \ge XX^*$, *cohyponormal* if X^* is hyponormal, and *quasinormal* if X commutes with X^*X, i.e., $X \cdot X^*X = X^*X \cdot X$. Each quasinormal operator $X \in \widetilde{\mathcal{M}}$ is hyponormal (see [13, Theorem 2.9]).

If $\mathcal{M} = \mathcal{B}(\mathcal{H})$ is the *-algebra of all bounded linear operators in $\mathcal{H}$ and $\tau = \mathrm{tr}$ is the canonical trace, then $\widetilde{\mathcal{M}}$ coincides with $\mathcal{B}(\mathcal{H})$ and $\widetilde{\mathcal{M}}_0$ and $\mathcal{F}(\mathcal{M})$ coincide with the ideals of compact operators and finite-dimensional operators in $\mathcal{H}$, respectively. We have

$$\mu_t(X) = \sum_{n=1}^{\infty} s_n(X) \chi_{[n-1,n)}(t), \quad t > 0,$$

where $\{s_n(X)\}_{n=1}^{\infty}$ is the sequence of s-numbers of the operator X (see [24, p. 46]) and χ_A is the indicator of the set $A \subset \mathbb{R}$. Then the space $L_p(\mathcal{M}, \tau)$ if the Schatten–von Neumann ideal $\mathfrak{S}_p$, $0 < p < \infty$.

Let (Ω, ν) be a space with measure and $\mathcal{M}$ be the von Neumann algebra of operators of multiplication by functions from $L_\infty(\Omega, \nu)$ in the space $L_2(\Omega, \nu)$. The algebra $\mathcal{M}$ does not contain nonzero compact operators if and only if the measure ν does not have atoms (see [1, Theorem 8.4]).

2. Lemmas on τ-measurable operators.

Lemma 2.1 (see [2, 21, 33]). *Let $X, Y \in \widetilde{\mathcal{M}}$. Then the following assertions hold*:

(1) $\mu_t(X) = \mu_t(|X|) = \mu_t(X^*)$ *for all* $t > 0$;
(2) *if* $|X| \le |Y|$, *then* $\mu_t(X) \le \mu_t(Y)$ *for all* $t > 0$;
(3) *if* $A, B \in \mathcal{M}$, *then* $\mu_t(AXB) \le \|A\|\|B\|\mu_t(X)$ *for all* $t > 0$;
(4) $\mu_{s+t}(XY) \le \mu_s(X)\mu_t(Y)$ *for all* $s, t > 0$;
(5) $\mu_{s+t}(X + Y) \le \mu_s(X) + \mu_t(Y)$ *for all* $s, t > 0$;
(6) $\mu_t(|X|^p) = \mu_t(X)^p$ *for all* $p > 0$ *and* $t > 0$;
(7) $\lim\limits_{t \to 0+} \mu_t(X) = \|X\|$ *for* $X \in \mathcal{M}$ *and* $\lim\limits_{t \to 0+} \mu_t(X) = \infty$ *for* $X \notin \mathcal{M}$.

Lemma 2.2 (see [20, p. 720]). *If $X, Y \in \widetilde{\mathcal{M}}^+$ and $Z \in \widetilde{\mathcal{M}}$, then the inequality $X \le Y$ implies $ZXZ^* \le ZYZ^*$.*

Lemma 2.3. *If $X, Y \in \widetilde{\mathcal{M}}$, then $|XY| = ||X|Y|$. In particular, if $X \in \mathcal{M}$ is an isometry (i.e., $X^*X = I$), then $|XY| = |Y|$.*

Proof. We have $|XY| = (Y^*X^*XY)^{1/2} = (Y^*|X|^2Y)^{1/2} = ||X|Y|$. $\square$

Lemma 2.4 (see [4, Proposition]). *If $X, Y \in \widetilde{\mathcal{M}}^+$, then $XY \in \widetilde{\mathcal{M}}_0 \Leftrightarrow X^{1/2}YX^{1/2} \in \widetilde{\mathcal{M}}_0 \Leftrightarrow Y^{1/2}XY^{1/2} \in \widetilde{\mathcal{M}}_0$.*

Lemma 2.5 (see [8, Theorem 1]). *Let $A \in \widetilde{\mathcal{M}}^+$, $B \in \widetilde{\mathcal{M}}^{\mathrm{sa}}$, and $-A \le B \le A$. Then there exists a unitary operator $S \in \mathcal{M}^{\mathrm{sa}}$ such that $2|B| \le A + SAS$.*

We also recall (see [12, Theorem 1]) that there exist operators $X \in \widetilde{\mathcal{M}}^{\mathrm{sa}}$ and $Y \in \widetilde{\mathcal{M}}^+$ such that $B = XY + YX$ and $A = X^2 + Y^2$. Examples of operators $A \in \widetilde{\mathcal{M}}^+$ and $B \in \widetilde{\mathcal{M}}^{\mathrm{sa}}$ with $-A \le B \le A$ can be found in [9]. Lemma 2.5 implies the following assertion.

Lemma 2.6 (see [15, Proposition 1.2]). *If $A \in \widetilde{\mathcal{M}}^+$, $B \in \widetilde{\mathcal{M}}^{\mathrm{sa}}$, and $-A \le B \le A$, then $B \prec\prec A$.*

Lemma 2.7 (see [3, 30, 31]). *If $X \in \widetilde{\mathcal{M}}^+$, $Y \in \widetilde{\mathcal{M}}^{\mathrm{sa}}$, and $XY \in (L_1+L_\infty)(\mathcal{M},\tau)$, then $X^tYX^{1-t} \prec\prec XY$ for all $0 < t < 1$.*

Lemma 2.8 (see [10, Theorem 3.5]). *Let $X, Y \in \widetilde{\mathcal{M}}$, X be hyponormal, and Y by cohyponormal. Then $\mu_t(XY) \ge \mu_t(YX)$ for all $t > 0$.*

3. Inequalities for permutations of τ-measurable operators.

Let τ be an exact, normal, semi-finite trace on the von Neumann algebra $\mathcal{M}$.

Theorem 3.1. *Let $A \in \widetilde{\mathcal{M}}$, $X_k, Y_k \in \widetilde{\mathcal{M}}^+$, and $X_k \le Y_k$, $k = 1, 2$. Then*

$$\mu_t(X_1^{1/2}AX_2^{1/2}) \le \mu_t(Y_1^{1/2}AY_2^{1/2}) \quad \forall t > 0.$$

Proof. By Lemma 2.2 we have $A^*X_1A \le A^*Y_1A$. Therefore, by items (1), (2), and (6) of Lemma 2.1 and the monotonicity of the real function $\lambda \mapsto \lambda^{1/2}$ ($\lambda \ge 0$) for all $t > 0$, we obtain

$$\mu_t(X_1^{1/2}A) = \mu_t(A^*X_1A)^{1/2} \le \mu_t(A^*Y_1A)^{1/2} = \mu_t(Y_1^{1/2}A).$$

Similarly, we obtain

$$\mu_t(X_2^{1/2}B^*) \le \mu_t(Y_2^{1/2}B^*)$$

for all $B \in \widetilde{\mathcal{M}}$ and $t > 0$. By item (1) of Lemma 2.1 we have

$$\mu_t(BX_2^{1/2}) = \mu_t((X_2^{1/2}B^*)^*) \le \mu_t((Y_2^{1/2}B^*)^*) = \mu_t(BY_2^{1/2})$$

for all $B \in \widetilde{\mathcal{M}}$ and $t > 0$. Replacing the operator A by $AX_2^{1/2}$ and the operator B by $Y_1^{1/2}$, we obtain for all $t > 0$ the inequalities

$$\mu_t(X_1^{1/2}AX_2^{1/2}) \le \mu_t(Y_1^{1/2}AX_2^{1/2}) \le \mu_t(Y_1^{1/2}AY_2^{1/2}),$$

which was required. $\square$

Proposition 3.1. *If operators $X, Y \in \widetilde{\mathcal{M}}$ are invertible and $X^{-1}, Y^{-1} \in \mathcal{M}_1$, then*

$$\mu_t(X^{-1} - Y^{-1}) \le \mu_t(X - Y) \quad \forall t > 0.$$

Moreover,

$$\mu_t(X^{-2} - Y^{-2}) \le 2\mu_{t/2}(X - Y) \quad \forall t > 0.$$

Proof. For all invertible $X, Y \in \widetilde{\mathcal{M}}$ we have

$$X^{-1} - Y^{-1} = X^{-1}(Y - X)Y^{-1} = Y^{-1}(Y - X)X^{-1}.$$

Therefore, by items (3) and (1) of Lemma 2.1, for all $t > 0$ we obtain

$$\mu_t(X^{-1} - Y^{-1}) = \mu_t(X^{-1}(Y - X)Y^{-1}) \le \|X^{-1}\|\|Y^{-1}\|\mu_t(Y - X) \le \mu_t(Y - X) = \mu_t(X - Y).$$

Since
$$\|X^{-1}+Y^{-1}\| \le \|X^{-1}\|+\|Y^{-1}\| \le 2,$$
$$X^{-2}-Y^{-2}=\frac{1}{2}\left((X^{-1}-Y^{-1})(X^{-1}+Y^{-1})+(X^{-1}+Y^{-1})(X^{-1}-Y^{-1})\right),$$
the inequality
$$\mu_t(X^{-2}-Y^{-2}) \le 2\mu_{t/2}(X-Y)$$
for all $t>0$ follows from items (3) and (5) of Lemma 2.1. The proposition is proved. □

If an operator $X \in \widetilde{\mathcal{M}}$ is invertible in $\widetilde{\mathcal{M}}$, then by item (4) of Lemma 2.1 we have
$$1=\mu_{2t}(I)=\mu_{2t}(XX^{-1}) \le \mu_t(X)\mu_t(X^{-1})$$
for all $t \in (0, 2^{-1}\tau(I))$. Therefore, $X, X^{-1} \notin \widetilde{\mathcal{M}}_0$ for $\tau(I)=+\infty$.

Proposition 3.2. *If $X \in \widetilde{\mathcal{M}}$ and $Y \in \mathcal{M}^{\mathrm{pr}}$, then*
$$\mu_t(YXY) \le \min\{\mu_t(XY),\ \mu_t(X \circ Y)\} \quad \forall t>0.$$

Proof. By item (3) of Lemma 2.1 for all $t>0$ we have
$$\mu_t(YXY) \le \|Y\|\mu_t(XY)=\mu_t(XY),$$
$$2\mu_t(YXY)=\mu_t(Y(XY+YX)Y) \le \|Y\|^2\mu_t(XY+YX)=\mu_t(XY+YX).$$
The proposition is proved. □

In particular, if
$$X \in \widetilde{\mathcal{M}}^+,$$
then
$$\mu_t(X^{1/2}YX^{1/2})=\mu_t(YXY), \quad \mu_t(X^{1/2}YX^{1/2}) \le \min\left\{\mu_t(XY),\ \mu_t(X \circ Y)\right\}$$
for all $t>0$. Note that for $X, Y \in \widetilde{\mathcal{M}}^+$, the inequality $\mu_t(X^{1/2}YX^{1/2}) \le \mu_t(XY)$ does not hold in the general case (see [3, p. 575]).

Theorem 3.2. *If $X, Y \in \widetilde{\mathcal{M}}$, then $\mu_t(XY)=\mu_t(|X||Y^*|)$ for all $t>0$.*

Proof. By Lemma 2.3 and item (1) of Lemma 2.1, for all $t>0$ we have
$$\mu_t(XY)=\mu_t(|XY|)=\mu_t(||X|Y|)=\mu_t(|X|Y)=\mu_t((|X|Y)^*)=\mu_t(Y^*|X|)=$$
$$=\mu_t(|Y^*|X||)=\mu_t(||Y^*||X||)=\mu_t(|Y^*||X|)=\mu_t((|Y^*||X|)^*)=$$
$$=\mu_t(|X||Y^*|).$$
The theorem is proved. □

Corollary 3.1. *If an operator $X \in \widetilde{\mathcal{M}}$ is nilpotent of order n and $m \ge n$, then $|X^{m-k}||X^{*k}|=0$ for all $k \in \{1,2,\ldots,m-1\}$.*

Proof. By the condition $X^n=0 \ne X^{n-1}$. We have
$$0=\mu_t(X^m)=\mu_t(X^{m-k}X^k)=\mu_t(|X^{m-k}||X^{*k}|)$$
for all $m \ge n$ and $t>0$. Therefore, $|X^{m-k}||X^{*k}|=0$ for all $k \in \{1,2,\ldots,m-1\}$. □

Theorem 3.2 and Lemma 2.7 imply the following assertion.

Corollary 3.2. *We have $|X|^t|Y^*||X|^{1-t} \prec\prec XY$ for all $0<t<1$ and $X, Y \in \widetilde{\mathcal{M}}$.*

Corollary 3.3. *Let $X, Y \in \widetilde{\mathcal{M}}$, where the operator X is hyponormal and the operator Y is cohyponormal. Then $\mu_t(|X||Y^*|) \ge \mu_t(|X^*||Y|)$ for all $t>0$.*

Proof. By Lemma 2.8 and item (1) of Lemma 2.1 for all $t > 0$ we have

$$\mu_t(|X||Y^*|) = \mu_t(XY) \geq \mu_t(YX) = \mu_t(|Y||X^*|) = \mu_t((|Y||X^*|)^*) = \mu_t(|X^*||Y|). \tag{1}$$

The proof is complete. □

Corollary 3.4. *Let operators $X, Y \in \widetilde{\mathcal{M}}$ be normal. Then $\mu_t(|X||Y^*|) = \mu_t(|X^*||Y|)$ for all $t > 0$.*

Proof. By [10, Corollary 3.6] we have the equality in (1). □

Theorem 3.3. *Let $X, Y \in \widetilde{\mathcal{M}}$, $XY \in (L_1 + L_\infty)(\mathcal{M}, \tau)$, X is hyponormal, and Y is cohyponormal. Then*

$$\lambda XY + (1-\lambda)YX \prec\prec XY \quad \forall 0 \leq \lambda \leq 1.$$

In particular, $X \circ Y \prec\prec XY$.

Proof. For all $t > 0$, due to Lemma 2.8 and the positive homogeneity and subadditivity of the functional

$$\Phi(A, t) = \int_0^t \mu_s(A)ds, \quad A \in (L_1 + L_\infty)(\mathcal{M}, \tau)$$

we obtain

$$\int_0^t \mu_s(\lambda XY + (1-\lambda)YX)ds \leq \lambda \int_0^t \mu_s(XY)ds + (1-\lambda)\int_0^t \mu_s(YX)ds \leq \int_0^t \mu_s(XY)ds.$$

The theorem is proved. □

Theorems 3.2 and 3.3 imply the following.

Corollary 3.5. *In conditions of Theorem* 3.3 *we have $\lambda XY + (1-\lambda)YX \prec\prec |X||Y^*|$.*

Proposition 3.3. *If $X, Y, A \in (L_1 + L_\infty)(\mathcal{M}, \tau)$ and $X, X - A \prec\prec Y$, then $X - \lambda A \prec\prec Y$ for all $0 \leq \lambda \leq 1$.*

Proof. The assertion follows from the positive homogeneity and the subadditivity of the functional

$$\Phi(A, t) = \int_0^t \mu_s(A)ds, \quad A \in (L_1 + L_\infty)(\mathcal{M}, \tau),$$

and the representation $X - \lambda A = (1-\lambda)X + \lambda(X - A)$. In particular, if $X, A \in \widetilde{\mathcal{M}}$ and $X - A \prec\prec X$, then $X - \lambda A \prec\prec X$ for all $0 \leq \lambda \leq 1$. □

Proposition 3.4. *If $X, Y \in \widetilde{\mathcal{M}}^{\mathrm{sa}}$ and $X^2 + Y^2 \in (L_1 + L_\infty)(\mathcal{M}, \tau)$, then*

$$X \circ Y \prec\prec \frac{1}{2}(X^2 + Y^2), \quad [X, Y] \prec\prec X^2 + Y^2.$$

Proof. Since $(X \pm Y)^2 \geq 0$ and $(X \pm iY)(X \mp iY) \geq 0$ with $i \in \mathbb{C}$, $i^2 = -1$, we have

$$-X^2 - Y^2 \leq XY + YX \leq X^2 + Y^2, \quad -X^2 - Y^2 \leq i(XY - YX) \leq X^2 + Y^2.$$

Now the assertions follow from Lemma 2.6. □

Since $XY = X \circ Y + \frac{1}{2}[X, Y]$, Proposition 3.4 implies the following assertion.

Corollary 3.6. *If $X, Y \in \widetilde{\mathcal{M}}^{\mathrm{sa}}$ and $X^2 + Y^2 \in (L_1 + L_\infty)(\mathcal{M}, \tau)$, then $XY \prec\prec X^2 + Y^2$.*

For a wide class of operators $X, Y \in \widetilde{\mathcal{M}}^{\mathrm{sa}}$ we have $\mu_t(XY) \leq \mu_t\left(\frac{X^2+Y^2}{2}\right)$ for all $t > 0$ (see [18, Lemma3.4]).

4. On the τ-compactness of products of τ-measurable operators.

Theorem 4.1. *Let operators $X, Y \in \widetilde{\mathcal{M}}^{\mathrm{sa}}$ be such that*

$$XY + YXY \in \widetilde{\mathcal{M}}_0, \quad Y^2 + Y \geq \lambda |Y|^p$$

with certain $0 < \lambda, p < +\infty$. Then $XY \in \widetilde{\mathcal{M}}_0$.

Proof. We have $XY = YX + A$ with

$$A = XY - YX = XY + YXY - (XY + YXY)^* \in \widetilde{\mathcal{M}}_0.$$

Then

$$XY + XY^2 - AY = XY + (XY - A)Y = XY + YXY \in \widetilde{\mathcal{M}}_0$$

and, since $AY \in \widetilde{\mathcal{M}}_0$, we have

$$XY + XY^2 \in \widetilde{\mathcal{M}}_0.$$

Therefore,

$$X(Y + Y^2)X = (XY + XY^2)X \in \widetilde{\mathcal{M}}_0.$$

By Lemma 2.2 and item (2) of Lemma 2.1, we obtain

$$X \cdot |Y|^p \cdot X \in \widetilde{\mathcal{M}}_0.$$

Since

$$\begin{aligned} \mu_t(X|Y|^{p/2})^2 &= \mu_t((X|Y|^{p/2})^*)^2 = \mu_t(|Y|^{p/2}X)^2 = \mu_t(||Y|^{p/2}X|)^2 = \mu_t(||Y|^{p/2}X|^2) \\ &= \mu_t(X|Y|^pX) \to 0 \quad \text{as } t \to +\infty, \end{aligned}$$

we obtain

$$X|Y|^{p/2} \in \widetilde{\mathcal{M}}_0.$$

Since

$$|X \cdot |Y|^{p/2}| = ||X| \cdot |Y|^{p/2}|$$

by Lemma 2.3, we have $|X| \cdot |Y|^{p/2} \in \widetilde{\mathcal{M}}_0$. Therefore,

$$|X| \cdot |Y| \in \widetilde{\mathcal{M}}_0$$

by Theorem 4.1 (see [10]). Let $X = U|X|$ and $Y = V|Y|$ be polar decompositions of the operators X and Y. Then $Y = |Y|V$ and $XY = U \cdot |X||Y| \cdot V \in \widetilde{\mathcal{M}}_0$. The theorem is proved. □

Corollary 4.1. *Let operators $X, Y \in \widetilde{\mathcal{M}}^{\mathrm{sa}}$ be such that*

$$XY - YXY \in \widetilde{\mathcal{M}}_0, \quad Y^2 - Y \geq \lambda |Y|^p$$

with certain $0 < \lambda, p < +\infty$. Then $XY \in \widetilde{\mathcal{M}}_0$.

Proof. The operators $X_1 = -X$ and $Y_1 = -Y$ satisfy all conditions of Theorem 4.1 and $X_1Y_1 = XY$. □

Proposition 4.1. *For operators $X \in \widetilde{\mathcal{M}}$ and $Y \in \widetilde{\mathcal{M}}^+$, the following conditions are equivalent:*

(i) $XY \in \widetilde{\mathcal{M}}_0$;
(ii) $XYX^* \in \widetilde{\mathcal{M}}_0$.

Proof. (ii)⇒(i). By items (1) and (6) of Lemma 2.1 we have

$$\mu_t(XY^{1/2})^2 = \mu_t((XY^{1/2})^*)^2 = \mu_t(Y^{1/2}X^*)^2 = \mu_t(|Y^{1/2}X^*|)^2 = \mu_t(|Y^{1/2}X^*|^2)$$
$$= \mu_t(XYX^*) \to 0 \quad \text{as } t \to +\infty.$$

Therefore, $XY^{1/2} \in \widetilde{\mathcal{M}}_0$ $XY = XY^{1/2} \cdot Y^{1/2} \in \widetilde{\mathcal{M}}_0$. □

Corollary 4.2. *For operators $X, Y \in \widetilde{\mathcal{M}}$ we have*

$$XY \in \widetilde{\mathcal{M}}_0 \quad \Leftrightarrow \quad |X||Y^*| \in \widetilde{\mathcal{M}}_0 \quad \Leftrightarrow \quad |X|^{1/2}|Y^*||X|^{1/2} \in \widetilde{\mathcal{M}}_0 \quad \Leftrightarrow \quad |Y^*|^{1/2}|X||Y^*|^{1/2} \in \widetilde{\mathcal{M}}_0.$$

Corollary 4.3. *Let $X \in \widetilde{\mathcal{M}}^{\mathrm{sa}}$ and $Y \in \widetilde{\mathcal{M}}$. If $XY \in \widetilde{\mathcal{M}}_0$, then $|Y^*|^{1/2}\, X\, |Y^*|^{1/2} \in \widetilde{\mathcal{M}}_0$.*

Proof. Let $X = X_+ - X_-$ be the Jordan decomposition of the operator $X \in \widetilde{\mathcal{M}}^{\mathrm{sa}}$, where $X_+, X_- \in \widetilde{\mathcal{M}}^+$ and $X_+X_- = 0$. Then $|X| = X_+ + X_-$ and

$$|Y^*|^{1/2}\, X_\pm\, |Y^*|^{1/2} \le |Y^*|^{1/2}\, |X|\, |Y^*|^{1/2}$$

by Lemma 2.2. If $XY \in \widetilde{\mathcal{M}}_0$, then $|Y^*|^{1/2}\, |X|\, |Y^*|^{1/2} \in \widetilde{\mathcal{M}}_0$ by Corollary 4.2. Now due to item (2) of Lemma 2.1 we have

$$|Y^*|^{1/2}\, X_\pm\, |Y^*|^{1/2} \in \widetilde{\mathcal{M}}_0^+,$$

and hence

$$|Y^*|^{1/2}\, X\, |Y^*|^{1/2} = |Y^*|^{1/2}\, X_+\, |Y^*|^{1/2} - |Y^*|^{1/2}\, X_-\, |Y^*|^{1/2} \in \widetilde{\mathcal{M}}_0.$$

The assertion is proved. □

Example 4.1. The condition $X, Y \in \widetilde{\mathcal{M}}^+$ is essential in Lemma 2.4 and the condition $Y \in \widetilde{\mathcal{M}}^+$ is essential in Proposition 4.1. We endow the von Neumann algebra $\mathcal{M} = \bigoplus_{n=1}^{\infty} \mathbb{M}_2(\mathbb{C})$ with the exact, normal, semi-finite trace $\tau = \bigoplus_{n=1}^{\infty} \mathrm{tr}_2$ and set

$$X = \bigoplus_{n=1}^{\infty} \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}, \quad Y = \bigoplus_{n=1}^{\infty} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Then $X \in \mathcal{M}^{\mathrm{pr}}$, $Y \in \mathcal{M}^{\mathrm{sa}}$, and $X^{1/2}YX^{1/2} = 0 \in \widetilde{\mathcal{M}}_0$, but the operators

$$XY = \bigoplus_{n=1}^{\infty} \begin{pmatrix} 1/2 & -1/2 \\ 1/2 & -1/2 \end{pmatrix}, \quad X \circ Y = \bigoplus_{n=1}^{\infty} \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix} \notin \widetilde{\mathcal{M}}_0.$$

Example 4.2 (theorem on lifting of idempotents; see [26, Proposition 7])**.** Let $\mathcal{M} = \mathcal{B}(\mathcal{H})$ and $\tau = \mathrm{tr}$ be the canonical trace, let operators $X \in \mathcal{M}$ and $Y = I - X$ be such that $XY \in \widetilde{\mathcal{M}}_0$. Then the representation $X = P + Z$ holds, where $P = P^2 \in \mathcal{M}$ and $Z \in \widetilde{\mathcal{M}}_0$.

Theorem 4.2. *Let $X, Y \in \mathcal{M}^+$, $n \in \mathbb{N}$, and $p_k > 0$, $q_k > 0$, $r > 0$, $k = 1, \ldots, n$. Then the following conditions are equivalent*:

(i) $XY \in \mathcal{F}(\mathcal{M})$;
(ii) $X^{p_1}Y^{q_1} \cdots X^{p_n}Y^{q_n} \in \mathcal{F}(\mathcal{M})$;
(iii) $X^{p_1}Y^{q_1} \cdots X^{p_n}Y^{q_n}X^r \in \mathcal{F}(\mathcal{M})$.

Proof. (i)⇒(ii), (iii). We have $XYX \in \mathcal{F}(\mathcal{M})$. By items (1) and (6) of Lemma 2.1 we obtain

$$\mu_t(XY^{1/2}) = \mu_t(XYX)^{1/2} \quad \forall t > 0;$$

therefore,

$$XY^{1/2} \in \mathcal{F}(\mathcal{M}).$$

Now
$$Y^{1/2}XY^{1/2} \in \mathcal{F}(\mathcal{M}).$$
By items (1) and (6) of Lemma 2.1 we have
$$\mu_t(X^{1/2}Y^{1/2}) = \mu_t(Y^{1/2}XY^{1/2})^{1/2} \quad \forall t > 0;$$
therefore,
$$X^{1/2}Y^{1/2} \in \mathcal{F}(\mathcal{M}).$$
Continuing this process, we obtain
$$X^{2^{-m}}Y^{2^{-m}} \in \mathcal{F}(\mathcal{M})$$
for all $m \in \mathbb{N}$. We choose m such that $2^{-m} < \min\{p_1, q_1\}$. Then
$$X^{p_1}Y^{q_1} = X^{p_1-2^{-m}} \cdot X^{2^{-m}}Y^{2^{-m}} \cdot Y^{q_1-2^{-m}} \in \mathcal{F}(\mathcal{M}).$$
The implications (ii)⇒(i) and (iii)⇒(i) can be verified by arguments similar to the proof of Theorem 4.1 (see [10]). □

Theorem 4.3. *Let operators $X, Y \in \mathcal{M}^{\mathrm{sa}}$ be such that $XY + YXY \in \mathcal{F}(\mathcal{M})$ and $Y^2 + Y \geq \lambda|Y|^p$ with certain $0 < \lambda, p < +\infty$. Then $XY \in \mathcal{F}(\mathcal{M})$.*

Proof. Repeating the arguing from the proof of Theorem 4.1, we obtain $X|Y|^{p/2} \in \mathcal{F}(\mathcal{M})$. Let $X = U|X|$ and $Y = V|Y|$ be the polar decompositions of the operators X and Y. Then $U, V \in \mathcal{M}^{\mathrm{sa}}$ and $UX = |X|$, $Y = |Y|V$. Since
$$|X||Y|^{p/2} = UX|Y|^{p/2} \in \mathcal{F}(\mathcal{M}),$$
we have $|X||Y| \in \mathcal{F}(\mathcal{M})$ by Theorem 4.2. Therefore, $XY = U \cdot |X||Y| \cdot V \in \mathcal{F}(\mathcal{M})$. The theorem is proved. □

Corollary 4.4. *Let operators $X, Y \in \mathcal{M}^{\mathrm{sa}}$ be such that $XY - YXY \in \mathcal{F}(\mathcal{M})$ and $Y^2 - Y \geq \lambda|Y|^p$ with certain $0 < \lambda, p < +\infty$. Then $XY \in \mathcal{F}(\mathcal{M})$.*

Proof. The operators $X_1 = -X$ and $Y_1 = -Y$ satisfy all condition of Theorem 4.3 and $X_1Y_1 = XY$. □

The proof of the following proposition is similar to the proof of Proposition 4.1.

Proposition 4.2. *For operators $X \in \mathcal{M}$ and $Y \in \mathcal{M}^+$ the following conditions are equivalent*:

(i) $XY \in \mathcal{F}(\mathcal{M})$;
(ii) $XYX^* \in \mathcal{F}(\mathcal{M})$.

Example 4.1 show that the positiveness condition of operator $Y \in \mathcal{M}$ is essential in Proposition 4.2.

Proposition 4.3. *Let an operator $X \in \widetilde{\mathcal{M}}$ be quasinormal and $X^n = X$ for a certain natural number $n \geq 2$. Then $X \in \mathcal{M}_1$ and the following conditions are equivalent*:

(i) $X \in \mathcal{F}(\mathcal{M})$;
(ii) $X \in \widetilde{\mathcal{M}}_0$.

Proof. We have $\mu_t(X) = \mu_t(X^n) = \mu_t(X)^n$ for all $t > 0$ due to [14, Theorem 2.4]. Therefore, $\mu_t(X) \in \{0, 1\}$ for all $t > 0$ and $X \in \mathcal{M}_1$ by item (7) of Lemma 2.1. The rest of the proof is obvious. □

Note that if $X \in \widetilde{\mathcal{M}}$ with $X^n = X$ for a certain natural number $n \geq 2$ and $X \notin \widetilde{\mathcal{M}}_0$, then $\mu_t(X) \geq 1$ for all $t > 0$ due to [11, Lemma 4.8]. The vector space $\mathcal{E}$ in $\widetilde{\mathcal{M}}$ is called the *symmetric space* on $(\mathcal{M}, \tau)$

if the conditions $X \in \mathcal{E}$, $Y \in \widetilde{\mathcal{M}}$, and $\mu(Y) \leq \mu(X)$ imply $Y \in \mathcal{E}$. For example, $\mathcal{M}$, $\mathcal{F}(\mathcal{M})$, $\widetilde{\mathcal{M}}_0$, $(L_1 + L_\infty)(\mathcal{M}, \tau)$, and $L_p(\mathcal{M}, \tau)$ for $0 < p < +\infty$. If $X \in \widetilde{\mathcal{M}}$ and $n \geq 2$, then by Theorem 3.2 we have

$$\mu_t(X^n) = \mu_t(X^{n-k}X^k) = \mu_t(|X^{n-k}||X^{*k}|)$$

for all $k \in \{1, 2, \ldots, n-1\}$ and $t > 0$. Therefore, $X^n \in \mathcal{E} \Leftrightarrow |X^{n-k}||X^{*k}| \in \mathcal{E}$ for all $k \in \{1, 2, \ldots, n-1\}$.

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