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Конференція присвячена обміну досвідом та досягненням в розвитку і застосуванні інтегральних методів математичного моделювання, алгоритмів та засобів розв'язування інтегральних рівнянь в наукових дослідженнях, освіті, проектних розробках і промислових технологіях та пов'язана з відзначенням 50-ї річниці заснування кафедри обчислювальної математики у Львівському університеті.

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3. Colton D., Kress R. Inverse Acoustic and Electromagnetic Scattering Theory, 2^{nd} edition.– Vol. **93** of Applied Mathematical Sciences.– Berlin: Springer-Verlag, 1998.
4. Ganesh M., Graham I. G. A high-order algorithm for obstacle scattering in three dimensions // J. Comput. Phys.– 2004.– **198**.– 1.– 211-242.
5. Kress R., Yaman F., Yapar A., Akduman I. Inverse scattering for an impedance cylinder buried in a dielectric cylinder // Inverse Probl. Sci. Eng.– 2009.– **17**.– 4.– 473-488.
6. Wienert L. Die numerische Approximation von Randintegraloperatoren für die Helmholtzgleichung im $\mathbb{R}^3$.– PhD thesis, University of Göttingen, 1990.

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INTEGRAL EQUATION METHODS IN OPTICAL WAVEGUIDE THEORY

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1. Introduction

Optical waveguides are regular dielectric rods, having various cross sectional shapes, and where generally the dielectric permittivity may vary in the waveguide's cross section [1]. The eigenvalue problems for natural modes (surface, leaky, and complex eigenmodes) of an inhomogeneous optical waveguide without a sharp boundary (by analogy with [2]) and for a step-index optical waveguide with smooth boundary of cross-section are formulated as problems for the set of time-harmonic Maxwell equations with partial radiation conditions ([3], [4]) at infinity in the cross-sectional plane. The original problems by integral equation methods are reduced to nonlinear spectral problems with Fredholm integral operators. Theorems on spectrum localization are proved, and then it is proved that the sets of all eigenvalues of the original problems can only be some sets of isolated points on the Reimann surface, ant it also proved that each eigenvalue depends continuously on the frequency and dielectric permittivity and can appear and disappear only at the boundary of the Reimann surface.

2. Step-index optical waveguide

Let the three-dimensional space be occupied by an isotropic source-free medium, and let the dielectric permittivity be prescribed as a positive real-valued function $\varepsilon = \varepsilon(x)$ independent of the longitudinal coordinate and equal to a constant $\varepsilon_\infty > 0$ outside a cylinder. In this section we consider the natural modes of an step-index optical fiber and suppose that the dielectric permittivity is equal to a constant $\varepsilon_+ > \varepsilon_\infty$ inside the cylinder. The axis of the cylinder is parallel to the longitudinal coordinate, and its cross section is a bounded domain Ω_i with a twice continuously differentiable boundary γ. The domain Ω_i is a subset of a circle with radius R_0. Denote by Ω_e the unbounded domain $\Omega_e = \mathbb{R}^2 \setminus \overline{\Omega}_i$. Denote by U the space of complex-valued continuous and continuously differentiable in $\overline{\Omega}_i$ and $\overline{\Omega}_e$, twice continuously differentiable in Ω_i

and Ω_e functions. Denote by Λ the Riemann surface of the function $\ln \chi_\infty(\beta)$, where $\chi_\infty = \sqrt{k^2\varepsilon_\infty - \beta^2}$. Here $k^2 = \omega^2\varepsilon_0\mu_0$, ω is a given radian frequency; ε_0, μ_0 are the free-space dielectric and magnetic constants, respectively. Denote by Λ_0 the principal ("proper") sheet of this Riemann surface, which is specified by the condition $\mathrm{Im}\chi_\infty(\beta) \geq 0$. A nonzero vector $\{\mathrm{E}, \mathrm{H}\} \in U^6$ is referred to as eigenvector (or eigenmode) of the problem corresponding to an eigenvalue $\beta \in \Lambda$ if the following relations are valid:

$$\mathrm{rot}_\beta \mathrm{E} = i\omega\mu_0 \mathrm{H}, \quad \mathrm{rot}_\beta \mathrm{H} = -i\omega\varepsilon_0\varepsilon \mathrm{E}, \quad x \in \mathbb{R}^2 \setminus \gamma, \tag{1}$$

$$\nu \times \mathrm{E}^+ = \nu \times \mathrm{E}^-, \quad x \in \gamma, \tag{2}$$

$$\nu \times \mathrm{H}^+ = \nu \times \mathrm{H}^-, \quad x \in \gamma, \tag{3}$$

$$\begin{bmatrix} \mathrm{E} \\ \mathrm{H} \end{bmatrix} = \sum_{l=-\infty}^{\infty} \begin{bmatrix} \mathrm{A}_l \\ \mathrm{B}_l \end{bmatrix} H_l^{(1)}(\chi_\infty r) \exp(il\varphi), \quad r \geq R_0. \tag{4}$$

Here differential operator $\mathbf{rot}_\beta$ is obtained from usual operator by replacing generating waveguide line derivative with $i\beta$ multiplication; and $H_l^{(1)}(z)$ is the Hankel function of the first kind and index l. The conditions (4) are the the partial radiation conditions.

Theorem 1. (See [5]). *The imaginary axis* $\mathbb{I}$ *and the real axis* $\mathbb{R}$ *of the sheet* Λ_0 *except the set* $G = \{\beta \in \mathbb{R} : k^2\varepsilon_\infty < \beta^2 < k^2\varepsilon_+\}$ *are free of the eigenvalues of the problem* (1)-(4). *Surface and complex eigenmodes correspond to real eigenvalues* $\beta \in G$ *and complex eigenvalues* $\beta \in \Lambda_0$, *respectively. Leaky eigenmodes correspond to complex eigenvalues* β *belonging to an "improper" sheet of* Λ *for which* $\mathrm{Im}\chi_\infty(\beta) < 0$.

We use the representation of the eigenvectors of problem (1)-(4) in the form of the single-layer potentials u and v:

$$\mathrm{E}_1 = \frac{i}{k^2\varepsilon - \beta^2}\left(\mu_0\omega\frac{\partial v}{\partial x_2} + \beta\frac{\partial u}{\partial x_1}\right),$$
$$\mathrm{E}_2 = \frac{-i}{k^2\varepsilon - \beta^2}\left(\mu_0\omega\frac{\partial v}{\partial x_1} - \beta\frac{\partial u}{\partial x_2}\right), \quad \mathrm{E}_3 = u, \tag{5}$$

$$\mathrm{H}_1 = \frac{i}{k^2\varepsilon - \beta^2}\left(\beta\frac{\partial v}{\partial x_1} - \varepsilon_0\varepsilon\omega\frac{\partial u}{\partial x_2}\right),$$
$$\mathrm{H}_2 = \frac{i}{k^2\varepsilon - \beta^2}\left(\beta\frac{\partial v}{\partial x_2} + \varepsilon_0\varepsilon\omega\frac{\partial u}{\partial x_1}\right), \quad \mathrm{H}_3 = v, \tag{6}$$

$$\begin{bmatrix} u(x) \\ v(x) \end{bmatrix} = \frac{i}{4}\int_\gamma H_0^{(1)}\left(\sqrt{k^2\varepsilon_{+/\infty} - \beta^2}\,|x-y|\right)\begin{bmatrix} f_{+/\infty}(y) \\ g_{+/\infty}(y) \end{bmatrix} dl(y), \quad x \in \Omega_{i/e}, \tag{7}$$

where unknown densities $f_{+/\infty}$ and $g_{+/\infty}$ belong to the space of Hölder continuous functions $C^{0,\alpha}$. The original problem (1)-(4) by single-layer potential representation is reduced [5] to a nonlinear eigenvalue problem for a set of singular integral equations at the boundary γ. This problem has the operator form

$$A(\beta)w \equiv (I + B(\beta))w = 0, \tag{8}$$

where I is the identical operator in the Banach space $W = (C^{0,\alpha})^4$ and $B(\beta) : W \to W$ is a compact operator consists particularly of the following boundary singular integral operators:

$$Lp = -\frac{1}{2\pi}\int_0^{2\pi} \ln|\sin\frac{t-\tau}{2}|p(\tau)d\tau, \quad t \in [0, 2\pi], \quad L : C^{0,\alpha} \to C^{1,\alpha}, \tag{9}$$

$$Sp = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{ctg}\frac{\tau - t}{2}p(\tau)d\tau + \frac{i}{2\pi}\int_0^{2\pi} p(\tau)d\tau, \quad t \in [0, 2\pi], \quad S : C^{0,\alpha} \to C^{0,\alpha}. \tag{10}$$

The original problem (1)-(4) is spectrally equivalent [5] to the problem (8). Namely, suppose that $w \in W$ is an eigenvector of the operator-valued function $A(\beta)$ corresponding to an eigenvalue $\beta \in \Lambda_0 \setminus D$, $D = \{\beta \in \mathbb{I}\} \bigcup \{\beta \in \mathbb{R} : \beta^2 < k^2\varepsilon_\infty\}$. Then using this vector we can construct the densities of the single-layer potential representation of an eigenmode $\{\mathrm{E}, \mathrm{H}\} \in U^6$ of the problem (1)-(4), corresponding to the same eigenvalue β. For other side, any eigenmode of the problem (1)-(4), corresponding to an eigenvalue $\beta \in \Lambda_0 \setminus D$ can be represented in the form of single-layer potentials. The densities of this potentials construct an eigenvector $w \in W$ of the operator-valued function $A(\beta)$ corresponding to the same eigenvalue β.

Theorem 2. (See [5]). *For each $\beta \in \{\beta \in \mathbb{R} : \beta^2 \geq k^2\varepsilon_+\}$ the operator $A(\beta)$ has the bounded inverse operator. The set of all eigenvalues β of the operator-valued function $A(\beta)$ can be only a set of isolated points on Λ. Each eigenvalue β depends continuously on $\omega > 0$, $\varepsilon_+ > 0$, and $\varepsilon_\infty > 0$ and can appear and disappear only at the boundary of Λ, i.e. at $\beta = \pm k\sqrt{\varepsilon_\infty}$ and at infinity.*

3. Inhomogeneous optical waveguide

In this section we consider the natural modes of an inhomogeneous optical fiber without a sharp boundary. Let the dielectric permittivity ε belongs to the space $C^2(\mathbb{R}^2)$ of twice continuously differentiable in $\mathbb{R}^2$ functions. Denote by ε_+ the maximum of the function ε in the domain Ω_i, let $\varepsilon_+ > \varepsilon_\infty > 0$. A nonzero complex vector $\{\mathrm{E}, \mathrm{H}\} \in (C^2(\mathbb{R}^2))^6$ is referred to as eigenvector (or eigenmode) of the problem corresponding to an eigenvalue $\beta \in \Lambda$ if the following relations are valid [6]:

$$\mathrm{rot}_\beta \mathrm{E} = i\omega\mu_0 \mathrm{H}, \quad \mathrm{rot}_\beta \mathrm{H} = -\, i\omega\varepsilon_0\varepsilon \mathrm{E}, \quad x \in \mathbb{R}^2, \tag{11}$$

$$\begin{bmatrix} \mathrm{E} \\ \mathrm{H} \end{bmatrix} = \sum_{l=-\infty}^{\infty} \begin{bmatrix} \mathrm{A}_l \\ \mathrm{B}_l \end{bmatrix} H_l^{(1)}(\chi_\infty r)\exp(il\varphi), \quad r \geq R_0. \tag{12}$$

Theorem 3. (See [6]). *The imaginary axis $\mathbb{I}$ and the real axis $\mathbb{R}$ of the sheet Λ_0 except the set $G = \left\{\beta \in \mathbb{R} : k^2\varepsilon_\infty < \beta^2 < k^2\varepsilon_+\right\}$ are free of the eigenvalues of the problem (11), (12). Surface and complex eigenmodes correspond to real eigenvalues $\beta \in G$ and complex eigenvalues $\beta \in \Lambda_0$, respectively. Leaky eigenmodes correspond to complex eigenvalues β belonging to an "improper" sheet of Λ for which $\mathrm{Im}\chi_\infty(\beta) < 0$.*

If vector $\{\mathrm{E},\mathrm{H}\} \in (C^2(\mathbb{R}^2))^6$ is an eigenvector of problem (11), (12) corresponding to an eigenvalue $\beta \in \Lambda$, then (see [6])

$$\mathrm{E}(x) = k^2 \int_{\Omega_i} (\varepsilon(y) - \varepsilon_\infty)\Phi(\beta; x, y)\mathrm{E}(y)dy+$$
$$+ \operatorname{grad}_\beta \int_{\Omega_i} \left(\mathrm{E}, \varepsilon^{-1}\operatorname{grad}\varepsilon\right)(y)\Phi(\beta; x, y)dy, \quad x \in \mathbb{R}^2, \tag{13}$$

$$\mathrm{H}(x) = -i\omega\varepsilon_0 \operatorname{rot}_\beta \int_{\Omega_i} (\varepsilon(y) - \varepsilon_\infty)\Phi(\beta; x, y)\mathrm{E}(y)dy, \quad x \in \mathbb{R}^2. \tag{14}$$

Using the integral representation (13) for $x \in \Omega_i$ we obtain a nonlinear eigenvalue problem for integral equation on the domain Ω_i. This problem has the operator form

$$A(\beta)\mathrm{F} \equiv (I - B(\beta))\mathrm{F} = 0, \tag{15}$$

where the operator $B(\beta) : (L_2(\Omega_i))^3 \to (L_2(\Omega_i))^3$ satisfies the right side of the integral representation (13) for $x \in \Omega_i$. For any $\beta \in \Lambda$ the operator $B(\beta)$ is compact [6].

It was proved in the paper [6] that the original problem (11), (12) is spectrally equivalent to problem (15). Namely, suppose that $\{\mathrm{E},\mathrm{H}\} \in (C^2(\mathbb{R}^2))^6$ is an eigenmode of problem (11), (12) corresponding to an eigenvalue $\beta \in \Lambda$. Then $\mathrm{F} = \mathrm{E} \in [L_2(\Omega_i)]^3$ is an eigenvector of the operator-valued function $A(\beta)$ corresponding to the same eigenvalue β. Suppose that $\mathrm{F} \in [L_2(\Omega_i)]^3$ is an eigenvector of the operator-valued function $A(\beta)$ corresponding to an eigenvalue $\beta \in \Lambda$, and also suppose that the same number β is not an eigenvalue of the following problem:

$$\left[\Delta + \left(k^2\varepsilon - \beta^2\right)\right] u = 0, \quad x \in \mathbb{R}^2, \quad u \in C^2(\mathbb{R}^2), \tag{16}$$

$$u = \sum_{l=-\infty}^{\infty} a_l H_l^{(1)}(\chi_\infty r) \exp(il\varphi), \quad r \geq R_0. \tag{17}$$

Let $\mathrm{E} = B(\beta)\mathrm{F}$ and $\mathrm{H} = (i\omega\mu_0)^{-1}\operatorname{rot}_\beta \mathrm{E}$ for $x \in \mathbb{R}^2$. Then $\{\mathrm{E},\mathrm{H}\} \in (C^2(\mathbb{R}^2))^6$, and $\{\mathrm{E},\mathrm{H}\}$ is an eigenvector of the original problem (11), (12) corresponding to the same eigenvalue β.

Theorem 4. (See [6]). *For each $\beta \in \{\beta \in \mathbb{R} : \beta^2 \geq k^2\varepsilon_+\}$ the operator $A(\beta)$ has the bounded inverse operator. The set of all eigenvalues β of the operator-valued function $A(\beta)$ can be only a set of isolated points on Λ. Each eigenvalue β depends continuously on $\omega > 0$, $\varepsilon_+ > 0$, and $\varepsilon_\infty > 0$ and can appear and disappear only at the boundary of Λ, i.e. at $\beta = \pm k\sqrt{\varepsilon_\infty}$ and at infinity.*

4. Numerical Methods

Galerkin methods for numerical calculations of the natural modes are proposed, the convergence of the methods is proved, and some results of numerical experiments are discussed in the book [7].

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1. Snyder A. W., Love D. Optical Waveguide Theory.– London: Chapman and Hall, 1983.
2. Colton D., Kress R. Time harmonic electromagnetic waves in an inhomogeneous medium // Proceedings of the Royal Society of Edinburgh.– 1990.– Vol. 116A.– 279-293.
3. Sveshnikov A. G. The Principle of Limit Absorption for a Waveguide // Dokl. AN SSSR.– 1951.– Vol. 80.– No 3.– 345-347. (In Russian).
4. Il'inskii A. S., Shestopalov Yu. V. Application of the Methods of Spectral Theory in the Problems of Wave Propagation.– Moscow: Moscow State University Press, 1989. (In Russian).
5. Karchevskii E. M. Mathematical Analysis and Numerical Modeling of the Guided Modes of the Step-Index Optical Fibers // SIAM Proc. in Applied Mathematics.– 2000.– Vol. 102.– 414-419.
6. Kartchevski E. M., Nosich A. I., Hanson G. W. Mathematical analysis of the generalized natural modes of an inhomogeneous optical fiber // SIAM J. Appl. Math.– 2005.– Vol. 65.– No 6.– 2033-2048.
7. Dautov R. Z., Karchevskii E. M. Integral Equation Methods and Exact Nonlocal Boundary Conditions in the Theory of Dielectric Waveguides Kazan: Kazan State University, 2009. (In Russian).

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TRANSFERENCE OF THE INITIAL CONDITION FOR LINEAR INTEGRO-DIFFERENTIAL EQUATION

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1. Preliminary result

Consider a linear integro-differential equation

$$\frac{\partial x(t,s)}{\partial t} = \int_a^b K(s,\tau)x(t,\tau)d\tau + f(t,s), \qquad (1)$$

where $x(t,s)$ is unknown function, $f(t,s)$ and $K(t,s)$ are given continuous functions of arguments t and s; $t \geq t_0$, $a < b$, $a \leq s \leq b$, a, b, t_0 are given nonnegative numbers. Let at $t = t_0$ such equality is valid

$$\int_a^b h_0(\tau)x(t_0,\tau)d\tau = g_0, \qquad (2)$$

where continuous function $h_0(\tau)$ and number g_0 are also given.