ISSN 0037–4466, Siberian Mathematical Journal, 2024, Vol. 65, No. 3, pp. 522–533. © Pleiades Publishing, Ltd., 2024.
Russian Text © The Author(s), 2024, published in Sibirskii Matematicheskii Zhurnal, 2024, Vol. 65, No. 3, pp. 455–468.

THE TRACE AND INTEGRABLE COMMUTATORS OF THE MEASURABLE OPERATORS AFFILIATED TO A SEMIFINITE VON NEUMANN ALGEBRA

A. M. Bikchentaev

UDC 517.983:517.986

Abstract—Assume that τ is a faithful normal semifinite trace on a von Neumann algebra $\mathscr{M}$, I is the unit of $\mathscr{M}$, $S(\mathscr{M},\tau)$ is the $*$-algebra of τ-measurable operators, and $L_1(\mathscr{M},\tau)$ is the Banach space of τ-integrable operators. We present a new proof of the following generalization of Putnam's theorem (1951): No positive self-commutator $[A^*,A]$ with $A \in S(\mathscr{M},\tau)$ is invertible in $\mathscr{M}$. If τ is infinite then no positive self-commutator $[A^*,A]$ with $A \in S(\mathscr{M},\tau)$ can be of the form $\lambda I + K$, where λ is a nonzero complex number and K is a τ-compact operator. Given $A, B \in S(\mathscr{M},\tau)$ with $[A,B] \in L_1(\mathscr{M},\tau)$ we seek for the conditions that $\tau([A,B]) = 0$. If $X \in S(\mathscr{M},\tau)$ and $Y = Y^3 \in \mathscr{M}$ with $[X,Y] \in L_1(\mathscr{M},\tau)$ then $\tau([X,Y]) = 0$. If $A^2 = A \in S(\mathscr{M},\tau)$ and $[A^*,A] \in L_1(\mathscr{M},\tau)$ then $\tau([A^*,A]) = 0$. If a partial isometry U lies in $\mathscr{M}$ and $U^n = 0$ for some integer $n \geq 2$ then U^{n-1} is a commutator and $U^{n-1} \in L_1(\mathscr{M},\tau)$ implies that $\tau(U^{n-1}) = 0$.

DOI: 10.1134/S0037446624030030

Keywords: Hilbert space, von Neumann algebra, normal trace, measurable operator, commutator, self-commutator, idempotent

1. Introduction

Given a complex Hilbert space $\mathscr{H}$, denote the identity operator on $\mathscr{H}$ by I and the Schatten–von Neumann ideals in $\mathscr{B}(\mathscr{H})$ by $\mathfrak{S}_p$, where $0 < p < +\infty$. An operator $X \in \mathscr{B}(\mathscr{H})$ is a *commutator* whenever $X = [A,B] = AB - BA$ for some $A, B \in \mathscr{B}(\mathscr{H})$. If $\dim \mathscr{H} < +\infty$ then the following conditions on $X \in \mathscr{B}(\mathscr{H})$ are equivalent:

(i) X has the zero diagonal in some basis for $\mathscr{H}$;

(ii) $\mathrm{tr}(X) = 0$;

(iii) X is a commutator.

The proof of (i)⇔(ii) is in [1, Chapter I, Problem 209] and the proof of (ii)⇔(iii) is in [2] or [3, Problem 230]. See [4, 5] for the further studies regarding (ii)⇔(iii). See [6, 7] for interesting applications of zero-trace matrices. As [8] shows, considering (ii)⇔(iii), we can choose nilpotent matrices A and B in $X = [A,B]$.

If $\mathscr{H}$ is a separable space with $\dim \mathscr{H} = +\infty$ then $X \in \mathscr{B}(\mathscr{H})$ is a commutator if and only if X cannot be expressed as $\lambda I + K$, where $\lambda \in \mathbb{C} \setminus \{0\}$ and $K \in \mathscr{B}(\mathscr{H})$ is a compact operator [9, Theorem 3; 3, Corollary to Problem 230]; in the nonseparable case the picture is similar, but instead of compact operators we have to use a maximal ideal $\mathbb{J}$ of $\mathscr{B}(\mathscr{H})$; see [9] and [10]. Thus, every compact operator $K \in \mathscr{B}(\mathscr{H})$ is of the form $K = [A,B]$ for some $A, B \in \mathscr{B}(\mathscr{H})$. Choosing K in the class $\mathfrak{S}_1$ of trace-class operators, we obtain $[A,B] \in \mathfrak{S}_1$ with $\mathrm{tr}([A,B]) \neq 0$. Curiously, for $X \in \mathfrak{S}_1$ we still have (i)⇔(ii); see [11, Corollary 1]. It is shown in [12] that a compact hermitian operator $X \in \mathscr{B}(\mathscr{H})$ is the self-commutator $[A^*,A]$ of a compact operator $A \in \mathscr{B}(\mathscr{H})$ if and only if (i) holds. For hermitian operators $X \in \mathscr{B}(\mathscr{H})$ Theorem 2 of [13] establishes that (i) ⇔ $\mathrm{tr}(X_+) = \mathrm{tr}(X_-)$, where $X_+ = (|X| + X)/2$ and $X_- = |X| - X_+$. In [14] the results of [13] are applied for $X \in \mathscr{B}(\mathscr{H})$ to show that

$$(i) \Leftrightarrow \mathrm{tr}(\mathrm{Re}(e^{i\theta}X)_+) = \mathrm{tr}(\mathrm{Im}(e^{i\theta}X)_-) \text{ for all } \theta,\ 0 \leq \theta < 2\pi.$$

†) Dedicated to Anatolii Nikolaevich Sherstnev (27.01.1938–25.05.2023).

Original article submitted November 21, 2023; revised November 21, 2023; accepted January 25, 2024.

Consider a von Neumann algebra $\mathscr{M}$ of operators on a Hilbert space $\mathscr{H}$ and a faithful normal semifinite trace τ on $\mathscr{M}$. Denote the $*$-algebra of all τ-measurable operators by $S(\mathscr{M},\tau)$ and the Banach space of all τ-integrable operators by $L_1(\mathscr{M},\tau)$. Assume that $\tau(I)=+\infty$ and take $A,B\in S(\mathscr{M},\tau)$. If $B=B^3$ or $B=C+X$, where $C=C^2\in S(\mathscr{M},\tau)$ and X is τ-compact then $[A,B]$ cannot be of the form $\lambda I+K$, where $\lambda\in\mathbb{C}\setminus\{0\}$ and K is τ-compact; see [15, Theorem 3] or [16, Theorem 3] and [17, Proposition 4] respectively. The commutators of τ-measurable operators are the values of inner *derivations* of $S(\mathscr{M},\tau)$ (see [18–20]).

Given $A,B\in S(\mathscr{M},\tau)$ and $[A,B]\in L_1(\mathscr{M},\tau)$, we look for the conditions that $\tau([A,B])=0$. If $\tau(I)<+\infty$ then $\tau([A,B])=0\Leftrightarrow \|I+z[A,B]\|_1\geq\tau(I)$ for all $z\in\mathbb{C}$; see [21, Theorem 4.8]. Some cases with $\tau([A,B])=0$ are indicated in [22–24].

If $U\in\mathscr{B}(\mathscr{H})$ is a nonunitary isometry and $(I-UU^*)\mathscr{H}$ is finite-dimensional then the canonical trace of $[U^*,U]=I-UU^*$ is also nonzero. If $U=X+iY$ is the Cartesian decomposition of U with $X,Y\in\mathscr{B}(\mathscr{H})^{\mathrm{sa}}$, then $[U^*,U]=2i[X,Y]$, i.e., there exist bounded selfadjoint operators whose commutator lies in $\mathfrak{S}_1$ and has nonzero canonical trace. However, if $X\in\mathscr{B}(\mathscr{H})^{\mathrm{sa}}$ and $Y\in\mathscr{B}(\mathscr{H})$ is a compact operator with $[X,Y]\in\mathfrak{S}_1$ then $\mathrm{tr}([X,Y])=0$ by [25, Lemma 1.3]. In [26, Lemma 8] for a normal operator $T\in\mathscr{B}(\mathscr{H})$ and $X\in\mathfrak{S}_2$ with $[T,X]\in\mathfrak{S}_1$ it is shown that $\mathrm{tr}([T,X])=0$. In [27] this result is generalized to certain nonnormal operators. In [28, Theorems 4 and 5] for $T\in\mathscr{B}(\mathscr{H})$ and $X\in\mathfrak{S}_2$ with $[T,X]\in\mathfrak{S}_1$ it is shown that $\mathrm{tr}([T,X])=0$ under either of the two conditions: (a) T^2 is normal; (b) T^n is normal for some integer $n>2$ and $[T^*,T]\in\mathfrak{S}_1$.

The main results of this article are obtained in the context of semifinite von Neumann algebras $\mathscr{M}$, but some of them are new even in the case of the algebra $\mathscr{M}=\mathscr{B}(\mathscr{H})$ endowed with the trace $\tau=\mathrm{tr}$. If $X\in S(\mathscr{M},\tau)$ and $Y=Y^3\in\mathscr{M}$ with $[X,Y]\in L_1(\mathscr{M},\tau)$ then $\tau([X,Y])=0$; see Theorem 2. We generalize the classical Putnam Theorem for bounded hyponormal operators [29] (see also [3, Problem 236]) to the case of τ-measurable unbounded hyponormal operators: No positive self-commutator $[A^*,A]$ with $A\in S(\mathscr{M},\tau)$ is invertible in $\mathscr{M}$; see Theorem 6. If $\tau(I)=+\infty$ then no positive self-commutator $[A^*,A]$ with $A\in S(\mathscr{M},\tau)$ can be of the form $\lambda I+K$, where $\lambda\in\mathbb{C}\setminus\{0\}$ and K is a τ-compact operator; see Theorem 7. If $A^2=A\in S(\mathscr{M},\tau)$ and $[A^*,A]\in L_1(\mathscr{M},\tau)$ then $\tau([A^*,A])=0$; see Theorem 8. Given a partial isometry $U\in\mathscr{M}$ with $U^n=0$ for some integer $n\geq 2$, the operator U^{n-1} is a commutator and $U^{n-1}\in L_1(\mathscr{M},\tau)$ implies that $\tau(U^{n-1})=0$; see Theorem 11. If $U\in L_1(\mathscr{M},\tau)$ and the projections $P=U^*U$ and $Q=UU^*$ are mutually orthogonal then $U^2=0$. Therefore, U is a commutator and $\tau(U)=0$; see Corollary 13.

2. Definitions and Notation

Denote a von Neumann algebra of operators on a Hilbert space $\mathscr{H}$ by $\mathscr{M}$; the lattice of projections ($P=P^2=P^*$) in $\mathscr{M}$ by $\mathscr{M}^{\mathrm{pr}}$; and the cone of positive elements of $\mathscr{M}$ by $\mathscr{M}^+$. Put $P^\perp=I-P$ for $P\in\mathscr{M}^{\mathrm{pr}}$. An operator $A\in\mathscr{M}$ is *unitary*, whenever $A^*A=AA^*=I$, while A is an *isometry* whenever $A^*A=I$, and A is a *partial isometry* whenever $A^*A\in\mathscr{M}^{\mathrm{pr}}$.

A mapping $\varphi:\mathscr{M}^+\to[0,+\infty]$ is a *trace* whenever $\varphi(X+Y)=\varphi(X)+\varphi(Y)$ and $\varphi(\lambda X)=\lambda\varphi(X)$ for all $X,Y\in\mathscr{M}^+$, where $\lambda\geq 0$; furthermore, $0\cdot(+\infty)\equiv 0$, and $\varphi(Z^*Z)=\varphi(ZZ^*)$ for all $Z\in\mathscr{M}$. A trace φ is

- *faithful* whenever $\varphi(X)>0$ for all $X\in\mathscr{M}^+$ with $X\neq 0$;
- *normal* whenever $X_i\nearrow X$ with $X_i,X\in\mathscr{M}^+$ implies that $\varphi(X)=\sup\varphi(X_i)$;
- *finite* whenever $\varphi(I)<+\infty$;
- *semifinite* whenever $\varphi(X)=\sup\{\varphi(Y):\ Y\in\mathscr{M}^+,\ Y\leq X,\ \varphi(Y)<+\infty\}$ for each $X\in\mathscr{M}^+$; see [30, Chapter V, Section 2].

An operator on $\mathscr{H}$, not necessarily bounded or densely defined, is *affiliated* to a von Neumann algebra $\mathscr{M}$ whenever it commutes with all unitary operators in the commutant $\mathscr{M}'$ of $\mathscr{M}$. Henceforth τ stands for a faithful normal semifinite trace on $\mathscr{M}$. A closed operator X affiliated to $\mathscr{M}$ whose domain $\mathscr{D}(X)$ is dense in $\mathscr{H}$ is τ-*measurable* whenever, given $\varepsilon>0$, there exists $P\in\mathscr{M}^{\mathrm{pr}}$ such that $P\mathscr{H}\subset\mathscr{D}(X)$ and $\tau(P^\perp)<\varepsilon$. The set $S(\mathscr{M},\tau)$ of all τ-measurable operators is a $*$-algebra under the taking of adjoint operators, multiplication by scalars, which is furnished with the strong addition and multiplication obtained as the closure of the ordinary operations [31, Chapter IX]. Given a family $\mathscr{L} \subset S(\mathscr{M}, \tau)$, denote by $\mathscr{L}^+$ and $\mathscr{L}^{\rm h}$ the positive and hermitian parts of $\mathscr{L}$. Denote by $\leq$ the partial order on $S(\mathscr{M}, \tau)^{\rm h}$ generated by the proper cone $S(\mathscr{M}, \tau)^+$. If $X \in S(\mathscr{M}, \tau)$ and $X = U|X|$ is the polar decomposition of X then $U \in \mathscr{M}$ and $|X| = \sqrt{X^*X} \in S(\mathscr{M}, \tau)^+$. An operator $A \in S(\mathscr{M}, \tau)$ is *hyponormal* whenever $A^*A \geq AA^*$. Recall that the formula $S_P = 2P - I$ establishes a bijection between the sets of idempotents ($P^2 = P$) and symmetries ($S^2 = I$) in $S(\mathscr{M}, \tau)$. Denote the commutator of operators $A, B \in S(\mathscr{M}, \tau)$ by $[A, B] = AB - BA$. The *self-commutator* of $A \in S(\mathscr{M}, \tau)$ is $[A^*, A] = A^*A - AA^*$. Two operators $A, B \in S(\mathscr{M}, \tau)$ *anticommute* provided that $AB = -BA$.

Denote by $\mu(t; X)$ the *function of singular values* of $X \in S(\mathscr{M}, \tau)$, meaning the nonincreasing right-continuous function $\mu(\cdot; X) : (0, +\infty) \to [0, +\infty)$ defined as

$$\mu(t; X) = \inf\{\|XP\| : P \in \mathscr{M}^{\rm pr},\ \tau(P^\perp) \leq t\}, \quad t > 0.$$

Lemma 1 [32]. *Take $X, Y \in S(\mathscr{M}, \tau)$, $A, B \in \mathscr{M}$, and unitary $U, V \in \mathscr{M}$. Then*
(i) $\mu(t; X) = \mu(t; |X|) = \mu(t; X^*) = \mu(t; UXV)$ *for all* $t > 0$;
(ii) *if* $|X| \leq |Y|$ *then* $\mu(t; X) \leq \mu(t; Y)$ *for all* $t > 0$;
(iii) $\mu(t; AXB) \leq \|A\|\,\|B\|\,\mu(t; X)$ *for all* $t > 0$;
(iv) $\mu(s + t; X + Y) \leq \mu(s; X) + \mu(t; Y)$ *for all* $s, t > 0$;
(v) $\mu(t; f(|X|)) = f(\mu(t; X))$ *for all continuous functions* $f : \mathbb{R}^+ \to \mathbb{R}^+$ *with* $f(0) = 0$ *and* $t > 0$.

Denote the Lebesgue measure on $\mathbb{R}$ by m. Given $0 < p < +\infty$, we can define the noncommutative Lebesgue L_p-space associated to $(\mathscr{M}, \tau)$ as

$$L_p(\mathscr{M}, \tau) = \{X \in S(\mathscr{M}, \tau) : \mu(\cdot; X) \in L_p(\mathbb{R}^+, m)\}$$

with the F-norm $\|X\|_p = \|\mu(\cdot; X)||_p$ for $X \in L_p(\mathscr{M}, \tau)$ which is a norm for $1 \leq p < +\infty$. Denote the unique extension of τ to a linear functional on the whole $L_1(\mathscr{M}, \tau)$ by the same letter τ.

If $\mathscr{M} = \mathscr{B}(\mathscr{H})$ and $\tau = {\rm tr}$ is the canonical trace then $S(\mathscr{M}, \tau)$ and $S_0(\mathscr{M}, \tau)$ coincide with $\mathscr{B}(\mathscr{H})$ and the ideal $\mathfrak{S}_\infty$ of compact operators on $\mathscr{H}$ respectively. We have

$$\mu(t; X) = \sum_{n=1}^{\infty} s_n(X)\chi_{[n-1,n)}(t), \quad t > 0,$$

where $\{s_n(X)\}_{n=1}^\infty$ is the sequence of s-numbers of the compact operator X and χ_A is the indicator of a set $A \subset \mathbb{R}$ [33, Chapter II]. Then the space $L_p(\mathscr{M}, \tau)$ is the Schatten–von Neumann ideal $\mathfrak{S}_p$ for $0 < p < +\infty$.

If $\mathscr{M}$ is abelian, i.e., commutative; then $\mathscr{M} \simeq L^\infty(\Omega, \Sigma, \nu)$ and $\tau(f) = \int_\Omega f\, d\nu$, where (Ω, Σ, ν) is a localizable measure space, the $*$-algebra $S(\mathscr{M}, \tau)$ coincides with the algebra of all measurable complex functions f on (Ω, Σ, ν) bounded beyond a set of finite measure. The function $\mu(t; f)$ coincides with a nonincreasing rearrangement of $|f|$; for the properties of rearrangements; see [34]. The algebra $\mathscr{M}$ lacks nonzero compact operators if and only if ν is atomless [35, Theorem 8.4].

3. The Main Results

Lemma 2 [31, Chapter IX, Theorem 2.13]. *If $X \in \mathscr{M}$ and $Y \in L_1(\mathscr{M}, \tau)$ then $XY, YX \in L_1(\mathscr{M}, \tau)$.*

Lemma 3 [36, Theorem 17]. *If $X, Y \in S(\mathscr{M}, \tau)$ and $XY, YX \in L_1(\mathscr{M}, \tau)$ then $\tau(XY) = \tau(YX)$.*

Theorem 1. *If $X \in L_1(\mathscr{M}, \tau)$ and $U \in \mathscr{M}$ is an isometry then $\tau(X) = \tau(UXU^*)$.*

PROOF. STEP 1. Observe that $\mu(t; X) = \mu(t; UXU^*)$ for all $t > 0$. Indeed, $U^*UXU^*U = X$ and

$$\begin{aligned}\mu(t; X) = \mu(t; U^*UXU^*U) &\leq \|U^*\|\|U\|\mu(t; UXU^*)\\ &= \mu(t; UXU^*) \leq \|U\|\|U^*\|\mu(t; X) = \mu(t; X)\end{aligned}$$

for all $t > 0$ by claim (iii) of Lemma 1 and the equalities $\|U^*\| = \|U\| = 1$. For $X \geq 0$ we infer from [37, Proposition 3.9(c)] that

$$\tau(X) = \int_0^{+\infty} \mu(t; X)\, dt = \int_0^{+\infty} \mu(t; UXU^*)\, dt = \tau(UXU^*).$$

STEP 2. For $X = X^*$ consider the Jordan decomposition $X = X_+ - X_-$. Then $X_+, X_- \in L_1(\mathscr{M}, \tau)^+$ and, since τ continues linearly to the whole space $L_1(\mathscr{M}, \tau)$, Step 1 yields

$$\tau(X) = \tau(X_+) - \tau(X_-) = \tau(UX_+U^*) - \tau(UX_-U^*) = \tau(UXU^*).$$

STEP 3. If $X \in L_1(\mathscr{M}, \tau)$ is an arbitrary operator and $X = \operatorname{Re} X + i \operatorname{Im} X$ is its Cartesian decomposition, then $\operatorname{Re} X = (X + X^*)/2$ and $\operatorname{Im} X = (X - X^*)/(2i)$ lie in $L_1(\mathscr{M}, \tau)^{\mathrm{h}}$ and, since τ continues linearly to the whole space $L_1(\mathscr{M}, \tau)$, Step 1 yields

$$\tau(X) = \tau(\operatorname{Re} X) + i\tau(\operatorname{Im} X) = \tau(U \cdot \operatorname{Re} X \cdot U^*) + i\tau(U \cdot \operatorname{Im} X \cdot U^*) = \tau(UXU^*). \quad \square$$

Corollary 1. *If $A \in S(\mathscr{M}, \tau)$ and $U \in \mathscr{M}$ is unitary with $A - UAU^* \in L_1(\mathscr{M}, \tau)$ then $U^*AU - A \in L_1(\mathscr{M}, \tau)$ and $\tau(U^*AU - A) = \tau(A - UAU^*)$.*

PROOF. We have $\tau(U^*AU - A) = \tau(U^*(A - UAU^*)U)$. $\square$

Corollary 2. *If $X \in S(\mathscr{M}, \tau)$ and $U \in \mathscr{M}$ is unitary with $U^2 = -I$ then $X - UXU^* \in L_1(\mathscr{M}, \tau) \Leftrightarrow [X, U] \in L_1(\mathscr{M}, \tau)$ and, furthermore, $\tau(X - UXU^*) = \tau([X, U]) = 0$.*

PROOF. Observe that $U^* = -U$. If $X - UXU^* \in L_1(\mathscr{M}, \tau)$ then

$$[X, U] = (X - UXU^*)U \in L_1(\mathscr{M}, \tau);$$

if $[X, U] \in L_1(\mathscr{M}, \tau)$ then

$$X - UXU^* = [X, U]U^* \in L_1(\mathscr{M}, \tau)$$

by Lemma 2. Under these conditions, Theorem 1 implies that

$$\begin{aligned}\tau(X - UXU^*) &= \tau(X + UXU) = \tau(U(X + UXU)U^*) = -\tau(U(X + UXU)U)\\ &= -\tau(UXU + X) = -\tau(X - UXU^*);\end{aligned}$$

therefore, $\tau(X - UXU^*) = 0$. Similarly,

$$\begin{aligned}\tau(XU - UX) &= \tau(U(XU - UX)U^*) = -\tau(U(XU - UX)U)\\ &= -\tau(-UX + XU) = -\tau(XU - UX),\end{aligned}$$

and so $\tau(XU - UX) = 0$. $\square$

Corollary 3. *If $U \in \mathscr{M}$ and $A \in S(\mathscr{M}, \tau)$ satisfies $[U^*, A] \in L_1(\mathscr{M}, \tau)$ then for the projection $P = UU^*$ we have*

$$\tau(U^*AU - A) = \tau(PA - UAU^*) = \tau(PAP - UAU^*).$$

PROOF. Lemma 3 with $X = U^*A - AU^*$ and $Y = U$ yields

$$\tau(U^*AU - A) = \tau((U^*A - AU^*)U) = \tau(U(U^*A - AU^*)) = \tau(PA - UAU^*),$$

and then

$$\tau(U^*AU - A) = \tau(U(U^*AU - A)U^*) = \tau(PAP - UAU^*)$$

by Theorem 1. Since $PA - UAU^*$ and $PAP - UAU^*$ lie in $L_1(\mathscr{M}, \tau)$, it follows that $PAP^{\perp} \in L_1(\mathscr{M}, \tau)$ and $\tau(PAP^{\perp}) = 0$. $\square$

Theorem 2. *If $X \in S(\mathscr{M}, \tau)$ and $Y = Y^3 \in \mathscr{M}$ with $[X, Y] \in L_1(\mathscr{M}, \tau)$ then $\tau([X, Y]) = 0$.*

Proof. Step 1. Suppose that $Y = Y^2 \in \mathscr{M}$. Since

$$2YXY - YX - XY = Y[X, Y] - [X, Y]Y \in L_1(\mathscr{M}, \tau),$$

the claim follows from the expansion

$$[X, Y] = Y(2YXY - YX - XY) - (2YXY - YX - XY)Y$$

and Lemma 3 for the pair of Y and $B = 2YXY - YX - XY$.

Step 2. Suppose that $Y = Y^3 \in \mathscr{M}$ and $Y = P - Q$ for two idempotents P and Q in $\mathscr{M}$ satisfying $PQ = QP = 0$; see [38, Proposition 1]. Then $Y^2 = P + Q$ is an idempotent and

$$[X, P] + [X, Q] = [X, Y^2] = [X, Y]Y + Y[X, Y] \in L_1(\mathscr{M}, \tau)$$

since $[X, P] - [X, Q] = [X, Y] \in L_1(\mathscr{M}, \tau)$ by assumption. The last two relations show that $[X, P], [X, Q] \in L_1(\mathscr{M}, \tau)$, and since τ continues linearly to the whole space $L_1(\mathscr{M}, \tau)$, Step 1 yields

$$\tau([X, Y]) = \tau([X, P]) - \tau([X, Q]) = 0 - 0 = 0. \quad \square$$

Corollary 4. *If $X \in S(\mathscr{M}, \tau)$ and $S \in \mathscr{M}$ with $S^2 = I$ then $X - SXS \in L_1(\mathscr{M}, \tau) \Leftrightarrow [X, S] \in L_1(\mathscr{M}, \tau)$ and, furthermore, $\tau(X - SXS) = \tau([X, S]) = 0$.*

Proof. If $X - SXS \in L_1(\mathscr{M}, \tau)$ then $[X, S] = (X - SXS)S \in L_1(\mathscr{M}, \tau)$; if $[X, S] \in L_1(\mathscr{M}, \tau)$ then $X - SXS = [X, S]S \in L_1(\mathscr{M}, \tau)$ by Lemma 2. Under these conditions

$$(XS - SX)S - S(XS - SX) = 2(X - SXS)$$

and Lemma 3 for the pair of $[X, S]$ and S, we have $\tau(X - SXS) = 0$. The equality $\tau([X, S]) = 0$ follows from Theorem 2 because $S^3 = S$. $\square$

Corollary 5. *If $A, B \in S(\mathscr{M}, \tau)^{\mathrm{h}}$ and $C = C^3 \in \mathscr{M}^{\mathrm{sa}}$ with $A - BC \in L_1(\mathscr{M}, \tau)$ then $[B, C] \in L_1(\mathscr{M}, \tau)$, while $\tau([B, C]) = 0$ and $\tau(A - BC) \in \mathbb{R}$.*

Proof. We have

$$[B, C] = (A - BC)^* - (A - BC) \in L_1(\mathscr{M}, \tau)$$

and $\tau([B, C]) = 0$ by Theorem 2. Furthermore,

$$\begin{aligned}\tau(A - BC) &= \tau(A - BC + CB - CB) = \tau(A - CB) - \tau([B, C]) \\ &= \tau(A - CB) = \overline{\tau(A - BC)},\end{aligned}$$

where the bar indicates complex conjugation; thus, $\tau(A - BC) \in \mathbb{R}$. $\square$

Theorem 3. *If $A, B, C \in S(\mathscr{M}, \tau)^{\mathrm{h}}$ and $A - BC \in L_1(\mathscr{M}, \tau)$ then $[B, C] \in L_1(\mathscr{M}, \tau)$. Moreover, if $A - AC \in L_1(\mathscr{M}, \tau)$ then $\tau(A - BC), \tau(A - AC) \in \mathbb{R}$ and $\tau([B, C]) = 0$.*

Proof. We have $[B, C] = (A - BC)^* - (A - BC) \in L_1(\mathscr{M}, \tau)$. Furthermore, if $A - AC \in L_1(\mathscr{M}, \tau)$ then $A - BC = A(I - C) + (A - B)C$ and $(A - B)C \in L_1(\mathscr{M}, \tau)$. Theorem 3.1 of [39] implies that $\tau(A(I - C)), \tau((A - B)C) \in \mathbb{R}$. Therefore, $\tau(A - BC) \in \mathbb{R}$.

Since $\tau(X^*) = \overline{\tau(X)}$ for all $X \in L_1(\mathscr{M}, \tau)$, we have

$$\begin{aligned}\tau([B, C]) &= \tau(A - CB - A + BC) = \tau(A - CB) - \tau(A - BC) \\ &= \overline{\tau(A - BC)} - \tau(A - BC) = 0,\end{aligned}$$

as required. $\square$

Theorem 4. *If $Y, P \in S(\mathscr{M}, \tau)$ with $P^2 = P$ and $X = [Y, P]$, while $S_P = 2P - I$; then*
(i) $S_P X = -X S_P$*;*
(ii) *if* $X^k, S_P X^k S_P \in L_1(\mathscr{M}, \tau)$ *for some odd* $k \in \mathbb{N}$ *then* $\tau(X^k) = \tau(S_P X^k S_P) = 0$*;*
(iii) *if* $P = P^*$ *then* $X^k \in L_1(\mathscr{M}, \tau) \Leftrightarrow S_P X^k S_P \in L_1(\mathscr{M}, \tau)$ *and, furthermore,* $[|X|, P] = 0$*.*

Proof. (ii): Assume that $k = 1$. From (i) we infer that $X = -S_P X S_P$, and Lemma 3 for the pair of $S_P X$ and S_P yields

$$\tau(S_P X S_P) = \tau(AB) = \tau(BA) = \tau(X);$$

i.e., $\tau(X) = \tau(-X) = -\tau(X) = 0$. For $k = 2n + 1 \geq 3$ from

$$X^2 = X S_P \cdot S_P X = -S_P X \cdot -X S_P = S_P X^2 S_P$$

we infer that

$$X^{2n+1} = \underbrace{X^2 \cdot X^2 \cdot \dots \cdot X^2}_{n} \cdot X = \underbrace{S_P X^2 S_P \cdot S_P X^2 S_P \cdot \dots \cdot S_P X^2 S_P}_{n} \cdot X$$
$$= S_P X^{2n} \cdot S_P X = -S_P X^{2n+1} S_P.$$

By Lemma 3 for the pair of $S_P X^{2n}$ and $X S_P$ we see that

$$\tau(S_P X^{2n+1} S_P) = \tau(AB) = \tau(BA) = \tau(X^{2n+1});$$

i.e., $\tau(X^{2n+1}) = -\tau(X^{2n+1}) = 0$.

(iii): If $P = P^*$ then $X^k \in L_1(\mathscr{M}, \tau) \Leftrightarrow S_P X^k S_P \in L_1(\mathscr{M}, \tau)$ by Lemma 2. Passing to the adjoints, by claim (i) we obtain $X^* S_P = -S_P X^*$ and $S_P X^* S_P = -X^*$. Therefore,

$$|X|^2 = X^* X = -S_P X^* S_P \cdot -S_P X S_P = S_P |X|^2 S_P$$

and $|X|^2 S_P = S_P |X|^2$, and so $|X|^2 P = P |X|^2$. Consequently, $|X| P = P |X|$ by the spectral theorem. □

Theorem 4.8 of [21] has the following corollary.

Corollary 6. *Suppose that* $\tau(I) = 1$*. Then*
(i) *under the conditions of Theorem* 3 *we have* $\|I + z[B, C]\|_1 \geq 1$ *for all* $z \in \mathbb{C}$*;*
(ii) *under the conditions of claim* (ii) *of Theorem* 4 *we have*

$$\|I + z X^{2n-1}\|_1 \geq 1, \quad \|I + z S_P X^{2n-1} S_P\|_1 \geq 1$$

for all $n \in \mathbb{N}$ *and* $z \in \mathbb{C}$*.*

Theorem 5. *Suppose that* $X, Y \in S(\mathscr{M}, \tau)^{\mathrm{h}}$ *and* $A = A^3 \in \mathscr{M}^{\mathrm{sa}}$*. If* $AX - YA \in L_1(\mathscr{M}, \tau)$ *then* $\tau(AX - YA) \in \mathbb{R}$*.*

Proof. Take $A = P - Q$, where $P, Q \in \mathscr{M}^{\mathrm{pr}}$ with $PQ = QP = 0$; see [38, Proposition 1]. Then $A^2 = P + Q$ is a projection. The operators

$$PXP - PYP = P(AX - YA)P, \quad QXQ - QYQ = -Q(AX - YA)Q$$

lie in $L_1(\mathscr{M}, \tau)^{\mathrm{h}}$ by Lemma 2. The operators

$$QXP + QYP = -Q(AX - YA)P, \quad PXQ + PYQ = (QXP + QYP)^*$$

also lie in $L_1(\mathscr{M}, \tau)$. By Lemma 3 for the pair of $I - A^2$ and $AX - YA$ and for the pair of $2A^2$ and

$$AX - A^2 YA - AXA^2,$$

since τ continues linearly to the whole space $L_1(\mathscr{M},\tau)$, we obtain

$$\begin{aligned}\tau(AX-YA)&=\tau(A^2(AX-YA)+(I-A^2)(AX-YA))\\&=\tau(A^2(AX-YA))+\tau((I-A^2)(AX-YA))\\&=\tau(A^2(AX-YA))+\tau((AX-YA)(I-A^2))\\&=\tau(2AX-A^2YA-AXA^2)=\tau(2A^3X-A^2YA-A^3XA^2)\\&=\tau(2A^2(AX-A^2YA-AXA^2))=\tau((AX-A^2YA-AXA^2)2A^2)\\&=\tau(AXA^2-A^2YA)=\tau((P-Q)X(P+Q)-(P+Q)Y(P-Q))\\&=\tau(PXP-PYP)+\tau(-QXQ+QYQ)+\tau(PXQ+PYQ)-\tau(QXP+QYP)\in\mathbb{R},\end{aligned}$$

because $\tau(PXP-PYP),\tau(QXQ-QYQ)\in\mathbb{R}$ and

$$\tau(PXQ+PYQ)=\tau(P(PXQ+PYQ))=\tau((PXQ+PYQ)P)=\tau(0)=0.$$

Similarly, $\tau(QXP+QYP)=0$. $\square$

Corollary 7. *Under the conditions of Theorem* 5 *we have*

$$[A,X+Y]\in L_1(\mathscr{M},\tau),\quad \tau([A,X+Y])=0.$$

Proof. Since $XA-AY=(AX-YA)^*\in L_1(\mathscr{M},\tau)$, it follows that

$$\tau(XA-AY)=\tau(AX-YA)\in\mathbb{R}.$$

Observe that

$$[A,X+Y]=AX-YA-(XA-AY).\quad\square$$

The next proposition generalizes the classical Putman Theorem for bounded hyponormal operators [29] (see also [3, Problem 236]) to the case of τ-measurable unbounded hyponormal operators.

Theorem 6. *No positive self-commutator A^*A-AA^* for $A\in S(\mathscr{M},\tau)$ is invertible in $\mathscr{M}$.*

Proof. Suppose that for some $A\in S(\mathscr{M},\tau)$ the operator A^*A-AA^* has an inverse in $\mathscr{M}$; i.e.,

$$A^*A-AA^*\ge\varepsilon I \tag{1}$$

for some $\varepsilon>0$. Multiplying both parts of (1) on the left by A and on the right by A^*, we obtain

$$A^2A^{*2}\le(AA^*)^2-\varepsilon AA^*.$$

Therefore, for each number $t>0$ we have

$$\begin{aligned}\mu(t;A^2)^2=\mu(t;A^{*2})^2=\mu(t;A^2A^{*2})&\le\mu(t;(AA^*)^2-\varepsilon AA^*)\\&\le\mu(t;(AA^*)^2)=\mu(t;AA^*)^2=\mu(t;A)^4\end{aligned}\tag{2}$$

by claims (ii) and (v) of Lemma 1.

Multiplying both sides of (1) on the left by A^* and on the right by A, we obtain

$$A^{*2}A^2\ge(A^*A)^2+\varepsilon A^*A.\tag{3}$$

Introduce the function $f(x)=x^2+\varepsilon x$ of $x\in\mathbb{R}^+$. Then for all $t>0$ we have

$$\begin{aligned}\mu(t;A^2)^2&=\mu(t;A^{*2}A^2)\ge\mu(t;(A^*A)^2+\varepsilon A^*A)=\mu(t;f(A^*A))\\&=f(\mu(t;A^*A))=\mu(t;A^*A)^2+\varepsilon\mu(t;A^*A)=\mu(t;A)^4+\varepsilon\mu(t;A)^2\end{aligned}\tag{4}$$

by (3) and claims (v) and (ii) of Lemma 1. Now (2) and (4) yield

$$\mu(t;A)^4\ge\mu(t;A^2)^2\ge\mu(t;A)^4+\varepsilon\mu(t;A)^2\quad\text{for all } t>0.$$

We arrive at a contradiction. $\square$

Note that the author obtained the claim of Theorem 6 by a different method in Theorem 2 of [15]; see also [16].

Theorem 7. *If $\tau(I) = +\infty$ then no positive self-commutator $A^*A - AA^*$ for $A \in S(\mathscr{M}, \tau)$ can be of the form $\lambda I + K$, where $\lambda \in \mathbb{C} \setminus \{0\}$ and K is a τ-compact operator.*

Proof. Suppose that $A^*A - AA^* = \lambda I + K \geq 0$ for some $A \in S(\mathscr{M}, \tau)$ with a suitable $\lambda > 0$ and $K \in S_0(\mathscr{M}, \tau)$. Assume without loss of generality that $\lambda = 1$. Then $A^*A - K = AA^* + I$. Since $\mu(t; A)$ is nonincreasing and $A \notin S_0(\mathscr{M}, \tau)$, we have the limit

$$\lim_{t \to +\infty} \mu(t; A) = a > 0.$$

Observe that $\mu(t; AA^* + I) = 1 + \mu(t; AA^*)$ for each real $t > 0$ since

$$\mu(t; X) = \inf\{s > 0 : \tau(P^{|X|}(s, +\infty)) \leq t\}$$

for each operator $X \in S(\mathscr{M}, \tau)$, where $|X| = \int_0^{+\infty} u P^{|X|}(du)$ is the spectral decomposition of $|X|$ and the infimum is attained [32, Proposition 2.2]. Therefore, for each $t > 0$ we have

$$1 + \mu(t; A)^2 = 1 + \mu(t; AA^*) = \mu(t; AA^* + I) = \mu(t; A^*A - K)$$
$$\leq \mu(t/2; A^*A) + \mu(t/2; K) = \mu(t/2; A)^2 + \mu(t/2; K)$$

by claims (iv) and (v) of Lemma 1. Passing in the resulting inequality

$$1 + \mu(t; A)^2 \leq \mu(t/2; A)^2 + \mu(t/2; K), \quad t > 0,$$

to the limit as $t \to +\infty$, we obtain $1 + a^2 \leq a^2 + 0 = a^2$, which is a contradiction. $\square$

Theorems 6 and 7 imply the following:

Corollary 8. *If $X, Y \in S(\mathscr{M}, \tau)^{\mathrm{h}}$ and $B := i[X, Y] \geq 0$ then*
(a) *B cannot be invertible in $\mathscr{M}$;*
(b) *for $\tau(I) = +\infty$ B cannot be of the form $\lambda I + K$, where $\lambda \in \mathbb{C} \setminus \{0\}$ and $K \in S_0(\mathscr{M}, \tau)$.*

Proof. It is easy to see that $B = \frac{1}{2}(A^*A - AA^*)$ for $A = X + iY$. $\square$

The self-commutator of an arbitrary operator $Y \in S(\mathscr{M}, \tau)$ is of the form $A - UAU^*$, where $A = Y^*Y$ and U is the partial isometry in the polar decomposition $Y = U|Y|$.

Theorem 8. *If $A \in S(\mathscr{M}, \tau)^+$ and $U \in \mathscr{M}$ is an isometry then $X := A - UAU^*$ is a self-commutator. For $\tau(I) < +\infty$ every self-commutator is of this form.*

Proof. If $X = A - UAU^*$ then $A^{1/2}U^* = (UA^{1/2})^*$ and $X = [A^{1/2}U^*, UA^{1/2}]$.

Suppose that $\tau(I) < +\infty$ and that $X \in S(\mathscr{M}, \tau)^{\mathrm{h}}$ is a self-commutator, meaning $X = Y^*Y - YY^*$ for some operator $Y \in S(\mathscr{M}, \tau)$. If $Y = V|Y|$ is the polar decomposition of Y then the partial isometry V "extends" to a unitary operator $U \in \mathscr{M}$ with the property $Y = U|Y|$; see the proof of Theorem 2 of [40]. Then $YY^* = U|Y|^2U^* = UY^*YU^*$, and we can choose $A := Y^*Y$. $\square$

Theorems 6–8 yields the following:

Corollary 9. *For $A \in S(\mathscr{M}, \tau)^+$ and an isometry $U \in \mathscr{M}$, put $X := A - UAU^* \geq 0$. Then*
(a) *X cannot be invertible in $\mathscr{M}$;*
(b) *for $\tau(I) = +\infty$, X cannot be of the form $\lambda I + K$, where $\lambda \in \mathbb{C} \setminus \{0\}$ and $K \in S_0(\mathscr{M}, \tau)$.*

Lemma 4. *If $A, B \in S(\mathscr{M}, \tau)$, while B is normal and $AB = BA$, then $[A^* - B^*, A - B] = [A^*, A]$.*

Proof. The Fuglede–Putnam Theorem for τ-measurable operators [41, Theorem 6] shows that $AB^* = B^*A$. Therefore,

$$BA^* = (AB^*)^* = (B^*A)^* = A^*B.$$

Note that for the algebra $LS(\mathscr{M})$ of all locally measurable operators affiliated to a von Neumann algebra $\mathscr{M}$ of type I or type III, the Fuglede–Putnam Theorem was established in [42, Theorem 1] and [43, Theorem 1]. $\square$

Theorem 9. *If $A^2 = A \in S(\mathscr{M}, \tau)$ and $[A^*, A] \in L_1(\mathscr{M}, \tau)$ then $\tau([A^*, A]) = 0$.*

Proof. For $A = A^2 \in S(\mathscr{M}, \tau)$ there exists a unique decomposition $A = P + Z$, where $P \in \mathscr{M}^{\mathrm{pr}}$ and the operator Z with $Z^2 = 0$ lies in $S(\mathscr{M}, \tau)$ and satisfies $ZP = 0$ and $PZ = Z$ [44, Theorem 2.23]. By assumption, the operator

$$Z + Z^* + Z^*Z - ZZ^* = [A^*, A] \tag{5}$$

lies in $L_1(\mathscr{M}, \tau)$. Since $Z^*P = (PZ)^* = Z^*$, the operators

$$Z^* - ZZ^* = [A^*, A]P, \quad Z + Z^*Z = [A^*, A]P^{\perp} \tag{6}$$

lie in $L_1(\mathscr{M}, \tau)$ as well. Therefore,

$$Z^* + Z^*Z = (Z + Z^*Z)^* \in L_1(\mathscr{M}, \tau)$$

and the first equality in (6) yields

$$ZZ^* + Z^*Z \in L_1(\mathscr{M}, \tau).$$

Then $Z^*Z \in L_1(\mathscr{M}, \tau)$ and the second equality in (6) yields $Z \in L_1(\mathscr{M}, \tau)$. By Lemma 3 with $X = P$ and $Y = Z$ we obtain

$$\tau(Z) = \tau(PZ) = \tau(ZP) = \tau(0) = 0.$$

Thus, $\tau(Z^*) = \overline{\tau(Z)} = 0$. Using the equalities

$$\tau(X) = \int_0^{+\infty} \mu(t; X)\, dt \quad (X \in S(\mathscr{M}, \tau)^+),$$

see [37, Proposition 3.9(c)], and claims (i) and (v) of Lemma 1, we find that

$$\begin{aligned}\tau(A^*A - AA^*) &= \tau(Z + Z^* + Z^*Z - ZZ^*) = \tau(Z) + \tau(Z^*) + \tau(Z^*Z) - \tau(ZZ^*) \\ &= 0 + 0 + \int_0^{+\infty} \mu(t; Z^*Z)\, dt - \int_0^{+\infty} \mu(t; ZZ^*)\, dt \\ &= \int_0^{+\infty} (\mu(t; Z)^2 - \mu(t; Z^*)^2)\, dt = \int_0^{+\infty} 0\, dt = 0.\end{aligned}$$

The proof of Theorem 9 is complete. □

Corollary 10. *If $X = X^3 \in S(\mathscr{M}, \tau)$, the operator $X^2 - X$ is hermitian, and $[X^*, X] \in L_1(\mathscr{M}, \tau)$, then $\tau([X^*, X]) = 0$.*

Proof. By [38, Proposition 1] we have $X = A - B$, where the idempotents $A = (X^2 + X)/2$ and $B = (X^2 - X)/2$ satisfy $AB = BA = 0$. Since $B = B^*$, Lemma 4 yields $[X^*, X] = [A^*, A]$, and then Theorem 8 applies. □

Corollary 11. *If $X \in S(\mathscr{M}, \tau)$ with $X^2 = I$ and $[X^*, X] \in L_1(\mathscr{M}, \tau)$ then $\tau([X^*, X]) = 0$.*

Proof. Given $A := (I + X)/2$, we have $A^2 = A$ and $X^*X - XX^* = 4(A^*A - AA^*)$. Thus, $\tau([X^*, X]) = 0$. □

Corollary 12. *Suppose that $\tau(I) = 1$ and $A \in S(\mathscr{M}, \tau)$. If $[A^*, A] \in L_1(\mathscr{M}, \tau)$ and $A^2 \in \{A, I\}$ then $\|I + z[A^*, A]\|_1 \geq 1$ for all $z \in \mathbb{C}$.*

Proposition 1. *If $X, Y \in S(\mathscr{M}, \tau)$ with $XY = \lambda YX$ for some $\lambda \in \mathbb{C}$ and $[X, Y] \in L_1(\mathscr{M}, \tau)$ then $\tau([X, Y]) = 0$.*

Proof. For $\lambda = 1$ the claim holds, and so we assume that $\lambda \neq 1$. We have

$$(\lambda - 1)YX = [X, Y] \in L_1(\mathscr{M}, \tau);$$

hence $YX, XY \in L_1(\mathscr{M}, \tau)$. Lemma 3 yields

$$\tau(YX) = \tau(XY) = \lambda\tau(YX) = 0,$$

and so $\tau([X, Y]) = 0$. $\square$

Some examples of operators $X, Y \in S(\mathscr{M}, \tau)$ with $XY = \lambda YX$ are given in claim (i) of Theorem 4; see also [45].

Theorem 10. *Consider $A, B \in S(\mathscr{M}, \tau)$ with $A^n = 0$ for some integer $n \geq 2$. For $k, m \in \mathbb{N}$ with $k + m \geq n$ the operator $A^k BA^m$ is a commutator and $A^k BA^m \in L_1(\mathscr{M}, \tau)$ implies that $\tau(A^k BA^m) = 0$.*

Proof. We have

$$A^k BA^m = A^k B \cdot A^m - A^m \cdot A^k B = [A^k B, A^m].$$

If $A^k BA^m \in L_1(\mathscr{M}, \tau)$ then Lemma 3 with $X = A^k B$ and $Y = A^m$ yields

$$\tau(A^k B \cdot A^m) = \tau(A^m \cdot A^k B) = \tau(0) = 0, \quad k + m \geq n.$$

Observe that for $2k \geq n$ the operators $[A^k, B]$ and A^k anticommute, while $A^k B + BA^k$ and A^k commute. In particular, in the case $\tau(I) = 1$ we have

$$\|I + zA^k BA^m\|_1 \geq 1$$

for all $z \in \mathbb{C}$ and $k, m \in \mathbb{N}$ with $k + m \geq n$. $\square$

Theorem 11. *If a partial isometry U lies in $\mathscr{M}$ and $U^n = 0$ for some integer $n \geq 2$ then U^{n-1} is a commutator and $U^{n-1} \in L_1(\mathscr{M}, \tau)$ implies that $\tau(U^{n-1}) = 0$.*

Proof. Since $U = UU^*U$ by [3, Corollary 3 to Problem 98], for $n \geq 2$ we have

$$U^{n-1} = U^{n-2} \cdot UU^*U = U^{n-1} \cdot U^*U - U^*U \cdot U^{n-1} = [U^{n-1}, U^*U];$$

if $n = 2$ then $U = U \cdot U^*U - U^*U \cdot U = [U, U^*U]$.

Assume that $U^{n-1} \in L_1(\mathscr{M}, \tau)$. For $n \geq 2$ Lemma 3 with $X = U^*U^{n-1}$ and $Y = U$ and the equality $U = UU^*U$ yield

$$0 = \tau(0) = \tau(U^*U^n) = \tau(U \cdot U^*U \cdot U^{n-2}) = \tau(U^{n-1});$$

if $n = 2$ then similarly

$$0 = \tau(0) = \tau(U^*U^2) = \tau(UU^*U) = \tau(U).$$

In particular, if $\tau(I) = 1$ then $\|I + zU^{n-1}\|_1 \geq 1$ for all $z \in \mathbb{C}$. $\square$

Corollary 13. *Given a partial isometry $U \in L_1(\mathscr{M}, \tau)$, if the projections $P = U^*U$ and $Q = UU^*$ are mutually orthogonal then $U^2 = 0$. Therefore, U is a commutator and $\tau(U) = 0$.*

Proof. Lemma 3 with $X = U$ and $Y = U^{*2}U$ shows that

$$0 = \tau(0) = \tau(QP) = \tau(UU^{*2}U) = \tau(U^{*2}U^2) = \tau(U^{2*}U^2) = \tau(|U^2|^2).$$

Since the trace τ is faithful, we infer that $|U^2|^2 = 0$; thus, $|U^2| = 0$ and $U^2 = 0$. $\square$

FUNDING

The research was carried out in the framework of the Development Program of the Scientific Educational Mathematical Center of the Volga Federal District (Agreement 075–02–2022–882).

CONFLICT OF INTEREST

As author of this work, I declare that I have no conflicts of interest.

References

1. Glazman I.M. and Lyubich Yu.I., *Finite-Dimensional Linear Analysis: A Systematic Presentation in Problem Form*, M.I.T., Cambridge and London (1974).
2. Albert A.A. and Muckenhoupt B., "On matrices of trace zeros," Michigan Math. J., vol. 4, no. 1, 1–3 (1957).
3. Halmos P.R., *A Hilbert Space Problem Book*, Springer, New York (1982) (Graduate Texts in Math.; vol. 19).
4. Johnson W.B., Ozawa N., and Schechtman G., "A quantitative version of the commutator theorem for zero trace matrices," Proc. Natl. Acad. Sci. USA, vol. 110, no. 48, 19251–19255 (2013).
5. Angel O. and Schechtman G., "The Hilbert–Schmidt version of the commutator theorem for zero trace matrices," Bull. Lond. Math. Soc., vol. 47, no. 4, 715–719 (2015).
6. Wang D.-G., Zhang J.J., and Zhuang G., "Coassociative Lie algebras," Glasg. Math. J., vol. 55, no. A, 195–215 (2013).
7. Ravichandran M. and Srivastava N., "Asymptotically optimal multi-paving," Int. Math. Res. Not., vol. 2021, no. 14, 10908–10940 (2021).
8. Robinson D.W., "Matrices with zero trace as commutators of nilpotents," Linear Multilinear Algebra, vol. 5, no. 1, 45–51 (1977).
9. Brown A. and Pearcy C., "Structure of commutators of operators," Ann. Math. (2), vol. 82, no. 1, 112–127 (1965).
10. Bikchentaev A., "Commutators in C^*-algebras and traces," Ann. Funct. Anal., vol. 14, no. 2, Paper no. 42; 14 pp. (2023).
11. Fan P., "On the diagonal of an operator," Trans. Amer. Math. Soc., vol. 283, no. 1, 239–251 (1984).
12. Fan P. and Fong C.K., "Which operators are the self-commutators of compact operators?," Proc. Amer. Math. Soc., vol. 80, no. 1, 58–60 (1980).
13. Fan P., Fong C.K., Che K., and Herrero D.A., "On zero-diagonal operators and traces," Proc. Amer. Math. Soc., vol. 99, no. 3, 445–451 (1987).
14. Fan P., Fong C.K., and Che K., "An intrinsic characterization for zero-diagonal operators," Proc. Amer. Math. Soc., vol. 121, no. 3, 803–805 (1994).
15. Bikchentaev A.M., "To the theory of τ-measurable operators affiliated to a semifinite von Neumann algebra. II," Matematika i Teor. Komp'yutern. Nauki, vol. 1, no. 2, 3–11 (2023).
16. Bikchentaev A.M., "Concerning the theory of τ-measurable operators affiliated to a semifinite von Neumann algebra. II," Lobachevskii J. Math., vol. 44, no. 10, 4507–4511 (2023).
17. Bikchentaev A.M., "Essentially invertible measurable operators affiliated to a semifinite von Neumann algebra and commutators," Sib. Math. J., vol. 63, no. 2, 224–232 (2022).
18. Weigt M., "Derivations of τ-measurable operators," in: *Operator Algebras, Operator Theory and Applications*, Birkhäuser, Basel (2010), 273–286 (Oper. Theory Adv. Appl.; vol. 195).
19. Ber A.F., Kudaybergenov K.K., and Sukochev F.A., "Derivations on Murray–von Neumann algebras," Russian Math. Surveys, vol. 74, no. 5, 950–952 (2019).
20. Huang J., Kudaybergenov K.K., and Sukochev F.A., "Ring derivations of Murray–von Neumann algebras," Linear Algebra Appl., vol. 672, no. 1, 28–52 (2023).
21. Bikchentaev A.M., "Convergence of integrable operators affiliated to a finite von Neumann algebra," Proc. Steklov Inst. Math., vol. 293, 67–76 (2016).
22. Bikchentaev A.M., "The trace and commutators of measurable operators affiliated to a von Neumann algebra," in: *Quantum Probability*, VINITI, Moscow (2018), 10–20 [Russian] (Itogi Nauki i Tekhniki; vol. 151).
23. Bikchentaev A.M., "Trace and commutators of measurable operators affiliated to a von Neumann algebra," J. Math. Sci. (New York), vol. 252, no. 1, 8–19 (2021).
24. Bikchentaev A.M. and Fawwaz Kh., "Differences and commutators of idempotents in C^*-algebras," Russian Math. (Iz. VUZ. Matematika), vol. 65, no. 8, 13–22 (2021).
25. Helton J. and Howe R., "Traces of commutators of integral operators," Acta Math., vol. 135, no. 3–4, 271–305 (1975).
26. Weiss G., "The Fuglede commutativity theorem modulo the Hilbert–Schmidt class and generating functions for matrix operators. I," Trans. Amer. Math. Soc., vol. 246, 193–209 (1978).
27. Kittaneh F., "On zero-trace commutators," Bull. Austral. Math. Soc., vol. 34, no. 1, 119–126 (1986).
28. Kittaneh F., "Some trace class commutators of trace zero," Proc. Amer. Math. Soc., vol. 113, no. 3, 655–661 (1991).
29. Putnam C.R., "On commutators of bounded matrices," Amer. J. Math., vol. 73, no. 1, 127–131 (1951).
30. Takesaki M., *Theory of Operator Algebras. I*. Operator Algebras and Non-Commutative Geometry V, Springer, Berlin (2002) (Encyclopaedia of Mathematical Sciences; vol. 124).
31. Takesaki M., *Theory of Operator Algebras. II*. Operator Algebras and Non-Commutative Geometry VI, Springer, Berlin (2003) (Encyclopaedia of Mathematical Sciences; vol. 125).
32. Fack T. and Kosaki H., "Generalized s-numbers of τ-measurable operators," Pacific J. Math., vol. 123, no. 2, 269–300 (1986).

33. Gokhberg I.Ts. and Krein M.G., *An Introduction to the Theory of Linear Nonselfadjoint Operators in Hilbert Space*, Amer. Math. Soc., Providence (1969).
34. Krein S.G., Petunin Yu.I., and Semenov E.M., *Interpolation of Linear Operators*, Amer. Math. Soc., Providence (1982).
35. Antonevich A.B., *Linear Functional Equations. Operator Approach*, Birkhäuser, Basel (1996).
36. Brown L.G. and Kosaki H., "Jensen's inequality in semifinite von Neumann algebra," J. Operator Theory, vol. 23, no. 1, 3–19 (1990).
37. Dodds P.G., Dodds T.K.-Y., and Pagter de B., "Noncommutative Köthe duality," Trans. Amer. Math. Soc., vol. 339, no. 2, 717–750 (1993).
38. Bikchentaev A.M. and Yakushev R.S., "Representation of tripotents and representations via tripotents," Linear Algebra Appl., vol. 435, no. 9, 2156–2165 (2011).
39. Bikchentaev A.M., "Concerning the theory of τ-measurable operators affiliated to a semifinite von Neumann algebra," Math. Notes, vol. 98, no. 3, 382–391 (2015).
40. Bikchentaev A.M., "Minimality of convergence in measure topologies on finite von Neumann algebras," Math. Notes, vol. 75, no. 3, 315–321 (2004).
41. Ber A., Chilin V., Sukochev F., and Zanin D., "Fuglede–Putnam theorem for locally measurable operators," Proc. Amer. Math. Soc., vol. 146, no. 4, 1681–1692 (2018).
42. Ahramovich M.V., Chilin V.I., and Muratov M.A., "Fuglede–Putnam theorem in the algebra of locally measurable operators," Indian J. Math., vol. 55, 13–20 (2013).
43. Ahramovich M.V., Muratov M.A., and Chilin V.I., "The Fuglede–Putnam theorem for locally measurable operators," Dinam. Sist. (Simferopol'), vol. 4(32), no. 1–2, 3–8 (2014).
44. Bikchentaev A.M., "On idempotent τ-measurable operators affiliated to a von Neumann algebra," Math. Notes, vol. 100, no. 4, 515–525 (2016).
45. Akhmadiev M., Alhasan H., Bikchentaev A., and Ivanshin P., "Commutators and hyponormal operators on a Hilbert space," J. Iran. Math. Soc., vol. 4, no. 1, 67–78 (2023).

Publisher's Note. Pleiades Publishing remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

A. M. Bikchentaev
Lobachevsky Institute of Mathematics and Mechanics
of Kazan (Volga Region) Federal University, Kazan, Russia
https://orcid.org/0000-0001-5992-3641
E-mail address: Airat.Bikchentaev@kpfu.ru