Structure of H-semiprime artinian algebras

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Let H be a Hopf algebra over the ground field k. This article aims at H-equivariant versions of classical results on the structure of rings satisfying the descending chain condition on right ideals. An earlier attempt [22, Th. 0.3] to deal with this problem was not quite satisfactory as it required some technical assumption about H.

An H-module algebra A is called *H-semiprime* if A contains no nonzero nilpotent H-stable ideals. If $A \neq 0$ and A has no H-stable ideals other than 0 and A itself, then A is said to be *H-simple*. Finally, A is *H-semisimple* if it is isomorphic to a direct product of finitely many H-simple H-module algebras.

Theorem 0.1. *Any H-semiprime right artinian left H-module algebra A is actually H-semisimple and quasi-Frobenius.*

There are several known cases of this theorem. One of those occurs when H is the universal enveloping algebra of a Lie algebra L. Fundamental results of Block [4] on L-semisimple rings have found important applications in the theory of modular Lie algebras (see [23]). In [22] the conclusion of Theorem 0.1 was verified, for example, for any Hopf algebra with cocommutative coradical and for any finite dimensional Hopf algebra. When H is arbitrary and A is module-finite over its center, this was done in [21]. A classical related result states that every finite dimensional Hopf algebra is Frobenius [11]. A similar question was studied for finite dimensional right coideal subalgebras of Hopf algebras (see [15], [20]). It is not clear whether A is always a Frobenius ring in Theorem 0.1 (which means that the socle and the head of A are isomorphic as either right or left A-modules).

The aim of [22] was to prove that any H-semiprime right noetherian left H-module algebra A has an artinian classical ring of fractions. When the antipode of H is bijective, a certain quotient ring $Q(A)$ was constructed there. It was shown that $Q(A)$ is H-semiprime and semiprimary. Moreover, to conclude that $Q(A)$ is a ring of fractions one needs only to know that $Q(A)$ is quasi-Frobenius. Unfortunately it is not proved yet whether $Q(A)$ is always right artinian, and Theorem 0.1 cannot be applied. Anyway, it brings us one step closer to settling the remaining unanswered question from [22].

Theorem 0.2. *Let A be an H-semiprime right artinian left H-module algebra. If H grows slower than exponentially then each H-equivariant right or left A-module is A-projective.*

The assumption on the growth of H actually used in the proof of this theorem is stated as follows: for every finite dimensional subspace $V \subset H$ and every real numbers $\alpha > 1$, $c > 0$ there exists an integer $n > 0$ such that

$$\dim V^n < c\,\alpha^n$$

where V^n is the subspace of H spanned by all products of elements in V having precisely n factors. This property is close to subexponential growth (cf. [10]).

A right A-module M is H-*equivariant* if M is equipped with a left H-module structure such that

$$h(va) = \sum_{(h)} (h_{(1)}v)(h_{(2)}a) \quad \text{for all } h \in H,\ v \in M,\ a \in A.$$

The compatibility of the two module structures means that M may be regarded as a left module over the smash product $A^{\mathrm{op}}\#H^{\mathrm{cop}}$. In a similar way, H-equivariant left A-modules are left $A\#H$-modules. Let ${}_H\mathcal{M}_A$ and ${}_{A\#H}\mathcal{M}$ denote, respectively, the categories of H-equivariant right and left A-modules.

Compared with my earlier results in [20] the modules in Theorem 0.2 are not assumed to be A-finite. This is a serious obstacle which demanded a completely different approach. If $M \in {}_H\mathcal{M}_A$ is A-finite, then all indecomposable projective right A-modules occur in a direct sum decomposition of M with equal multiplicities. I do not know whether this remains true in general. Nevertheless, Theorem 5.2 establishes the freeness of some special equivariant modules.

Theorem 0.3. *Suppose that H is a cosemisimple Hopf algebra growing slower than exponentially. If A is any right artinian left H-module algebra, then its Jacobson radical $J(A)$ is stable under the action of H. In particular, A is semisimple whenever A is H-semiprime.*

For an arbitrary H-module algebra A the question concerning the H-stability of $J(A)$ was raised by Linchenko, Montgomery and Small [13]. It is intimately related to the question of Cohen and Fischman [5] about semiprimeness of $A\#H$, and to the question of Bahturin and Linchenko [1] as to whether A satisfies a polynomial identity whenever so does the subalgebra of invariants A^H. Positive answers are known under restrictions on either A or H. For example, Linchenko and Montgomery [14] verified that the prime radical $P(A)$ of A is H-stable when H is finite dimensional cosemisimple and A satisfies a polynomial identity. On the other hand, Linchenko [12] proved that $J(A)$ is H-stable when H is involutory, A is finite dimensional, and either $\operatorname{char} k = 0$ or $\operatorname{char} k > \dim A$. Theorem 0.3 shows that the questions on H-stability of $P(A)$ and $J(A)$ are also meaningful when H is cosemisimple, but infinite dimensional. I do not know whether the assumption on the growth of H is necessary. However, numeric estimates are essential in the proof of Theorem 0.2 on which Theorem 0.3 depends.

A crucial idea of the proofs is borrowed from a paper of Donkin [7]. One essential difference is that Donkin works within tensor categories, whereas in our case the categories $\mathcal{M}_A$ and ${}_A\mathcal{M}$, respectively, of right and left A-modules are module categories over the tensor category $\mathcal{M}^H$ of right H-comodules. Given a finitely generated right A-module V and a finite dimensional right H-comodule U, we consider a decomposition $U^{\otimes n} \otimes V \cong G_n \oplus W_n$ in $\mathcal{M}_A$ where G_n is a projective module, while W_n has no nonzero projective direct summands. It turns out that, with a suitable choice of U, the size of W_n gets relatively small compared with the size of G_n as $n \to \infty$. If $\dim A < \infty$, the k-dimension of finitely generated right A-modules measures their size. Otherwise different numeric characteristics must be looked at. The precise statement is given in Proposition 1.10.

Theorem 0.1 is proved in section 2, Theorems 0.2 and 0.3 in section 4. All results remain valid when k is an arbitrary commutative ring provided that H is the union of a directed family $\mathcal{F}$ of subcoalgebras such that each $C \in \mathcal{F}$ is a finitely generated projective k-module (see the Remark at the end of section 2). A reduction to the case of H-simple module algebras has been already accomplished in [22]. Apart from this, only basic properties of the tensoring operation in $\mathcal{M}_A$ are used from previous work. The antipode $S : H \to H$ is not assumed to be bijective. This does incur a few complications since one has to be careful about the module and comodule structures in use. Lemmas 1.1 and 3.2 are critical in this respect.

I am most grateful to Akira Masuoka for drawing my attention to Donkin's paper. Under an assumption on the growth of H Donkin proved that H, regarded as a right comodule over a finite dimensional factor Hopf algebra $\overline{H}$, is isomorphic to a direct sum of copies of $\overline{H}$. Theorem 5.4 strengthens this result.

1. Big projective summands

Let H be a Hopf algebra over the ground field k with the counit $\varepsilon : H \to k$, the comultiplication $\Delta : H \to H \otimes H$ and the antipode $S : H \to H$. Let $\mathcal{F}$ be the set of all finite dimensional subcoalgebras of H. We denote by $\mathcal{M}_R$ and ${}_R\mathcal{M}$, respectively, the categories of right and left modules over a ring R, by $\mathcal{M}^C$ and ${}^C\mathcal{M}$ those of right and left comodules over a coalgebra C. The comodule structure map $U \to U \otimes C$ (respectively, $U \to C \otimes U$) is symbolically written as $u \mapsto \sum u_{(0)} \otimes u_{(1)}$ (respectively, $u \mapsto \sum u_{(-1)} \otimes u_{(0)}$).

Let further A be a left H-module algebra. For $U \in \mathcal{M}^H$ and $V \in \mathcal{M}_A$ the *twisted action* of A on $U \otimes V$ has been defined in [22] by the rule

$$(u \otimes v)a = \sum_{(u)} u_{(0)} \otimes v\big(S(u_{(1)})a\big), \qquad u \in U,\ v \in V,\ a \in A.$$

If k is equipped with the trivial right H-comodule structure, then the canonical k-linear bijection $k \otimes V \cong V$ is an isomorphism in $\mathcal{M}_A$, and so too is

$$U \otimes (U' \otimes V) \cong (U \otimes U') \otimes V \quad \text{for } U, U' \in \mathcal{M}^H.$$

This means that $\mathcal{M}_A$ is a left module category over the tensor category $\mathcal{M}^H$ (a formal definition of a module category is given in [17]). In particular, for each integer $n > 0$ the nth iteration of the endofunctor $U \otimes ?$ of $\mathcal{M}_A$ is isomorphic with $U^{\otimes n} \otimes ?$ where $U^{\otimes n}$ denotes the nth tensor power of U.

Let $U_{\rm triv}$ denote U with the trivial right H-comodule structure $u \mapsto u \otimes 1$. In the A-module $U_{\rm triv} \otimes V$ we have $(u \otimes v)a = u \otimes va$. We will always regard H as a right H-comodule with respect to Δ. In particular, $H \otimes V$ is a right A-module with respect to the twisted action of A. Below we repeat a part of [22, Lemma 1.2]:

Lemma 1.1. *If $M \in {}_H\mathcal{M}_A$ then the map $H \otimes M \to M$, $h \otimes v \mapsto hv$, is A-linear and $U \otimes M \cong U_{\rm triv} \otimes M$ in $\mathcal{M}_A$ for any $U \in \mathcal{M}^H$.*

Lemma 1.2. *Let $U \in \mathcal{M}^H$ and $V \in \mathcal{M}_A$. If V is either free or projective, then so too is $U \otimes V$. More precisely, $U \otimes V \cong A^{dm}$ when $\dim U = d$ and $V \cong A^m$.*

Proof. Lemma 1.1 applied with $M = A$ shows that $U \otimes A \cong U_{\mathrm{triv}} \otimes A$ in $\mathcal{M}_A$. As the latter module is a direct sum of d copies of A, the conclusion is true for $V = A$. Since $U\otimes ?$ is an additive functor and the projective modules are precisely the direct summands of free modules, the rest is clear. $\square$

Let us assume further that A is right artinian. Then each finitely generated right A-module V has finite length $\mathrm{lng}(V)$. Denote by $\mathrm{Max}\, A$ the finite set of all maximal ideals of A. Put

$$r(V) = \max\{r_P(V) \mid P \in \mathrm{Max}\, A\} \quad \text{where } r_P(V) = \frac{\mathrm{lng}(V/VP)}{\mathrm{lng}(A/P)}.$$

We will also need a different numeric characteristic of V. Let $V \cong G \oplus W$ in $\mathcal{M}_A$ where G is a projective right A-module, while W has no nonzero projective direct summands. By the Krull-Schmidt Theorem such a decomposition is unique up to isomorphism. Put

$$\mu(V) = \min\{r_P(G) \mid P \in \mathrm{Max}\, A\}.$$

For each integer $l > 0$ we denote by V^l the direct sum of l copies of V. Clearly $r_P(V^l) = r_P(V)l$ for all $P \in \mathrm{Max}\, A$.

Lemma 1.3. *Let $n \geq 0$ and $l > 0$ be two integers.*

(i) *$r(V) \leq n/l$ if and only if there exists an epimorphism $A^n \to V^l$ in $\mathcal{M}_A$.*

(ii) *$\mu(V) \geq n/l$ if and only if A^n is a direct summand of V^l.*

Proof. Since $r(V^l) = r(V)l$ and $\mu(V^l) = \mu(V)l$, we can reduce the proof to the case $l = 1$, replacing V with V^l. By Nakayama's Lemma the A-module V is n-generated if and only if so are the A/P-modules V/VP for all $P \in \mathrm{Max}\, A$. Since each ring A/P is simple artinian, the latter condition can be rewritten as $r_P(V) \leq n$ for all P, i.e. $r(V) \leq n$. This proves (i).

Each projective right A-module G is a direct sum of indecomposable projectives. Moreover, the multiplicity with which the projective cover in $\mathcal{M}_A$ of a simple right A/P-module occurs in such a decomposition of G is equal to

$$\mathrm{lng}(G/GP) = r_P(G)\,\mathrm{lng}(A/P).$$

It follows that A^n is a direct summand of G if and only if $r_P(G) \geq r_P(A^n) = n$ for all $P \in \mathrm{Max}\, A$. Thus A^n is a direct summand of V if and only if $\mu(V) \geq n$. $\square$

Lemma 1.4. *If $F \cong A^m$ is a free right A-module of rank m, then*

$$r(F \oplus V) = m + r(V) \quad \text{and} \quad \mu(F \oplus V) = m + \mu(V).$$

Proof. The first equality follows from the fact that

$$r_P(F \oplus V) = r_P(F) + r_P(V) = m + r_P(V)$$

for all $P \in \operatorname{Max} A$. If $V \cong G \oplus W$ where G is a projective module and W has no nonzero projective direct summands, then $F \oplus V \cong (F \oplus G) \oplus W$ is a similar decomposition. Since $r_P(F \oplus G) = m + r_P(G)$, the second equality in the statement of the lemma is also clear. □

Lemma 1.5. *We have* $\mu(V)\operatorname{lng}(A) \le \operatorname{lng}(V) \le r(V)\operatorname{lng}(A)$.

Proof. Let $r(V) = n/l$ for some integers $n \ge 0$ and $l > 0$. By Lemma 1.3 there exists an epimorphism $A^n \to V^l$. Hence

$$\operatorname{lng}(A)n = \operatorname{lng}(A^n) \ge \operatorname{lng}(V^l) = \operatorname{lng}(V)l,$$

which proves one inequality. The second inequality is proved similarly. □

Lemma 1.6. *We have* $r(U \otimes V) \le \dim(U)r(V)$ *and* $\mu(U \otimes V) \ge \dim(U)\mu(V)$ *for* $U \in \mathcal{M}^H$.

Proof. Let $r(V) = n/l$ and $d = \dim U$. Any epimorphism $A^n \to V^l$ in $\mathcal{M}_A$ gives rise to an epimorphism $U \otimes A^n \to U \otimes V^l \cong (U \otimes V)^l$. By Lemma 1.2 $U \otimes A^n \cong A^{dn}$, and so Lemma 1.3 yields $r(U \otimes V) \le dn/l$. The second inequality is proved similarly. □

For a subcoalgebra C of H and an ideal I of A we define another ideal

$$I_C = \{a \in A \mid Ca \subset I\}.$$

In particular, I_H is the largest H-stable ideal of A contained in I. Since H is the sum of subcoalgebras in $\mathcal{F}$, we have $I_H = \bigcap_{C \in \mathcal{F}} I_C$.

Lemma 1.7. *If A is H-simple, then there exists $C \in \mathcal{F}$ such that $P_C = 0$ for all $P \in \operatorname{Max} A$.*

Proof. Given any ideal I of A, the family of ideals $\{I_C \mid C \in \mathcal{F}\}$ has a minimal element. Since the correspondence $C \mapsto I_C$ reverses inclusions, there exists $C \in \mathcal{F}$ such that $I_D = I_C$ for all $D \in \mathcal{F}$ with $C \subset D$, and then $I_H = I_C$. If $I \ne A$, then $I_H = 0$ by the H-simplicity of A, so that $I_C = 0$. Let $P_1, \ldots, P_n$ be all maximal ideals of A. For each i we can find $C_i \in \mathcal{F}$ such that $(P_i)_{C_i} = 0$. Now $C = \sum C_i$ satisfies the required condition. □

Corollary 1.8. *If A is H-simple, then H has a right coideal U of finite dimension such that*

(i) *$A(UI) = A$ for each nonzero right ideal I of A,*

(ii) *$U \otimes V$ is a generator in $\mathcal{M}_A$ for each $V \in \mathcal{M}_A$ satisfying* $\operatorname{Hom}_A(V, A) \ne 0$.

Proof. In view of Lemma 1.1 each A-linear map $\varphi : V \to A$ gives rise to an A-linear map $\theta : U \otimes V \to A$ obtained as the composite

$$U \otimes V \xrightarrow{\mathrm{id}\otimes\varphi} H \otimes A \longrightarrow A$$

where the second map is afforded by the H-module structure on A. For $a \in A$ the composite θ_a of θ and the left multiplication by a is again a morphism $U \otimes V \to A$ in $\mathcal{M}_A$. We have $\operatorname{Im}\theta = U\varphi(V)$ and $\sum_{a \in A} \operatorname{Im}\theta_a = A \cdot \operatorname{Im}\theta$. If $A \cdot \operatorname{Im}\theta = A$, then

A is an epimorphic image in $\mathcal{M}_A$ of a direct sum of copies of $U \otimes V$. Thus (ii) is a consequence of (i).

Take $U = C$ where C satisfies the conclusion of Lemma 1.7. As we have seen, the composite $U \otimes I \hookrightarrow H \otimes A \to A$ is a morphism in $\mathcal{M}_A$. Hence its image UI is a right ideal of A. It follows that $A(UI)$ is a two-sided ideal. If $A(UI) \neq A$, then $UI \subset P$ for some $P \in \operatorname{Max} A$, and so $I \subset P_C = 0$. This proves (i). □

Lemma 1.9. *Suppose that I is a right ideal of a semiperfect ring R with the property that $RI = R$. Then R has a right ideal T such that $T \subset I$, T is an $\mathcal{M}_R$-direct summand of R, and T is a generator in $\mathcal{M}_R$.*

Proof. Let $\sigma : R \to R/I$ be the projection. By [9, Th. 11.1.1] there is a decomposition $R = T \oplus T'$ where T, T' are right ideals of R such that $T \subset \operatorname{Ker} \sigma$ and the restriction $T' \to R/I$ of σ is a projective cover in $\mathcal{M}_R$. Thus $T \subset I$ and $T' \cap I \subset T'J$ where J denotes the Jacobson radical of R. It follows that $I = T + (T' \cap I) \subset T + J$, and therefore $RI \subset RT + J$. We deduce that $RT + J = R$, whence $RT = R$ by Nakayama's Lemma. There exist finitely many elements $a_1, \ldots, a_n \in R$ such that $\sum a_i T = R$. Since the map $T^n \to R$ defined by the rule $(t_1, \ldots, t_n) \mapsto \sum a_i t_i$ is an epimorphism in $\mathcal{M}_R$, we conclude that T is a generator in $\mathcal{M}_R$. □

Recall that a right A-module V is said to be *torsionless* if for each $0 \neq v \in V$ there exists $\varphi \in \operatorname{Hom}_A(V, A)$ with $\varphi(v) \neq 0$. In other words, this means that V embeds as a submodule in a direct product of copies of A.

Proposition 1.10. *Let U be as in Corollary* 1.8*. Given any finitely generated torsionless module $V \in \mathcal{M}_A$, consider the A-modules $V_n = U^{\otimes n} \otimes V$ for all integers $n > 0$. We have*

$$r(V_n) - \mu(V_n) \leq (\dim U - c)^n \, r(V)$$

where

$$c = 1/\big(t \operatorname{lng}(A)\big), \qquad t = \max\{\operatorname{lng}(A/P) \mid P \in \operatorname{Max} A\}.$$

If $V_n \cong G_n \oplus W_n$ in $\mathcal{M}_A$ where G_n is a projective module, while W_n has no nonzero projective direct summands, then

$$\frac{r(W_n)}{r(V_n)} \leq \frac{(\dim U - c)^n}{c\,(\dim U)^{n-1}}.$$

Proof. First we choose $\varphi_1, \ldots, \varphi_q \in \operatorname{Hom}_A(V, A)$ proceeding as follows. If $V \neq 0$, let $\varphi_1 : V \to A$ be any nonzero A-linear map and $K_1 = \operatorname{Ker} \varphi_1$. If $K_1 \neq 0$, the assumption that V is torsionless enables us to find φ_2 such that $\varphi_2(K_1) \neq 0$. Continuing in this way, let $K_i = K_{i-1} \cap \operatorname{Ker} \varphi_i$; if $K_i \neq 0$, we take any φ_{i+1} with the property that $\varphi_{i+1}(K_i) \neq 0$. Since $\operatorname{lng}(V) < \infty$ and the sequence of A-submodules $V = K_0 \supset K_1 \supset \cdots$ is strictly descending, we must get $K_q = 0$ at some step. Note that φ_i induces an isomorphism of K_{i-1}/K_i with a right ideal of A. This entails $\operatorname{lng}(K_{i-1}/K_i) \leq \operatorname{lng}(A)$, and therefore $\operatorname{lng}(V) \leq \operatorname{lng}(A)\, q$. Since $\operatorname{lng}(V)$ is an obvious upper bound for the number of elements generating V, Lemma 1.3 shows that $r(V) \leq \operatorname{lng}(V)$. Thus

$$q \geq \operatorname{lng}(V)/\operatorname{lng}(A) \geq r(V)/\operatorname{lng}(A).$$

By (i) of Corollary 1.8 and Lemma 1.9 for each $i = 1, \dots, q$ there is a right ideal T_i of A such that $T_i \subset U\,\varphi_i(K_{i-1})$, T_i is a direct summand of A, and T_i is a projective generator in $\mathcal{M}_A$. Since each indecomposable projective right A-module has to be a direct summand of T_i, we get $r_P(T_i) > 0$, whence $r_P(T_i) \geq 1/\operatorname{lng}(A/P) \geq 1/t$ for any $P \in \operatorname{Max} A$. For each i choose any right A-linear retraction $\pi_i : A \to T_i$.

Define $\theta_i : U \otimes V \to A$ by the rule $\theta_i(u \otimes v) = u \cdot \varphi_i(v)$ for $u \in U$ and $v \in V$. It has been observed in the proof of Corollary 1.8 that θ_i is A-linear. Let

$$\psi : U \otimes V \to T_1 \oplus \cdots \oplus T_q$$

denote the A-linear map having $\pi_i\theta_i : U \otimes V \to T_i$ as its ith component. By our construction $T_i \subset \theta_i(U \otimes K_{i-1})$. Hence $\pi_i\theta_i$ restricts to a surjection $U \otimes K_{i-1} \to T_i$, while $U \otimes K_i \subset \operatorname{Ker} \pi_i\theta_i$. An easy downward induction on i now shows that ψ maps $U \otimes K_{i-1}$ onto the submodule $T_i \oplus \cdots \oplus T_q$ of $T_1 \oplus \cdots \oplus T_q$. Taking $i = 1$, we deduce that ψ is an epimorphism in $\mathcal{M}_A$. Since $T_1 \oplus \cdots \oplus T_q$ is a projective A-module, it has to be isomorphic with a direct summand of $V_1 = U \otimes V$. It follows that

$$r_P(G_1) \geq r_P(T_1 \oplus \cdots \oplus T_q) = \sum_{i=1}^{q} r_P(T_i) \geq \frac{q}{t} \geq \frac{r(V)}{t \operatorname{lng}(A)} = c\,r(V)$$

for all $P \in \operatorname{Max} A$. In other words, $\mu(V_1) \geq c\,r(V)$. By Lemma 1.6 $r(V_1) \leq d\,r(V)$ where we put $d = \dim U$. Hence the inequality

$$r(V_n) - \mu(V_n) \leq (d-c)^n r(V)$$

holds for $n = 1$. We next employ induction on n. Suppose that the inequality above is true for some n. Let $\mu(V_n) = m/l$ for some integers $m \geq 0$ and $l > 0$. By Lemma 1.3 $V_n^l \cong A^m \oplus V'$ for some $V' \in \mathcal{M}_A$. Now

$$r(V') = l\,r(V_n) - m = l\big(r(V_n) - \mu(V_n)\big) \leq l(d-c)^n\,r(V)$$

by Lemma 1.4. Since $V_{n+1} \cong U \otimes V_n$, Lemma 1.2 allows us to write

$$V_{n+1}^l \cong U \otimes V_n^l \cong (U \otimes A^m) \oplus (U \otimes V') \cong A^{dm} \oplus (U \otimes V').$$

Hence $l\,r(V_{n+1}) = dm + r(U \otimes V')$ and $l\,\mu(V_{n+1}) = dm + \mu(U \otimes V')$, again by Lemma 1.4. Since V is finitely generated and torsionless, V embeds in a free A-module, say F. Then V_n embeds in $U^{\otimes n} \otimes F$. By Lemma 1.2 the latter A-module is also free. It follows that V_n^l and V' are submodules of a free module too. In particular, V' is torsionless. Applying the first part of the proof with V' in place of V, we get

$$r(V_{n+1}) - \mu(V_{n+1}) = \frac{r(U \otimes V') - \mu(U \otimes V')}{l} \leq \frac{(d-c)r(V')}{l}\,.$$

This enables us to carry out the inductive step.

Next, $r_P(V_n) = r_P(G_n) + r_P(W_n) \geq \mu(V_n) + r_P(W_n)$ for all $P \in \operatorname{Max} A$. Taking the maximum over all P, we get $r(V_n) \geq \mu(V_n) + r(W_n) \geq \mu(V_n)$. Thus

$$r(W_n)/r(V_n) \leq \big(r(V_n) - \mu(V_n)\big)/\mu(V_n).$$

Since $\mu(V_n) \geq d^{n-1}\mu(V_1) \geq cd^{n-1}r(V)$ by Lemma 1.6, the second inequality in the statement of the proposition is also clear. $\square$

Note that U in Corollary 1.8 can be replaced with any larger finite dimensional right coideal. Hence Proposition 1.10 holds for any larger U as well.

2. The quasi-Frobenius property

We continue to assume that A is a right artinian left H-module algebra. Since all right ideals of A are finitely generated, we may apply [22, Lemma 4.2]:

Lemma 2.1. *If A is H-semiprime, then A is H-semisimple. Moreover, if $P_H = 0$ for some $P \in \operatorname{Max} A$, then A is H-simple.*

Lemma 2.2. *If A is H-simple, then A is also $S(H)$-simple.*

Proof. Let V be any simple right ideal of A, and denote by P its right annihilator in A. Corollary 1.8 shows that there exists $U \in \mathcal{M}^H$ such that $U \otimes V$ is a generator in $\mathcal{M}_A$. In particular, $U \otimes V$ is a faithful A-module. However, any element $a \in A$ satisfying $S(H)a \subset P$ annihilates $U \otimes V$. Hence $P_{S(H)} = 0$. Replacing H with its Hopf subalgebra $S(H)$ in Lemma 2.1, we get the conclusion. □

Lemma 2.3. *If A is H-simple, then there exists a right coideal U of H such that $\dim U < \infty$ and $U \otimes V$ is a faithful A-module for every $0 \neq V \in \mathcal{M}_A$.*

Proof. In view of Lemma 2.2 A is $S(H)$-simple. Hence by Lemma 1.7 there exists a finite dimensional subcoalgebra $C' \subset S(H)$ such that $P_{C'} = 0$ for all $P \in \operatorname{Max} A$. Now take any $C \in \mathcal{F}$ such that $C' \subset S(C)$ and put $U = C$. Consider the annihilator I of V in A. If $a \in A$ annihilates $C \otimes V$, then

$$\sum_{(c)} c_{(1)} \otimes v\big(S(c_{(2)})a\big) = (c \otimes v)a = 0,$$

and therefore $v\big(S(c)a\big) = \sum_{(c)} \varepsilon(c_{(1)})\, v\big(S(c_{(2)})a\big) = 0$ for all $c \in C$ and $v \in V$. This shows that $S(C)a \subset I$, yielding $C'a \subset I$. If $V \neq 0$, then $I \neq A$. In this case I is contained in a maximal ideal P of A, whence $a \in P_{C'}$. We conclude that $a = 0$. □

Lemma 2.4. *Let $V \in \mathcal{M}_A$ be finitely generated and $U \in \mathcal{M}^H$ finite dimensional. Put $V_n = U^{\otimes n} \otimes V$. If V_1 is a faithful A-module, then $\operatorname{lng}(V_n) \geq (\dim U)^{n-1}$ for all $n > 0$.*

Proof. By the faithfulness of V_1 there is an embedding $A \to V_1^l$ for some $l > 0$. Furthermore, it is possible to take $l \leq \operatorname{lng}(A)$. Now $U^{\otimes(n-1)} \otimes A \cong A^{(\dim U)^{n-1}}$ embeds in $V_n^l \cong U^{\otimes(n-1)} \otimes V_1^l$. Comparison of lengths completes the proof:

$$(\dim U)^{n-1} \operatorname{lng}(A) = \operatorname{lng}(U^{\otimes(n-1)} \otimes A) \leq \operatorname{lng}(V_n^l) = \operatorname{lng}(V_n)\, l \leq \operatorname{lng}(V_n) \operatorname{lng}(A).$$

□

Lemma 2.5. *Suppose that A is H-simple. Given any $0 \neq V \in \mathcal{M}_A$, there exists $U' \in \mathcal{M}^H$ such that $\operatorname{Hom}_A(U' \otimes V, A) \neq 0$.*

Proof. Since the Jacobson radical J of A is nilpotent, we have $VJ \neq V$. Since V/VJ is a semisimple module, V has a simple factor module. Thus it suffices to prove the lemma assuming that V is simple. Take a finite dimensional right coideal U of H large enough, so that the conclusions of both Proposition 1.10 and Lemma 2.3 are true. Let $V \cong A/R$ for some maximal right ideal R of A. Put

$$V_n = U^{\otimes n} \otimes V, \qquad F_n = U^{\otimes n} \otimes A, \qquad R_n = U^{\otimes n} \otimes R.$$

We may identify the right A-module R_n with a submodule of F_n. Then $V_n \cong F_n/R_n$. By Lemma 1.2 F_n is free. We are going to show that

$$\lim_{n\to\infty}\big(r(F_n) - r(R_n)\big) = +\infty. \tag{$*$}$$

Since $r(F_n) = \operatorname{lng}(F_n)/\operatorname{lng}(A)$, while $\mu(R_n) \le \operatorname{lng}(R_n)/\operatorname{lng}(A)$ by Lemma 1.5, we have

$$r(F_n) - \mu(R_n) \ge \frac{\operatorname{lng}(F_n) - \operatorname{lng}(R_n)}{\operatorname{lng}(A)} = \frac{\operatorname{lng}(V_n)}{\operatorname{lng}(A)}.$$

Lemma 2.4 and Proposition 1.10 yield

$$\begin{aligned} r(F_n) - r(R_n) &= r(F_n) - \mu(R_n) - r(R_n) + \mu(R_n) \\ &\ge \frac{(\dim U)^{n-1}}{\operatorname{lng}(A)} - (\dim U - c)^n\, r(R). \end{aligned}$$

Since $\dim U - c < \dim U$, the claim $(*)$ is now clear.

Thus $r(F_n) - r(R_n) \ge 1$ for sufficiently large n. Fix such an n, and put $r = r(F_n)$. Then $F_n \cong A^r$, while there exists an epimorphism $\varphi : A^{r-1} \to R_n$ in $\mathcal{M}_A$. Let W be any simple right ideal of A. By Schur's Lemma $D = \operatorname{End}_A W$ is a skew field. Since W is a simple module over a simple artinian factor ring of A, we have $\dim_D W < \infty$. Applying the functor $\operatorname{Hom}_A(?, W)$ to the inclusion $R_n \to F_n$ and to φ, we get linear maps of finite dimensional D-vector spaces

$$W^r \cong \operatorname{Hom}_A(F_n, W) \to \operatorname{Hom}_A(R_n, W) \hookrightarrow \operatorname{Hom}_A(A^{r-1}, W) \cong W^{r-1}.$$

Since $\dim_D W^{r-1} < \dim_D W^r$, the first of the two maps above has a nonzero kernel. As W is a submodule of A, this shows that there exists a nonzero A-linear map $F_n \to A$ vanishing on R_n. Thus $\operatorname{Hom}_A(V_n, A) \ne 0$, and we may take $U' = U^{\otimes n}$. □

Proof of Theorem 0.1. In view of Lemma 2.1 it suffices to consider the case where A is H-simple. There exists a nonzero $\mathcal{M}_A$-injective object $M \in {}_H\mathcal{M}_A$ (indeed, by [22, Lemma 1.1] we may take $M = \operatorname{Hom}_k(H, E)$ where E is any nonzero injective in $\mathcal{M}_A$). By Lemma 2.5 $\operatorname{Hom}_A(U' \otimes M, A) \ne 0$ for a suitable $U' \in \mathcal{M}^H$. Now Corollary 1.8 applied to $V = U' \otimes M$ shows that $U \otimes M$ is a generator in $\mathcal{M}_A$ with another $U \in \mathcal{M}^H$. In view of Lemma 1.1 the A-module $U \otimes M$ is a direct sum of copies of M, whence M itself is a generator in $\mathcal{M}_A$. Then A embeds in M^p for some integer $p > 0$ as an $\mathcal{M}_A$-direct summand. The injectivity of M^p in $\mathcal{M}_A$ forces A to be right selfinjective. By the Eilenberg-Nakayama Theorem [8, Th. 18] A is quasi-Frobenius. □

Remark. If we consider algebras over an arbitrary commutative base ring k and A is H-simple, then λA coincides with either A or 0 for each $\lambda \in k$. It follows that the kernel $\mathfrak{p}$ of the canonical map $k \to A$ is a prime ideal of k and A may be viewed as an algebra over K, the field of fractions of the domain $k/\mathfrak{p}$. Now $K \otimes H$ is a Hopf algebra over K and A is a left $K \otimes H$-module algebra. Thus we are able to make a reduction to the case where the base ring is a field. However, the assumption about the family of subcoalgebras $\mathcal{F}$ is needed in Lemma 2.1.

3. Module and comodule structures

Let A be an arbitrary left H-module algebra. The proof of Theorem 0.2 will utilize additional structures available on the A-modules $U \otimes V$. Denote by $\mathcal{M}_A^H$ the category of right (H, A)-Hopf modules where we regard A as an H-comodule algebra with respect to the trivial coaction of H (see [6], [25]). Thus the objects of $\mathcal{M}_A^H$ carry a right A-module structure and a right H-comodule structure such that the action of A commutes with the coaction of H. In a similar category ${}^H\mathcal{M}_A$ the objects are equipped with a left H-comodule structure instead of right one. Given any coalgebra C, we will view $\mathrm{Hom}(C, A)$ as an algebra with respect to the convolution multiplication (see [16], [24]). The formula

$$x\xi = \sum_{(x)} x_{(0)}\xi(Sx_{(1)}), \qquad x \in N,\ \xi \in \mathrm{Hom}(H, A).$$

defines a right $\mathrm{Hom}(H, A)$-module structure on any object $N \in \mathcal{M}_A^H$. Similarly, any object $N \in {}^H\mathcal{M}_A$ is a right $\mathrm{Hom}(H, A)$-module with respect to the action

$$x\xi = \sum_{(x)} x_{(0)}\xi(x_{(-1)}).$$

Given $U \in {}^H\mathcal{M}$, let $U_S \in \mathcal{M}^H$ denote U with the right comodule structure defined by the rule $u \mapsto \sum_{(u)} u_{(0)} \otimes Su_{(-1)}$. If $N \in {}^H\mathcal{M}_A$, then retaining the same A-module structure and transforming the H-comodule structure, we get an object $N_S \in \mathcal{M}_A^H$. There is a commutative diagram

$$\begin{tikzcd}
\mathcal{M}_A^H \arrow[r] & \mathcal{M}_{\mathrm{Hom}(H,A)} \arrow[r, "\text{pullback by } \tau"] & \mathcal{M}_A \\
{}^H\mathcal{M}_A \arrow[r] \arrow[u, "?_S"] & \mathcal{M}_{\mathrm{Hom}(H,A)} \arrow[u, "\text{pullback by } \mathrm{Hom}(S^2, \mathrm{id})"'] &
\end{tikzcd}$$

where the functor represented by the second vertical arrow transforms the action of $\mathrm{Hom}(H, A)$ by composing one with the endomorphism of that algebra induced by the bialgebra endomorphism $S^2 : H \to H$. The action of A on $\mathrm{Hom}(H, A)$-modules comes from the algebra homomorphism

$$\tau : A \to \mathrm{Hom}(H, A)$$

such that $\tau(a)(h) = ha$ for $a \in A$ and $h \in H$. The composites $\mathcal{T} : \mathcal{M}_A^H \to \mathcal{M}_A$ and $\mathcal{T}' : {}^H\mathcal{M}_A \to \mathcal{M}_A$ of functors shown in the diagram give *twisted action* of A on objects of $\mathcal{M}_A^H$ and ${}^H\mathcal{M}_A$, while the forgetful functors recover the original action which we call *plain action.*

Lemma 3.1. *Any* $\mathrm{Hom}(H, A)$*-submodule* Z *of an object* $N \in {}^H\mathcal{M}_A$ *is in fact an* ${}^H\mathcal{M}_A$*-subobject.*

Proof. Define an algebra homomorphism $\delta : A \to \mathrm{Hom}(H, A)$ by $\delta(a)(h) = \varepsilon(h)a$ for $a \in A$ and $h \in H$ (recall that ε is the counit of H). The plain action of A on N coincides with that induced by δ. Hence Z is stable under this action. We may also identify the dual algebra H^* with a subalgebra of $\mathrm{Hom}(H,A)$. The H^*-module structure on N corresponds to the H-comodule structure, so that N is a rational H^*-module in the terminology of [24, Section 2.1]. As Z is stable under the action of H^*, it is stable under the coaction of H. $\square$

Given $U\in\mathcal{M}^H$ and $N\in\mathcal{M}_A^H$, we will consider $U\otimes N$ as an object of $\mathcal{M}_A^H$ with respect to the tensor product of comodule structures and the plain action of A on the second tensorand. This makes $\mathcal{M}_A^H$ into a left module category over $\mathcal{M}^H$. Furthermore, $\mathcal{T}(U\otimes N)=U\otimes\mathcal{T}(N)$ in $\mathcal{M}_A$. We may view any $V\in\mathcal{M}_A$ as an object of $\mathcal{M}_A^H$ with respect to the trivial coaction of H. Hence $U\otimes V$ becomes an object of $\mathcal{M}_A^H$ with the comodule and module structures

$$u\otimes v\mapsto\sum_{(u)}(u_{(0)}\otimes v)\otimes u_{(1)},\qquad (u\otimes v)a=u\otimes va.$$

The twisted action of A on $U\otimes V$ coincides with that introduced in section 1.

In a similar way ${}^H\mathcal{M}_A$ is a right module category over the tensor category ${}^H\mathcal{M}$, and we may view $V\otimes U$ as an object of ${}^H\mathcal{M}_A$ for $U\in{}^H\mathcal{M}$ and $V\in\mathcal{M}_A$. For any $N\in{}^H\mathcal{M}_A$ the canonical k-linear bijection $(N\otimes U)_S\to U_S\otimes N_S,\ x\otimes u\mapsto u\otimes x$, is an isomorphism in $\mathcal{M}_A^H$. In particular,

$$(V\otimes U)_S\cong U_S\otimes V\quad\text{in }\mathcal{M}_A^H.$$

In the sequel we assume tacitly that each object N of either $\mathcal{M}_A^H$ or ${}^H\mathcal{M}_A$ is regarded as a right A-module with respect to the twisted action of A. Let N_{triv} denote N with the trivial right coaction of H and the plain action of A. Thus we have $N_{\mathrm{triv}}\in\mathcal{M}_A^H$. The twisted action of A on N_{triv} coincides with the plain one. In the next lemma H is regarded as a right H-comodule with respect to Δ.

Lemma 3.2. *For any $N\in{}^H\mathcal{M}_A$ there is an isomorphism $\Phi:H\otimes N_S\to H\otimes N_{\mathrm{triv}}$ in $\mathcal{M}_A^H$. If N_{triv} is a semisimple A-module, then $H\otimes Z_S$ is an $\mathcal{M}_A^H$-direct summand of $H\otimes N_S$ for any subobject $Z\subset N$.*

Proof. This is an obvious generalization of the fact that $H\otimes U_S\cong H\otimes U_{\mathrm{triv}}$ in $\mathcal{M}^H$ for any $U\in{}^H\mathcal{M}$. Define Φ and its inverse by the rules

$$\Phi(h\otimes x)=\sum_{(x)}hS(x_{(-1)})\otimes x_{(0)},\qquad \Phi^{-1}(h\otimes x)=\sum_{(x)}hx_{(-1)}\otimes x_{(0)}$$

where $h\in H,\ x\in N$. It is straightforward to check that Φ commutes with the plain action of A and with the coaction of H. If Z is a subobject of N, then Φ maps $H\otimes Z_S$ onto $H\otimes Z_{\mathrm{triv}}$. Semisimplicity of N_{triv} ensures that Z_{triv} is a direct summand of N_{triv}, whence $H\otimes Z_{\mathrm{triv}}$ is a direct summand of $H\otimes N_{\mathrm{triv}}$. $\square$

Lemma 3.3. *Let $B=\mathrm{Hom}(C,A)$ where $C\in\mathcal{F}$, and let $\varphi:A\to B$ be the algebra homomorphism defined by the rule $\varphi(a)(c)=S^2(c)a$ for $a\in A$ and $c\in C$. Identifying C^* with a subalgebra of B, we have $B=C^*\varphi(A)$. In particular, B is right module-finite over $\varphi(A)$.*

Proof. There is an algebra isomorphism $B\cong C^*\otimes A$ such that $\xi\otimes a$ with $\xi\in C^*$ and $a\in A$ corresponds to $\xi*\delta(a)$ where $*$ stands for the multiplication in B and

$\delta : A \to B$ is defined by the rule $\delta(a)(c) = \varepsilon(c)a$ for $c \in C$. Thus it suffices to check that $\delta(A) \subset C^*\varphi(A)$. Having fixed $a \in A$, pick any basis $a_1, \ldots, a_n$ for the vector subspace $S(C)a \subset A$. There are $\eta_1, \ldots, \eta_n \in C^*$ such that $S(c)a = \sum_{i=1}^n \eta_i(c)a_i$ for all $c \in C$. Then

$$\varepsilon(c)(a) = \sum_{(c)} S^2(c_{(2)})S(c_{(1)})a = \sum_{(c)} \sum_{i=1}^{n} \eta_i(c_{(1)})\, S^2(c_{(2)})a_i$$
$$= \sum_{i=1}^{n} \sum_{(c)} \eta_i(c_{(1)})\, \varphi(a_i)(c_{(2)}),$$

which shows that $\delta(a) = \sum_{i=1}^n \eta_i * \varphi(a_i) \in C^*\varphi(A)$, as required. □

Lemma 3.4. *Suppose that $V \in \mathcal{M}_A$ and $U \in {}^C\mathcal{M}$ where $C \in \mathcal{F}$. Let W be an A-submodule of $V \otimes U$, and let Z be the smallest ${}^H\mathcal{M}_A$-subobject of $V \otimes U$ such that $W \subset Z$. If A is right artinian, then $r(Z) \le \dim(C)r(W)$.*

Proof. Since $V \otimes U$, regarded as a comodule, belongs to the full subcategory ${}^C\mathcal{M}$ of ${}^H\mathcal{M}$, the action of $\mathrm{Hom}(H, A)$ on $V \otimes U$ factors through an action of the algebra $B = \mathrm{Hom}(C, A)$. It follows from Lemma 3.1 that $Z = WB$. Furthermore, A acts on $U \otimes V$ via the homomorphism φ defined in Lemma 3.3. Let $r(W) = n/l$ for some integers $n \ge 0$ and $l > 0$. By Lemma 1.3 the A-module W^l is n-generated, whence so too is the B-module Z^l. Put $d = \dim C$. By Lemma 3.3 B is generated by d elements as a $\varphi(A)$-module with respect to right multiplications. Hence there exists an epimorphism $A^{dn} \to Z^l$ in $\mathcal{M}_A$, i.e. $r(Z) \le dn/l$. □

4. Projectivity and semisimplicity

Lemma 2.1 reduces the proof of Theorem 0.2 to the case where A is H-simple. So we assume in this section that A is an H-simple right artinian left H-module algebra. Starting from Lemma 4.2 we also require that H grow slower than exponentially.

Lemma 4.1. *There exists $U \in {}^H\mathcal{M}$ such that $\dim U < \infty$ and $U_S \otimes V$ is a generator in $\mathcal{M}_A$ for each $0 \ne V \in \mathcal{M}_A$.*

Proof. Since A is quasi-Frobenius by Theorem 0.1, each simple right A-module embeds in A as a right ideal. It follows that $\mathrm{Hom}_A(V, A) \ne 0$ for each $0 \ne V \in \mathcal{M}_A$. By Lemma 2.2 A is $S(H)$-simple. Hence there exists a finite dimensional right coideal $T \subset S(H)$ which satisfies the conclusions of Corollary 1.8. Let U be any left coideal of H such that $\dim U < \infty$ and $T \subset S(U)$. Then $S(U)$ is a right coideal of H for which Corollary 1.8 remains true. In particular, $S(U) \otimes V$ is a generator in $\mathcal{M}_A$ for each $0 \ne V \in \mathcal{M}_A$. Since the map $S : U_S \to S(U)$ is an epimorphism in $\mathcal{M}^H$, the right A-module $S(U) \otimes V$ is an epimorphic image of $U_S \otimes V$, and the conclusion is clear. □

Lemma 4.2. *Given a simple $V \in \mathcal{M}_A$, there exists a finite dimensional $U' \in {}^H\mathcal{M}$ such that $V \otimes U'$ has an ${}^H\mathcal{M}_A$-subobject Z and an A-submodule W satisfying the following properties:*

(a) $W \subset Z$, (b) $Z \ne V \otimes U'$, (c) $(V \otimes U')/W$ *is $\mathcal{M}_A$-projective.*

Proof. First take U as in Lemma 4.1 and put $V_n = V \otimes U^{\otimes n} \in {}^H\mathcal{M}_A$. For any $N \in {}^H\mathcal{M}_A$ we have $N_S = N$ as right A-modules with respect to the twisted action of A. In particular, $V_n \cong (V \otimes U^{\otimes n})_S \cong (U_S)^{\otimes n} \otimes V$ in $\mathcal{M}_A$.

Let $V_n = G_n \oplus W_n$ be a decomposition in $\mathcal{M}_A$ where G_n is a projective A-module, while W_n has no nonzero projective direct summands. Consider the ${}^H\mathcal{M}_A$-subobject Z_n of V_n generated by W_n. We have $U \in {}^C\mathcal{M}$ for some $C \in \mathcal{F}$. Then $C^n \in \mathcal{F}$ as well, and $U^{\otimes n} \in {}^{C^n}\mathcal{M}$. By Lemma 3.4 and Proposition 1.10

$$\frac{r(Z_n)}{r(V_n)} \le \dim(C^n)\frac{r(W_n)}{r(V_n)} \le \dim(C^n)\,\frac{\dim U}{c}\left(1 - \frac{c}{\dim U}\right)^n.$$

It follows from the assumption on the growth of H and the inequalities above that there exists n such that $r(Z_n)/r(V_n) < 1$, and therefore $Z_n \ne V_n$. We may take $U' = U^{\otimes n}$, $Z = Z_n$, $W = W_n$ for such an n. □

Lemma 4.3. *$H \otimes V$ is projective in $\mathcal{M}_A$ for any $V \in \mathcal{M}_A$.*

Proof. Suppose first that V is simple. Let $N = V \otimes U'$ where U' is taken as in Lemma 4.2. Note that N_{triv} is a sum of copies of V, hence semisimple in $\mathcal{M}_A$. Pick Z and W satisfying the conclusion of Lemma 4.2. By Lemma 3.2 $H \otimes Z_S$ is a direct summand of $H \otimes N_S$ in $\mathcal{M}_A^H$, and therefore in $\mathcal{M}_A$ (with respect to the twisted action of A). Note that W is an A-submodule of Z_S since $N_S = N$ in $\mathcal{M}_A$. Hence $H \otimes W$ is an A-submodule of $H \otimes Z_S$, which implies that $H \otimes Z_S/W$ is an $\mathcal{M}_A$-direct summand of $H \otimes N_S/W$.

Any complementary summand is isomorphic with $H \otimes (N/Z)_S$, and therefore with $H \otimes (N/Z)_{\mathrm{triv}}$ by Lemma 3.2. The latter is a direct sum of copies of $H \otimes V$ since $(N/Z)_{\mathrm{triv}}$ is an isotypic semisimple A-module of type V. Furthermore, V occurs in $(N/Z)_{\mathrm{triv}}$ at least once since $Z \ne N$. It follows that $H \otimes V$ is an $\mathcal{M}_A$-direct summand of $H \otimes N_S/W$. The latter is projective in $\mathcal{M}_A$ by Lemma 1.2 since so is $N_S/W = N/W$. This entails the desired conclusion.

If now V is an arbitrary finitely generated module, then V has a composition series. In this case $H \otimes V$ has a finite chain of submodules all whose factors are projective A-modules by the first part of the proof. Projectivity of $H \otimes V$ is immediate.

Suppose that V is an arbitrary module. Then $H \otimes V = \varinjlim H \otimes V'$ where V' runs over the finitely generated submodules of V. Since each $H \otimes V'$ is flat in $\mathcal{M}_A$, so too is $H \otimes V$. Every right artinian ring is semiprimary, hence left and right perfect. In particular, so is A. It follows that all flat A-modules are projective by a well-known theorem of Bass [2]. □

Lemma 4.4. *Any object $M \in {}_H\mathcal{M}_A$ is an $\mathcal{M}_A$-direct summand of $H \otimes V$ for a suitable $V \in \mathcal{M}_A$. Hence M is projective in $\mathcal{M}_A$.*

Proof. It follows from Lemma 1.1 that $H \otimes M$, as an A-module, is a direct sum of copies of M. So we may take $V = M$ and apply Lemma 4.3. □

Lemma 4.4 completes the proof of Theorem 0.2 for objects of ${}_H\mathcal{M}_A$. Note that A^{op} is a left H^{cop}-module algebra and there is a category equivalence ${}_{A\#H}\mathcal{M} \approx {}_{H^{\mathrm{cop}}}\mathcal{M}_{A^{\mathrm{op}}}$. If $S : H \to H$ is bijective, then the bialgebra H^{cop} is a Hopf algebra. In this case ${}_A\mathcal{M}$-projectivity of left $A\#H$-modules follows from Lemma 4.4 as well. Without the bijectivity assumption we can complete the proof of Theorem 0.2 for objects of ${}_{A\#H}\mathcal{M}$ repeating the same steps as in case of ${}_H\mathcal{M}_A$. Below we describe necessary alterations briefly.

For $V \in {}_A\mathcal{M}$ and $U \in \mathcal{M}^H$ the twisted action of A on $V \otimes U$ is defined by

$$a(v \otimes u) = \sum_{(u)} (u_{(1)}a)v \otimes u_{(0)}, \qquad a \in A,\ v \in V,\ u \in U.$$

These module structures make $\otimes$ into a functor ${}_A\mathcal{M} \times \mathcal{M}^H \to {}_A\mathcal{M}$ with respect to which ${}_A\mathcal{M}$ is a right module category over the tensor category $\mathcal{M}^H$.

Lemma 4.5. *If $M \in {}_{A\#H}\mathcal{M}$ then the map $M \otimes H \to M$, $v \otimes h \mapsto S(h)v$, is A-linear and $M \otimes U \cong M \otimes U_{\mathrm{triv}}$ in ${}_A\mathcal{M}$ for any $U \in \mathcal{M}^H$.*

Proof. The k-linear transformation Φ of $M \otimes U$ such that $\Phi(v \otimes u) = \sum_{(u)} u_{(1)}v \otimes u_{(0)}$ has the inverse given by $v \otimes u \mapsto \sum_{(u)} S(u_{(1)})v \otimes u_{(0)}$. Moreover, Φ is an isomorphism of left A-modules $M \otimes U_{\mathrm{triv}} \to M \otimes U$ since

$$\Phi(av \otimes u) = \sum_{(u)} (u_{(1)}a)(u_{(2)}v) \otimes u_{(0)} = \sum_{(u)} a \cdot (u_{(1)}v \otimes u_{(0)}) = a \cdot \Phi(v \otimes u)$$

for all $v \in M$, $u \in U$ and $a \in A$. Now take $U = H$. The counit $\varepsilon : H \to k$ induces an A-linear map $M \otimes H_{\mathrm{triv}} \to M$. The composite of the latter with Φ^{-1} is an A-linear map $M \otimes H \to M$ which sends $v \otimes h$ to $\sum_{(h)} \varepsilon(h_{(1)})S(h_{(2)})v = S(h)v$. $\square$

In particular, Lemma 4.5 applies to $M = A$. Hence an analog of Lemma 1.2: *If V is either free or projective in ${}_A\mathcal{M}$, then so too is $V \otimes U$.* We have proved already that A is quasi-Frobenius. In particular, A is left artinian. So the quantities $r(V)$ and $\mu(V)$ introduced in section 1 make sense for finitely generated left A-modules. Analogs of Lemmas 1.3–1.6 are straightforward. By Lemma 2.2 A is $S(H)$-simple. This entails a strengthened version of Lemma 1.7: *there exists $C \in \mathcal{F}$ such that $P_{S(C)} = 0$ for all $P \in \operatorname{Max} A$.* Given a right coideal U of H and a morphism $\varphi : V \to A$ in ${}_A\mathcal{M}$ the composite

$$V \otimes U \xrightarrow{\varphi \otimes \mathrm{id}} A \otimes H \xrightarrow{\text{the map from Lemma 4.5}} A$$

is a morphism in ${}_A\mathcal{M}$. In particular, its image $S(U) \cdot \varphi(V)$ is a left ideal of A. Proceeding as in Corollary 1.8 we conclude that H has a right coideal U of finite dimension such that

(i) $\big(S(U)I\big)A = A$ *for each nonzero left ideal I of A,*

(ii) *$V \otimes U$ is a generator in ${}_A\mathcal{M}$ for each $V \in {}_A\mathcal{M}$ satisfying* $\operatorname{Hom}_A(V, A) \neq 0$.

Now all ingredients are available to obtain an analog of Proposition 1.10 for $V \in {}_A\mathcal{M}$ and $V_n = V \otimes U^{\otimes n}$.

We may regard $V \otimes U$ with $V \in {}_A\mathcal{M}$ and $U \in \mathcal{M}^H$ as an object of ${}_A\mathcal{M}^H$. There are left $\operatorname{Hom}(H, A)$-module structures on objects of ${}_A\mathcal{M}^H$, and an analog of Lemma 3.1 holds for ${}_A\mathcal{M}^H$. Both ${}_A\mathcal{M}^H$ and ${}_A\mathcal{M}$ are right module categories over $\mathcal{M}^H$. The composite functor

$$\mathcal{T} : {}_A\mathcal{M}^H \longrightarrow {}_{\operatorname{Hom}(H,A)}\mathcal{M} \xrightarrow{\text{pullback by } \tau} {}_A\mathcal{M}.$$

intertwines the functors $? \otimes U$ on ${}_A\mathcal{M}^H$ and ${}_A\mathcal{M}$. Given $N \in {}_A\mathcal{M}^H$, let $N_{\mathrm{triv}} \in {}_A\mathcal{M}^H$ denote N with the same action of A and the trivial coaction of H.

Lemma 4.6. *For any $N \in {}_A\mathcal{M}^H$ there is an isomorphism $\Phi : N \otimes H \to N_{\mathrm{triv}} \otimes H$ in ${}_A\mathcal{M}^H$. If N_{triv} is a semisimple A-module, then $Z \otimes H$ is an ${}_A\mathcal{M}^H$-direct summand of $N \otimes H$ for any subobject $Z \subset N$.*

Proof. The map Φ and its inverse are defined by

$$\Phi(x \otimes h) = \sum_{(x)} x_{(0)} \otimes x_{(1)}h, \qquad \Phi^{-1}(x \otimes h) = \sum_{(x)} x_{(0)} \otimes S(x_{(1)})h.$$

□

Compared with Lemma 3.2 here we do not need to transform the comodule structure by S. If $B = \mathrm{Hom}(C, A)$ where $C \in \mathcal{F}$ and $\tau : A \to B$ is defined by the formula $\tau(a)(c) = ca$ for $a \in A$ and $c \in C$, then $B = \tau(A)C^*$ according to [22, Lemma 2.1]. This entails ${}_A\mathcal{M}^H$-versions of Lemmas 3.4 and 4.2. Proceeding as in Lemma 4.3, we conclude that *$V \otimes H$ is projective in ${}_A\mathcal{M}$ for any $V \in {}_A\mathcal{M}$*. The isomorphism $M \otimes H \cong M \otimes H_{\mathrm{triv}}$ of Lemma 4.5 shows that *each $M \in {}_{A\#H}\mathcal{M}$ is an ${}_A\mathcal{M}$-direct summand of a suitable $V \otimes H$*. Thus the conclusion of Theorem 0.2 is established also for objects of ${}_{A\#H}\mathcal{M}$.

Lemma 4.7. *If H is cosemisimple then A is semisimple.*

Proof. The cosemisimplicity means that $H = k \oplus C$ for some subcoalgebra C of H. Hence $H \otimes V \cong (k \otimes V) \oplus (C \otimes V)$ in $\mathcal{M}_A$ for any $V \in \mathcal{M}_A$, so that $V \cong k \otimes V$ is a direct summand of the A-module $H \otimes V$. Lemma 4.3 implies that V is projective. Thus all right A-modules are projective, yielding the conclusion. □

Proof of Theorem 0.3. Let H be a cosemisimple Hopf algebra growing slower than exponentially and A a right artinian left H-module algebra. It remains only to make a reduction to the case of H-simple module algebras where Lemma 4.7 applies. For each $P \in \operatorname{Max} A$ the right artinian H-module algebra A/P_H has a maximal ideal P/P_H containing no nonzero H-stable ideals of A/P_H. By Lemma 2.1 A/P_H is H-simple, hence semisimple by Lemma 4.7. This shows that

$$J(A) \subset \bigcap_{P \in \operatorname{Max} A} P_H \subset \bigcap_{P \in \operatorname{Max} A} P = J(A).$$

Thus we must have equalities everywhere above. In particular, $J(A)$ is an intersection of H-stable ideals. □

5. A strengthened version of Donkin's Theorem

Freeness of certain projective modules can be derived from the following fact:

Lemma 5.1. *Suppose that R is a right perfect ring. Let F be a free right R-module of infinite rank. If G is a direct summand of F such that F is isomorphic to a direct summand of G^n for some integer $n > 0$, then $G \cong F$. In particular, $G \cong F$ whenever $G^n \cong F$.*

Proof. Each projective right R-module is a direct sum of indecomposable projectives; the multiplicities with which indecomposable projectives occur are independent of a choice of such a decomposition. The hypothesis of the lemma implies that each indecomposable projective occurs with the same multiplicity in F and in G. □

Remark. Lemma 5.1 holds, more generally, when R is any semilocal ring. Indeed, under the same assumptions about F and G we have $G/GJ \cong F/FJ$ where J stands for the Jacobson radical of R. Hence $G \cong F$ by [3] or [18].

Theorem 5.2. *Suppose that H is a Hopf algebra growing slower than exponentially and A is an H-simple right artinian left H-module algebra. Put*

$$l = \gcd\{\operatorname{lng}(A/P) \mid P \in \operatorname{Max} A\}$$

Then $H \otimes V^l$ is a free right A-module for any $V \in \mathcal{M}_A$. Moreover, $H \otimes V$ is free when either H is infinite dimensional or V is not finitely generated. A similar conclusion holds for the left A-modules $W^l \otimes H$ and $W \otimes H$ where $W \in {}_A\mathcal{M}$.

Proof. Denote $G = H \otimes V$. By Lemma 4.3 G is projective in $\mathcal{M}_A$. Further arguments depend on whether H is finite dimensional or not.

Case 1. Assume that $\dim H = \infty$. Put $F = H \otimes A$. By Lemma 1.1 $F \cong H_{\text{triv}} \otimes A$ in $\mathcal{M}_A$. Hence F is a free A-module of infinite rank. Suppose first that V is a cyclic right A-module. Then G is an epimorphic image of F, so that it has to be a direct summand of F. Let U be as in Lemma 4.1. Since $U_S \otimes V$ is a generator in $\mathcal{M}_A$, there exists an epimorphism $U_S \otimes V^p \to A$ for a suitable integer $p > 0$. Tensoring with H, we get an epimorphism $H \otimes U_S \otimes V^p \to F$. Lemma 3.2 shows that

$$H \otimes U_S \otimes V^p \cong H \otimes U_{\text{triv}} \otimes V^p$$

is a direct sum of dp copies of G where $d = \dim U$. Hence F is a direct summand of G^{dp}, and Lemma 5.1 applies.

Next we obtain the conclusion for finitely generated right A-modules. Any short exact sequence $0 \to V' \to V \to V'' \to 0$ in $\mathcal{M}_A$ gives rise to an exact sequence $0 \to H \otimes V' \to H \otimes V \to H \otimes V'' \to 0$; when $H \otimes V'$ and $H \otimes V''$ are both free, so too is $H \otimes V$. This enables us to proceed by induction on the number of generators.

Finally, if V is an arbitrary right A-module, we apply Zorn's Lemma to the ordered set of all pairs (X, B) where X is a submodule of V such that $H \otimes X$ is a free A-module, and B is a basis for the latter. This argument is similar to one from [19, Prop. 1].

Case 2. Assume that $d = \dim H < \infty$. In this case Proposition 1.10 can be applied with $U = H$. Since A is quasi-Frobenius, A is an injective cogenerator in $\mathcal{M}_A$. Hence any right A-module V is torsionless. If V is finitely generated, we obtain

$$\lim_{n\to\infty} \frac{r(V_n) - \mu(V_n)}{d^n} = 0$$

where $V_n = H^{\otimes n} \otimes V$. However, $U \otimes H \cong U_{\text{triv}} \otimes H$ in $\mathcal{M}^H$ for any $U \in \mathcal{M}^H$. Hence $H^{\otimes n}$ is isomorphic in $\mathcal{M}^H$ to a direct sum of d^{n-1} copies of H, and therefore $V_n \cong G^{d^{n-1}}$ in $\mathcal{M}_A$. It follows that $r(V_n) = d^{n-1} r(G)$, $\mu(V_n) = d^{n-1}\mu(G)$, and

$$\frac{r(V_n) - \mu(V_n)}{d^n} = \frac{r(G) - \mu(G)}{d} \quad \text{for all } n > 0.$$

We conclude that $r(G) = \mu(G)$. In other words, the maximum value of $r_P(G)$ over all $P \in \operatorname{Max} A$ equals the minimum value, i.e. $r_P(G) = r(G)$ for all P. It follows that $r(G)\lg(A/P) \in \mathbb{Z}$ for all P, whence $r(G)l \in \mathbb{Z}$. Put $m = r(G)l$. The equalities $r_P(G^l) = m = r_P(A^m)$ show that each indecomposable projective right A-module occurs in G^l and in A^m with equal multiplicities. This entails $G^l \cong A^m$.

If V is not finitely generated, we apply Zorn's Lemma to the set of pairs (X, B) where X is a submodule of V such that $H \otimes X^l$ is a free A-module, and B is a basis for the latter. Thus G^l is free. The functor $H \otimes ?$ embeds the submodule lattice of V into that of G. Hence G is a module of infinite length, and so too is G^l. By Lemma 5.1 $G \cong G^l$. □

Lemma 5.3. *Any finite dimensional H-simple left H-module algebra A is Frobenius.*

Proof. There is an ideal K of finite codimension in H which annihilates A. Therefore A has a right comodule structure over the coalgebra $(H/K)^*$ dual to the finite dimensional algebra H/K. We may regard A as a right H°-comodule algebra where H° stands for the finite dual of H (see [16]). The H-simplicity of A means that A has no nontrivial ideals stable under the coaction of H°. Now the desired conclusion follows from [20, Th. 4.2]. □

Theorem 5.4. *Suppose that H is a Hopf algebra growing slower than exponentially and C is a finite dimensional right H-module factor coalgebra of H. Then H, as an object of $\mathcal{M}^C$, is isomorphic to a direct sum of copies of C. A similar conclusion holds in ${}^C\mathcal{M}$ provided the antipode $S : H \to H$ is bijective.*

Proof. We have $C = H/I$ where I is a coideal and a right ideal of H. The dual $A = C^*$ is a left H-module algebra with respect to the action $\rightharpoonup$ defined by

$$(h \rightharpoonup \xi)(c) = \xi(ch), \qquad h \in H,\ \xi \in A,\ c \in C.$$

We may identify A with the subalgebra of H^* consisting of linear functions vanishing on I. Since 1 is a grouplike element of H, the assignment $\xi \mapsto \xi(1)$ defines an algebra homomorphism $A \to k$. Hence

$$P = \{\xi \in A \mid \xi(1) = 0\}$$

is a maximal ideal of A such that $A/P \cong k$. If $\xi \in P_H$, then $h \rightharpoonup \xi \in P$, and therefore $\xi(h) = (h \rightharpoonup \xi)(1) = 0$, for all $h \in H$. Thus $P_H = 0$. Since $\dim A = \dim C < \infty$, the algebra A is right artinian. By Lemma 2.1 A is H-simple.

Now Theorem 5.2 applies with $l = 1$. Let $W = k$ denote the simple left A-module with annihilator P. Recall that A acts on $W \otimes H$ by the rule

$$\xi \cdot (w \otimes h) = \sum_{(h)} (h_{(2)} \rightharpoonup \xi) \cdot w \otimes h_{(1)} = \sum_{(h)} \xi(h_{(2)}) w \otimes h_{(1)}$$

where $w \in W$, $h \in H$, $\xi \in A$. Under the canonical k-linear bijection $W \otimes H \cong H$, the corresponding action of A on H is given by $\xi \rightharpoonup h = \sum_{(h)} \xi(h_{(2)})h_{(1)}$. The latter makes H into a rational left A-module with the corresponding right C-comodule structure $H \to H \otimes C$ given by $h \mapsto \sum_{(h)} h_{(1)} \otimes \pi(h_{(2)})$ where $\pi : H \to C$ stands for the projection. By Theorem 5.2 H is free in ${}_A\mathcal{M}$. Next, $C \cong A^*$ as a rational left A-module. Since A is Frobenius by Lemma 5.3, we have $A^* \cong A$ in ${}_A\mathcal{M}$. Hence H is ${}_A\mathcal{M}$-isomorphic to a direct sum of copies of C. It remains to observe that any isomorphism between two rational A-modules is an isomorphism between the corresponding C-comodules.

Let $V = k$ be the simple right A-module with annihilator P. Then $H \otimes V \cong H$; the action of A on H corresponding to the twisted action on $H \otimes V$ is

$$h \leftharpoondown \xi = \sum_{(h)} \xi(Sh_{(2)})h_{(1)}, \qquad h \in H,\ \xi \in A.$$

This action makes H into a rational right A-module with the corresponding comodule structure $H \to C \otimes H$ given by $h \mapsto \sum_{(h)} \pi(Sh_{(2)}) \otimes h_{(1)}$. Let ${}_SH$ denote H with the latter left C-comodule structure. We will regard H as a left C-comodule with respect to $h \mapsto \sum_{(h)} \pi(h_{(1)}) \otimes h_{(2)}$. Since there are isomorphisms $C \cong A^* \cong A$ also in $\mathcal{M}_A$, we deduce from Theorem 5.2 that ${}_SH$ is isomorphic to a direct sum of copies of C. The map S is a morphism ${}_SH \to H$ in ${}^C\mathcal{M}$. Hence ${}_SH \cong H$ in ${}^C\mathcal{M}$ whenever S is bijective, which yields the second conclusion. □

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