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MATHEMATICAL ANALYSIS OF THE GENERALIZED NATURAL MODES OF AN IHHOMOGENEOUS OPTICAL FIBER*

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Abstract. The eigenvalue problem for generalized natural modes of an inhomogeneous optical fiber without a sharp boundary is formulated as a problem for the set of time-harmonic Maxwell equations with the Reichardt condition at infinity in the cross-sectional plane. The generalized eigenvalues (including, as subsets, the well-known guided and leaky modes) of this problem are the complex propagation constants on a logarithmic Riemann surface. A theorem on spectrum localization is proved, and then the original problem is reduced to a nonlinear spectral problem with a compact integral operator. It is proved that the set of all eigenvalues of the original problem can only be a set of isolated points on the Riemann surface, and it is also proved that each eigenvalue depends continuously on the frequency and refraction index and can appear and disappear only at the boundary of the Riemann surface.

Key words. electromagnetic theory, optical fiber, waveguides, eigenvalue problem, guided modes

AMS subject classifications. 35P30, 45C05, 65R20, 78A50

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1. Introduction.

Optical fibers are dielectric waveguides (DWs), i.e., regular dielectric rods, having various cross sectional shapes, and where generally the refractive index of the dielectric may vary in the waveguide's cross section. Although existing technologies often result in a refractive index that is anisotropic, frequently it is possible to assume that the fiber is isotropic, which is the case investigated in this work. The study of the source-free electromagnetic fields, called *natural modes*, that can propagate on DWs necessitates that longitudinally the rod extend to infinity. Since often DWs are not shielded, the medium surrounding the waveguide transversely forms an unbounded domain, typically taken to be free space. This fact plays an extremely important role in the mathematical analysis of natural waveguide modes, and brings into consideration a variety of possible formulations. Each different formulation can be cast as an eigenvalue problem for the set of time-harmonic Maxwell equations, but they differ in the form of the condition imposed at infinity in the cross-sectional plane, and hence in the functional class of the natural-mode field. As we discuss below, this also restricts the localization of the eigenvalues in the complex plane of the eigenparameter.

Historically, the first DWs to be studied were step-index waveguides having circular cross section; interior to the waveguide, the refractive index was either homo-

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geneous or coaxial-layered. In these cases, by using separation of variables, modal eigenvalue problems are easily reduced to families of transcendental dispersion equations associated with the azimuthal indices (see, e.g., [1], [2]). All questions concerning discreteness and existence of the natural-mode spectrum are settled "automatically" due to general results from the theory of complex variables and the analytic properties of cylindrical functions with integer indices and complex arguments.

For these circular cross section DWs the first class of natural modes to be studied were *purely guided modes*, which have real-valued wavenumbers. The fields of the guided modes are confined near to the waveguide, decaying exponentially transversely away from the waveguide, so that they belong to the space L_2 in the whole cross-sectional plane. Corresponding eigenvalue problems are self-adjoint. Later it was discovered that the guided modes of a circular DW can turn into (i.e., be analytically continued as) so-called *leaky-wave modes*, existing on the "improper" sheet of a square-root Riemann surface, with the wavenumbers migrating off the real axis of the "proper" sheet onto the "improper" sheet as some parameters of the structure vary [3]. It was noticed that leaky modes can be studied as solutions of a more general eigenvalue problem, without cross-sectional field confinement, due to some relaxed, although never explicitly formulated, condition at infinity.

Although leaky waves exist on an "improper" Riemann sheet, they have considerable physical importance in wave excitation and fiber discontinuity problems. In particular, it is known that the electromagnetic fields existing on a dielectric waveguide can be represented as a discrete sum of bound modes (which are the mentioned guided modes generated by the eigenvalues of the propagation constant on the real axis of the "proper" sheet) and a continuous sum (i.e., integral) of so-called *radiation modes* (whose physical sense still causes discussions) [1], [2], [4]. It has been shown that, although leaky waves are not themselves a part of a "proper" spectral field representation, in many cases the continuum of the radiation modes may be approximated by a discrete sum of leaky modes [5], representing the near field of a source-excited fiber. Often the leaky-wave sum can be reduced to a single term, providing a concise analytical form for the near-zone radiation field. Furthermore, various features in the far-field radiation pattern of a real, finite-length, source-driven fiber can be interpreted in terms of leaky-wave excitation. In addition to source-driven waveguides, leaky modes on longitudinally invariant fibers are important in the analysis of radiation and mode-conversion effects associated with waveguide discontinuities such as fiber splices [6], radiation from waveguide bends [7], and radiation from anisotropic fibers [8], [9]. Some properties of leaky modes on dielectric waveguides, and, in particular, dielectric fibers, are presented in [2], [3], [5], [6], [7], [8], [9], [10], [11], [12].

In addition to leaky modes, it was discovered that on the "proper" sheet, but off its real axis, one can also find other generalized eigenvalues (modal wavenumbers) [13] known as *complex modes*. Analogous results were obtained numerically for gradient-index DWs of arbitrary cross section [14]. These complex modes are often important in near-field fiber discontinuity problems and mode-matching analysis. It is important to note that all of these known types of natural modes can transform (be continued) one into another, following variation of some geometrical or material parameter or frequency. Due to the presence of the two-dimensional unbounded domain and the resulting Green's functions represented as Hankel functions, it is easy to see that the dispersion equations contain logarithmic as well as square-root–type branch points.

If the cross section is not circular, the study of the natural modes encounters both methodological and numerical problems. In [15] an elliptic DW was studied by using expansions in terms of Mathieu functions. However, in that work as well as in other studies of waveguides having complicated cross sections, or of multirod waveguides, the modal problems are reduced not to transcendental equations but to infinite matrix equations or integral equations (IEs). Hence, it is necessary to base the analysis on the theory of operator-functions depending on parameters. Once again, by restricting the desired field behavior in the cross-sectional plane, one arrives at different formulations of the eigenvalue problem in terms of the transverse condition at infinity; eigenvalue localization and the function class of the natural mode field are tied up with this condition.

In recent years, research on the natural modes of arbitrarily shaped DWs has been focused on the development of efficient and reliable computational methods. For instance, in [16] the eigenvalue problem for the natural modes of arbitrary DWs was studied by splitting the differential operator into self-adjoint and perturbation parts and using a discretization in terms of the eigenfunctions of the self-adjoint operator. This enabled the authors to develop a very efficient numerical technique, although its convergence was not proven.

In the papers on numerical methods for DWs, the mathematical grounding of the methods was frequently neglected; however, useful insight into the encountered difficulties and modal behavior has been discussed (e.g., see [17], [18]). The most rigorous efforts were connected with IE formulations. Within this class the domain IE method has the attractive advantage of being applicable to cross-sectionally inhomogeneous (and, in fact, anisotropic) DWs [19], [20]. A problem with domain IEs is that they are strongly singular, which previously prevented their use in a mathematical study of the spectrum of the eigenvalues, with the exception of [21] for the purely guided modes of an inhomogeneous DW. For real-valued propagation constants it was proven in [21] that the operator of the domain IE is semi-Fredholm.

A rigorous mathematical study of an arbitrary-shaped DW was performed in [22] within the guided (proper) mode formulation. This enabled the authors to make extensive use of the theory of unbounded self-adjoint operators. For example, by using the min-max principle, they proved the existence of guided modes, the number of which is finite and depends on frequency. However, generalized natural modes having complex valued propagation constants cannot be studied by this approach.

The above considerations give a new thrust to the idea of elaborating a generalized formulation of the modal eigenproblem in order to bring together all the possible natural-mode solutions. All of the known natural-mode solutions (i.e., guided modes, leaky modes, complex modes) satisfy the Reichardt condition [23] at infinity. The wavenumbers may be generally considered on the appropriate logarithmic Riemann surface. The Reichardt condition in this problem is connected with the fact that the wavenumber may be complex. For real wavenumbers on the principal ("proper") sheet of this Riemann surface, one can reduce the Reichardt condition to either the Sommerfeld radiation condition or to the condition of exponential decay. The Reichardt condition may be considered as a generalization of the Sommerfeld radiation condition and can be applied for complex wavenumbers. This condition may also be considered as the continuation of the Sommerfeld radiation condition from a part of the real axis of the complex parameter (wavenumber) to the appropriate logarithmic Riemann surface.

During recent years the Reichardt condition has been widely used for statements of various wave propagation problems [24], [25], [26]. By using the Reichardt condition, the problems on generalized modes of microstrip and slot lines on a cylindrical substrate were investigated in [27], [28], [29]. Tensor Green's functions of generic open waveguides with compact cross sections were analyzed in [30] by using Fourier transforms and IE techniques in the transform domain. It was shown that the complex-valued poles of analytic continuations of the Green's functions satisfy a certain eigenvalue problem. Their residues can be interpreted as the generalized natural modes. In this case, the eigenvalue problem should be formulated with the Reichardt condition at infinity. Reducing Maxwell's equations to an IE and converting the latter to a Fredholm second kind equation enabled the proof of some important properties of the spectrum of the generalized modes. Furthermore, in [31], [32] a similar formulation was applied to study generalized guided modes in DWs, and a numerical algorithm was developed based on a Galerkin discretization in terms of a trigonometric basis.

In this paper we extend the approach of [30], [31], [32] to the analysis of generalized natural modes of arbitrary-cross-section DWs having inhomogeneous (although continuous) refractive index. Here, we use the model of DW without a sharp boundary, as was proposed in [33]. Such an approach enables one to reduce the original problem to a nonlinear spectral problem with a compact integral operator, and was originally introduced in [34] and used in [35]. We present a unified and rigorous theory of generalized natural modes in terms of the Reichardt condition at infinity.

The rest of this paper is organized as follows. Physical assumptions, basic equations, and notation are presented in section 2. In section 3 we formulate the modal eigenvalue problem as a problem for the set of time-harmonic Maxwell equations with the Reichardt condition at infinity in the cross-sectional plane. The eigenvalues of this problem are the complex propagation constants of the natural modes, and we introduce a classification of modal eigenvalues in terms of their location on the logarithmic Riemann surface. In section 4 we prove a theorem on localization of eigenvalues, where it is established that there exists a domain free of eigenvalues on this surface. In section 5 we investigate the eigenvalues as functions of frequency and refractive index, and we reduce the original problem to a nonlinear spectral problem with a compact integral operator. Using general results from the spectral theory of operator-valued functions [36], we prove that the set of all eigenvalues of the original problem can only be a set of isolated points on the logarithmic Riemann surface, and also we prove that each eigenvalue depends continuously on frequency and refractive index, and can appear and disappear only at the boundary of the logarithmic Riemann surface.

2. Basic relations. We consider the generalized natural modes of the regular DW shown in Figure 1. Let the three-dimensional space $\{(x_1, x_2, x_3) : -\infty < x_1, x_2, x_3 < \infty\}$ be occupied by an isotropic source-free medium, and let the refractive index be prescribed as a positive real-valued function $n = n(x_1, x_2)$ independent of the longitudinal coordinate x_3 and equal to a constant n_∞ outside a cylinder. The axis of the cylinder is parallel to the x_3-axis, and its cross section is a bounded domain Ω with a Lipschitz boundary on the plane $\mathbb{R}^2 = \{(x_1, x_2) : -\infty < x_1, x_2 < \infty\}$. Denote by Ω_∞ the unbounded domain $\Omega_\infty = \mathbb{R}^2 \setminus \overline{\Omega}$, and denote by n_+ the maximum of the function n in the domain Ω, where $n_+ > n_\infty$. Let the function n belong to the space of real-valued twice continuously differentiable functions in $\mathbb{R}^2$.

The modal problem can be formulated as a vector eigenvalue problem for the set of harmonic Maxwell equations, assuming that electric and magnetic field vectors

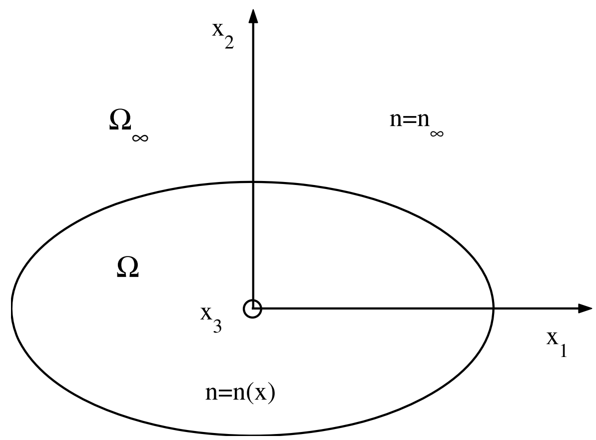

FIG. 1. *Geometry of a dielectric waveguide.*

have the form

$$\mathcal{E}(x, x_3, t) = \mathrm{Re}\left(\mathrm{E}(x)\exp\left(i\beta x_3 - i\omega t\right)\right), \tag{1}$$

$$\mathcal{H}(x, x_3, t) = \mathrm{Re}\left(\mathrm{H}(x)\exp\left(i\beta x_3 - i\omega t\right)\right). \tag{2}$$

Here $x = (x_1, x_2)$, $\omega > 0$ is the radian frequency, β is the complex-valued modal wavenumber (or propagation constant), and E and H are complex amplitudes of $\mathcal{E}$ and $\mathcal{H}$. For the sake of clarity, we note that, unlike in [22], we consider the propagation constant β as an unknown complex parameter and $\omega > 0$ as a given parameter. Such a choice seems to be commonly adopted in the fiber optics and microwave research communities due to the easy control of frequency.

For the fields of the form (1), (2), the set of Maxwell equations becomes

$$\mathrm{Rot}_\beta \mathrm{E} = i\omega\mu_0 \mathrm{H}, \quad x \in \mathbb{R}^2, \tag{3}$$

$$\mathrm{Rot}_\beta \mathrm{H} = -i\omega\varepsilon_0 n^2 \mathrm{E}, \quad x \in \mathbb{R}^2. \tag{4}$$

Here ε_0, μ_0 are the free-space dielectric and magnetic constants, respectively, and

$$\mathrm{Rot}_\beta \mathrm{E} = \begin{bmatrix} \partial E_3/\partial x_2 - i\beta E_2 \\ i\beta E_1 - \partial E_3/\partial x_1 \\ \partial E_2/\partial x_1 - \partial E_1/\partial x_2 \end{bmatrix}. \tag{5}$$

By $C^2(\mathbb{R}^2)$ denote the space of twice continuously differentiable in $\mathbb{R}^2$ complex-valued functions. We shall be seeking nonzero solutions $[\mathrm{E}, \mathrm{H}]$ of set (3), (4) in the space $(C^2(\mathbb{R}^2))^6$.

Let F be a three-dimensional vector-function,

$$\mathrm{F} = \begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix} \in \left(C^2(\mathbb{R}^2)\right)^3,$$

and let $u \in C^2(\mathbb{R}^2)$ be a scalar function. By definition, set

$$\mathrm{Div}_\beta \mathrm{F} = \frac{\partial F_1}{\partial x_1} + \frac{\partial F_2}{\partial x_2} + i\beta F_3, \tag{6}$$

$$\Delta u = \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2}, \tag{7}$$

$$\mathrm{Grad}_\beta u = \begin{bmatrix} \partial u/\partial x_1 \\ \partial u/\partial x_2 \\ i\beta u \end{bmatrix}, \quad \mathrm{grad} u = \begin{bmatrix} \partial u/\partial x_1 \\ \partial u/\partial x_2 \\ 0 \end{bmatrix}, \tag{8}$$

$$\mathrm{grad}_2 u = \begin{bmatrix} \partial u/\partial x_1 \\ \partial u/\partial x_2 \end{bmatrix}, \quad \mathbf{F} = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}. \tag{9}$$

By direct calculation it is easy to obtain the following equations:

$$\mathrm{Div}_\beta \left(\mathrm{Grad}_\beta u\right) = \Delta u - \beta^2 u, \tag{10}$$

$$\mathrm{Div}_\beta \left(\mathrm{Rot}_\beta \mathrm{F}\right) = 0, \tag{11}$$

$$\mathrm{Div}_\beta \left(u\mathrm{F}\right) = u\mathrm{Div}_\beta \mathrm{F} + \left(\mathrm{F}, \mathrm{grad} u\right), \tag{12}$$

$$\mathrm{Rot}_\beta \left(\mathrm{Grad}_\beta u\right) = 0, \tag{13}$$

$$\mathrm{Rot}_\beta \left(\mathrm{Rot}_\beta \mathrm{F}\right) = -\Delta \mathrm{F} + \beta^2 \mathrm{F} + \mathrm{Grad}_\beta \left(\mathrm{Div}_\beta \mathrm{F}\right), \tag{14}$$

where

$$(\mathrm{F}, \mathrm{L}) = \sum_{i=1}^{3} F_i L_i. \tag{15}$$

LEMMA 2.1. *If* $[\mathrm{E}, \mathrm{H}]$ *is a solution of the set* (2.3), (2.4), *then for* $x \in \mathbb{R}^2$

$$\mathrm{Rot}_\beta \left(\mathrm{Rot}_\beta \mathrm{E}\right) = k^2 n^2 \mathrm{E}, \tag{16}$$

$$\mathrm{Rot}_\beta \left(n^{-2} \mathrm{Rot}_\beta \mathrm{H}\right) = k^2 \mathrm{H}, \tag{17}$$

$$\mathrm{Div}_\beta \left(n^2 \mathrm{E}\right) = 0, \tag{18}$$

$$\mathrm{Div}_\beta \left(\mathrm{H}\right) = 0, \tag{19}$$

where $k^2 = \varepsilon_0 \mu_0 \omega^2$.

Proof. Applying the Rot_β operator to both sides of (3) and (4), we obtain (16), (17). Applying the Div_β operator to both sides of (3) and (4) and using (11), we obtain (18), (19). ◻

LEMMA 2.2. *If* $[\mathrm{E}, \mathrm{H}]$ *is a solution of the set* (2.3), (2.4), *then*

$$\mathrm{Div}_\beta \left(\left(n^2 - n_\infty^2\right) \mathrm{E}\right) = n_\infty^2 (\mathrm{E}, n^{-2} \mathrm{grad} n^2), \quad x \in \mathbb{R}^2. \tag{20}$$

Proof. Using (12) leads to

$$\mathrm{Div}_\beta \left(\left(n^2 - n_\infty^2\right) \mathrm{E}\right) = \left(n^2 - n_\infty^2\right) \mathrm{Div}_\beta \mathrm{E} + \left(\mathrm{E}, \mathrm{grad} \left(n^2 - n_\infty^2\right)\right), \quad x \in \mathbb{R}^2. \tag{21}$$

Taking into account (18) and (12), we arrive at

$$-\mathrm{Div}_\beta \mathrm{E} = (\mathrm{E}, n^{-2} \mathrm{grad} n^2), \quad x \in \mathbb{R}^2. \tag{22}$$

Combining (21) and (22), we obtain (20). ◻

Lemma 2.3. *If* [E, H] *is a solution of the set* (2.3), (2.4)*, then*

$$\left[\Delta + \left(k^2 n_\infty^2 - \beta^2\right)\right] \left[\begin{array}{c} \mathrm{E} \\ \mathrm{H} \end{array} \right] = 0, \quad x \in \Omega_\infty. \tag{23}$$

Proof. The function n is equal to a constant n_∞ in the domain Ω_∞. Therefore we obtain (23) from (16)–(19) and (14). ◻

3. Reichardt condition. Because the domain Ω_∞ is unbounded, to have the problem formulation complete we have to specify the behavior of E and H at infinity. This can be done in various ways; for the problem under consideration the most general condition is the Reichardt condition [23], as discussed below. Denote by Ω_R a circle $\Omega_R = \left\{x \in \mathbb{R}^2 : |x| \le R\right\}$, and by Γ_R the boundary of Ω_R.

Definition 3.1. *Let R_0 be a large positive constant such that $\Omega \subset \Omega_{R_0}$. We say that functions* E *and* H *satisfy the Reichardt condition if the functions* E *and* H *can be represented for all $x \in \mathbb{R}^2 \setminus \Omega_{R_0}$ as*

$$\left[\begin{array}{c} \mathrm{E} \\ \mathrm{H} \end{array} \right] = \sum_{l=-\infty}^{\infty} \left[\begin{array}{c} \mathrm{A}_l \\ \mathrm{B}_l \end{array} \right] H_l^{(1)}\left(\chi r\right) \exp\left(il\varphi\right), \tag{24}$$

where $H_l^{(1)}$ is the Hankel function of the first kind and index l (see, e.g., [37]*),* (r, φ) *are the polar coordinates of the point x, and $\chi = \sqrt{k^2 n_\infty^2 - \beta^2}$. The series in* (3.1) *should converge uniformly and absolutely.*

Definition 3.2. *By Λ denote the Riemann surface of the function* $\ln \chi(\beta)$*. A nonzero vector* [E, H] $\in (C^2(\mathbb{R}^2))^6$ *is referred to as a* generalized eigenvector *(or eigenmode) of the problem* (2.3), (2.4)*, and* (3.1) *corresponding to an eigenvalue $\beta \in \Lambda$ if the relations of problem* (2.3), (2.4)*, and* (3.1) *are valid.*

In order to discuss the Reichardt condition in more detail, we need to analyze the Riemann surface Λ and consider the different types of modes that are possible.

3.1. Riemann surface Λ. The Hankel functions $H_l^{(1)}\left(\chi(\beta) r\right)$ are many-valued functions of the variable β. If we want to consider these functions as holomorphic functions, it is seen that β should be considered on the set Λ, which is the Riemann surface of the function $\ln \chi(\beta)$. This is due to the fact that Hankel functions can be represented as

$$H_l^{(1)}\left(\chi r\right) = c_l^{(1)}\left(\chi r\right) \ln\left(\chi r\right) + R_l^{(1)}\left(\chi r\right), \tag{25}$$

where $c_l^{(1)}\left(\chi r\right)$ and $R_l^{(1)}\left(\chi r\right)$ are holomorphic single-valued functions (see, e.g., [37]). The Riemann surface Λ is infinitely sheeted, with each sheet having two branch points, $\beta = \pm k n_\infty$. More precisely, due to the branching of $\chi(\beta)$ itself, we consider an infinite number of logarithmic branches Λ_m, $m = 0, \pm 1, \dots$, each consisting of two square-root sheets of the complex variable β: $\Lambda_m^{(1)}$ and $\Lambda_m^{(2)}$. By $\Lambda_0^{(1)}$ denote the principal ("proper") sheet of Λ, which is specified by the conditions

$$-\pi/2 < \arg \chi(\beta) < \frac{3\pi}{2}, \quad \mathrm{Im}\left(\chi(\beta)\right) \ge 0, \quad \beta \in \Lambda_0^{(1)}. \tag{26}$$

The "improper" sheet $\Lambda_0^{(2)}$ is specified by the conditions

$$-\pi 2 < \arg \chi(\beta) < \frac{3\pi}{2}, \quad \mathrm{Im}\left(\chi(\beta)\right) < 0, \quad \beta \in \Lambda_0^{(2)}. \tag{27}$$

Denote also the whole real axis of $\Lambda_0^{(1)}$ as $R_0^{(1)}$, and that of $\Lambda_0^{(2)}$ as $R_0^{(2)}$. All the other pairs of sheets $\Lambda_{m\neq 0}^{(1),(2)}$ differ from $\Lambda_0^{(1),(2)}$ by a shift in $\arg\chi(\beta)$ equal to $2\pi m$, and satisfy the conditions

$$\begin{aligned} &-\pi/2 + 2\pi m < \arg\chi(\beta) < \frac{3\pi}{2} + 2\pi m, \quad \mathrm{Im}\,(\chi(\beta)) \geq 0, \quad \beta \in \Lambda_m^{(1)}, \\ &-\pi/2 + 2\pi m < \arg\chi(\beta) < \frac{3\pi}{2} + 2\pi m, \quad \mathrm{Im}\,(\chi(\beta)) < 0, \quad \beta \in \Lambda_m^{(2)}. \end{aligned} \tag{28}$$

Hence, on $\Lambda_0^{(1)}$ there is only a pair of branch-cuts dividing it from $\Lambda_0^{(2)}$; they run along the real axis at $|\beta| < kn_\infty$ and along the imaginary axis. On $\Lambda_0^{(2)}$, additionally, there is a pair of branch-cuts dividing it from $\Lambda_{\pm 1}^{(2)}$; they run along the real axis at $|\beta| > kn_\infty$.

3.2. Purely guided, complex, and leaky-wave modes. Denote a set of points on the real axis $R_0^{(1)}$ of the sheet $\Lambda_0^{(1)}$ by G, that is, the union of two intervals:

$$G = \{\beta \in R_0^{(1)} : kn_\infty < |\beta| < kn_+\}. \tag{29}$$

By $C_0^{(1)}$ denote the set

$$C_0^{(1)} = \{\beta \in \Lambda_0^{(1)} : \mathrm{Re}\beta \neq 0\} \setminus R_0^{(1)}. \tag{30}$$

Propagation constants β of purely guided modes, complex modes, and leaky-wave modes belong to sets $G \subset \Lambda_0^{(1)}$, $C_0^{(1)} \subset \Lambda_0^{(1)}$, and $\Lambda_0^{(2)} \setminus R_0^{(2)}$, respectively.

If $-\pi/2 < \arg\chi < 3\pi/2$, then the large-argument asymptotic forms of the Hankel functions of the first kind are known (see, e.g., [37]) to be

$$H_l^{(1)}(\chi r) = \sqrt{\frac{2}{\pi\chi r}} \exp\left[i\left(\chi r - \frac{l\pi}{2} - \frac{\pi}{4}\right)\right]\left[1 + O\left(\frac{1}{\chi r}\right)\right], \quad r \to \infty. \tag{31}$$

Hence, if $-\pi/2 < \arg\chi < 3\pi/2$, $\mathrm{Im}(\chi) \neq 0$, and a function $[\mathrm{E}, \mathrm{H}]$ satisfies the Reichardt condition, then this function satisfies the following condition at infinity:

$$\begin{bmatrix} \mathrm{E} \\ \mathrm{H} \end{bmatrix} = \exp(i\chi r)\, O\left(\frac{1}{\sqrt{r}}\right), \quad r \to \infty. \tag{32}$$

It is easy to see that for purely guided and complex modes, $\mathrm{Im}(\chi) > 0$. Therefore corresponding eigenmodes $[\mathrm{E}, \mathrm{H}]$ decay at infinity as $\exp(-\mathrm{Im}(\chi) r) r^{-1/2}$. Eigenvectors $[\mathrm{E}, \mathrm{H}]$ of leaky-wave modes grow at infinity as $\exp(-\mathrm{Im}(\chi) r) r^{-1/2}$ because $\mathrm{Im}(\chi) < 0$ for them.

3.3. Radiation modes. By D denote the set

$$D = \{\beta \in \Lambda_0^{(1)} : \mathrm{Re}\beta = 0\} \bigcup \{\beta \in R_0^{(1)} : |\beta| < kn_\infty\}. \tag{33}$$

The continuous spectrum of radiation modes belongs to domain D, and each radiation mode can be expressed as (see [1])

$$\begin{bmatrix} \mathrm{E} \\ \mathrm{H} \end{bmatrix} = \sum_{l=-\infty}^{\infty} \begin{bmatrix} \mathrm{A}_l \\ \mathrm{B}_l \end{bmatrix} H_l^{(1)}(\chi r) \exp(il\varphi) + \sum_{l=-\infty}^{\infty} \begin{bmatrix} \mathrm{C}_l \\ \mathrm{D}_l \end{bmatrix} H_l^{(2)}(\chi r) \exp(il\varphi), \tag{34}$$

where $x \in \mathbb{R}^2 \setminus \Omega_{R_0}$ and $H_l^{(2)}$ is the Hankel function of the second kind and index l (see, e.g., [37]).

If $-\pi/2 < \arg \chi < 3\pi/2$, then the large-argument asymptotic forms of the Hankel functions of the second kind are known (see, e.g., [37]) to be

$$H_l^{(2)}(\chi r) = \sqrt{\frac{2}{\pi \chi r}} \exp\left[-i\left(\chi r - \frac{l\pi}{2} - \frac{\pi}{4}\right)\right]\left[1 + O\left(\frac{1}{\chi r}\right)\right], \quad r \to \infty. \tag{35}$$

It is easy to see that for radiation modes $\mathrm{Im}(\chi) = 0$, and that the radiation modes satisfy the following condition at infinity:

$$\begin{bmatrix} \mathrm{E} \\ \mathrm{H} \end{bmatrix} = O\left(\frac{1}{\sqrt{r}}\right), \quad r \to \infty. \tag{36}$$

The Reichardt condition (24) for all functions which satisfy (23) and all $\beta \in D$ is equivalent to the Sommerfeld condition

$$\left(\frac{\partial}{\partial r} - i\chi\right)\begin{bmatrix} \mathrm{E} \\ \mathrm{H} \end{bmatrix} = o\left(\frac{1}{\sqrt{r}}\right), \quad r \to \infty, \tag{37}$$

a fact which was proven in [38]. Therefore, radiation modes do not satisfy the Reichardt condition (24). In section 4 we will prove that the set D is free of the eigenvalues of problem (3), (4), and (24). In section 5, using the Reichardt condition (24), we will reduce problem (3), (4), and (24) to a problem with a purely point spectrum. Therefore, in this work we will not investigate the continuous spectrum of radiation modes.

3.4. Mode notation. The eigenvectors corresponding to the eigenvalues $\beta \in R_0^{(1)}$ such that $|\beta| < kn_\infty$ and satisfying the Sommerfeld condition (37) do not exist in a "passive" DW (i.e., when $\mathrm{Im} n^2 = 0$), which we investigate in this paper. However, if the waveguide is "active," i.e., if $\mathrm{Im} n^2 < 0$, then such modes, radiating to $r \to \infty$ (i.e., satisfying the Sommerfeld condition (37)) and propagating along x_3 without attenuation, may exist. In contrast, the eigenvectors corresponding to the eigenvalues $\beta \in G \subset R_0^{(1)}$ satisfy the condition of exponential decay at infinity. We suggest calling all natural modes generated by the real-axis eigenvalues *eigenmodes*, and, to distinguish between them, calling the first ones *radiating eigenmodes* and the second *guided-wave eigenmodes*. Note, however, that our radiating eigenmodes should not be confused with the "radiation modes" discussed in the previous section. Note that the condition (24) leads to a non–self-adjoint problem in general, which becomes self-adjoint if $\beta \in G$, i.e., for the guided-wave eigenmodes.

If $\beta \in \Lambda_0^{(1),(2)}$ but off $R_0^{(1)}$, then the corresponding modes will be called *quasi eigenmodes*: they consist of the exponentially decaying *"proper" complex quasi eigenmodes* if $\beta \in C_0^{(1)}$, the exponentially growing *leaky-wave quasi eigenmodes* if $\beta \in \Lambda_0^{(2)} \setminus R_0^{(2)}$, and the exponentially growing *"anti-guided" quasi eigenmodes* if $\beta \in R_0^{(2)}$ such that $|\beta| > kn_\infty$.

For all $m \neq 0$, $l = 0, \pm 1, \pm 2, \ldots$, and $\beta \in \bigcup_{m \neq 0} (\Lambda_m^{(1)} \bigcup \Lambda_m^{(2)})$ we have

$$H_l^{(1)}(\chi \exp(i2\pi m) r) = \alpha_l^{(m)} H_l^{(1)}(\chi r) + \gamma_l^{(m)} H_l^{(2)}(\chi r), \qquad \alpha_l^{(m)}, \gamma_l^{(m)} \neq 0. \tag{38}$$

All of the modes whose wavenumbers are located on the higher-order pairs of sheets $\Lambda_{m \neq 0}^{(1),(2)}$ will be collectively called *pseudoeigenmodes* because, according to (31), (35), and (38), they are composed of a sum of incoming and outgoing cylindrical waves. Another justification of this terminology is that all of the possible eigenmodes in a "passive" DW are solutions of a self-adjoint problem, whereas quasi eigenmodes and pseudoeigenmodes satisfy non–self-adjoint problems.

The eigenvalues β on Λ possess a symmetry which is a consequence of equivalency between positive and negative directions along the longitudinal axis x_3 and time t (see [33]). Namely, if β is an eigenvalue and $[\mathrm{E}, \mathrm{H}]$ is a corresponding generalized eigenvector, then $-\beta$ is also an eigenvalue, with the generalized eigenvector given by $[-\mathrm{E}, \mathrm{H}]$. Further, because $\operatorname{Im} \omega = 0$ and $\operatorname{Im} n = 0$, the complex-conjugate numbers $\pm\overline{\beta}$ are eigenvalues as well, with the eigenvectors given by $\left[\mp\overline{\mathrm{E}}, -\overline{\mathrm{H}}\right]$. All these facts can be easily verified by direct substitution into (3), (4), and (24). We shall call the above-mentioned modes *forward*, *backward*, *conjugate*, and *backward-conjugate* modes, respectively.

4. Localization of the eigenvalues.

THEOREM 4.1. *The sets* $B = \{\beta \in R_0^{(1)} : |\beta| \geq kn_+\}$ *and* D *are free of the eigenvalues of problem* (2.3), (2.4), *and* (3.1).

Proof. Suppose that conditions (3), (4), and (24) are satisfied for some $[\mathrm{E}, \mathrm{H}] \in (C^2(\mathbb{R}^2))^6$ and $\beta \in B$. Multiplying both sides of (17) by $\overline{\mathrm{H}}$, integrating over $\mathbb{R}^2$, and using (31), we obtain

$$k^2 \int_{\mathbb{R}^2} |\mathrm{H}|^2 dx = \int_{\mathbb{R}^2} \left(\mathrm{Rot}_\beta \left(\frac{1}{n^2}\mathrm{Rot}_\beta \mathrm{H}\right), \overline{\mathrm{H}}\right) dx \tag{39}$$
$$= \int_{\mathbb{R}^2} \left(\frac{1}{n^2}\mathrm{Rot}_\beta \mathrm{H}, \overline{\mathrm{Rot}_\beta \mathrm{H}}\right) dx$$
$$\geq \frac{1}{n_+^2} \int_{\mathbb{R}^2} \left(\mathrm{Rot}_\beta \mathrm{H}, \overline{\mathrm{Rot}_\beta \mathrm{H}}\right) dx$$
$$= \frac{1}{n_+^2} \int_{\mathbb{R}^2} \left(\mathrm{Rot}_\beta \left(\mathrm{Rot}_\beta \mathrm{H}\right), \overline{\mathrm{H}}\right) dx.$$

Combining this with (19) and (14), we obtain

$$k^2 \int_{\mathbb{R}^2} |\mathrm{H}|^2 dx \geq \frac{1}{n_+^2} \int_{\mathbb{R}^2} \left(-\Delta \mathrm{H} + \beta^2 \mathrm{H}, \overline{\mathrm{H}}\right) dx \tag{40}$$
$$= \frac{1}{n_+^2} \int_{\mathbb{R}^2} |\mathrm{grad}\, \mathrm{H}|^2 dx + \frac{\beta^2}{n_+^2} \int_{\mathbb{R}^2} |\mathrm{H}|^2 dx.$$

Therefore, we have

$$\left(\beta^2 - k^2 n_+^2\right) \int_{\mathbb{R}^2} |\mathrm{H}|^2 dx + \int_{\mathbb{R}^2} |\mathrm{grad}\, \mathrm{H}|^2 dx \leq 0. \tag{41}$$

Hence, if $\beta \in B$ and $|\beta| > kn_+$, then $\mathrm{H} = 0$ for $x \in \mathbb{R}^2$, and

$$\mathrm{E} = \frac{-1}{(i\omega\varepsilon_0 n^2)} \mathrm{Rot}_\beta \mathrm{H} = 0 \tag{42}$$

for $x \in \mathbb{R}^2$. If $\beta \in B$ and $|\beta| = kn_+$, then the function H is equivalent to a constant in $\mathbb{R}^2$, but if H satisfies (24), then it must vanish at infinity for all $\beta \in B$. Therefore, if $\beta \in B$ and $|\beta| = kn_+$, then $\mathrm{H} = 0$ for $x \in \mathbb{R}^2$, and $\mathrm{E} = 0$ for $x \in \mathbb{R}^2$. Therefore the vector $[\mathrm{E}, \mathrm{H}]$ is not an eigenvector of problem (3), (4), and (24) if $\beta \in B$.

Now suppose that conditions (3), (4), and (24) are satisfied for some $[\mathrm{E}, \mathrm{H}] \in (C^2(\mathbb{R}^2))^6$ and $\beta \in D$. Multiplying both sides of (16) by $\overline{\mathrm{E}}$, integrating over Ω_R where $R \geq R_0$, and using (14) and (18), we obtain

$$\text{(43)} \quad k^2 \int_{\Omega_R} n^2 |\mathrm{E}|^2 dx = \int_{\Omega_R} \left(\mathrm{Rot}_\beta \left(\mathrm{Rot}_\beta \mathrm{E}\right), \overline{\mathrm{E}}\right) dx$$

$$= \int_{\Omega_R} \left(-\Delta \mathrm{E} + \beta^2 \mathrm{E} + \mathrm{Grad}_\beta \left(\mathrm{Div}_\beta \mathrm{E}\right), \overline{\mathrm{E}}\right) dx$$

$$= \int_{\Omega_\mathrm{R}} |\mathrm{grad}\, \mathrm{E}|^2 dx - \int_{\Gamma_R} \left(\frac{\partial \mathrm{E}}{\partial |x|}, \overline{\mathrm{E}}\right) dx + \beta^2 \int_{\Omega_\mathrm{R}} |\mathrm{E}|^2 dx$$

$$- \int_{\Omega_R} |\mathrm{Div}_\beta \mathrm{E}|^2 \, dx.$$

For all $\beta \in D$ the number β^2 is real, and therefore we have

$$\text{(44)} \quad \mathrm{Im} \int_{\Gamma_R} \left(\frac{\partial \mathrm{E}}{\partial |x|}, \overline{\mathrm{E}}\right) dx = 0, \quad R \geq R_0.$$

If we combine this with (24), we obtain

$$\text{(45)} \quad 2\pi\chi R \sum_{l=-\infty}^{\infty} \mathrm{Im} \left[H_l^{(2)}(\chi R)\, H_l^{(1)'}(\chi R)\right] |\mathrm{A}_l|^2 = 0, \quad R \geq R_0.$$

We also have

$$\text{(46)} \quad \mathrm{Im} \left[H_l^{(2)}(\chi R)\, H_l^{(1)'}(\chi R)\right] = \frac{2}{\pi \chi R}, \quad l = 0, \pm 1, \pm 2, \dots,$$

which leads to $A_l = 0$ for all l and any $R \geq R_0$. Hence $\mathrm{E} = 0$ for $r \geq R_0$. Under the assumption of the smoothness of the function n, we have $\mathrm{E} = 0$ for $x \in \Omega_{R_0}$ (see [39, p. 190]) and

$$\text{(47)} \quad \mathrm{H} = \frac{1}{(i\omega\mu_0)} \mathrm{Rot}_\beta \mathrm{E} = 0$$

for $x \in \mathbb{R}^2$. Therefore the vector $[\mathrm{E}, \mathrm{H}]$ is not an eigenvector of problem (3), (4), and (24) if $\beta \in D$. The proof of the theorem is complete. □

5. Discreteness and dependence of the eigenvalues on parameters. Now we shall prove that the set of all eigenvalues of problem (3), (4), and (24) can be only a set of isolated points on Λ. We shall also investigate the behavior of eigenvalues β of the problem (3), (4), and (24) as functions of parameters $n_\infty \in \mathbb{R}_+$ and $\omega \in \mathbb{R}_+$, where $\mathbb{R}_+$ is the set of all positive numbers, $\mathbb{R}_+ = \{x > 0\}$. We will use general results of the theory of operator-valued functions [36]. The results in [36] were obtained for operators of the form $I + \mathcal{B}(\beta)$, where I is the identity operator and the operator $\mathcal{B}(\beta)$ is compact for all β. Therefore we shall reduce the problem (3), (4), and (24) to a nonlinear spectral problem with a compact integral operator.

LEMMA 5.1. *Suppose that* $[\mathrm{E},\mathrm{H}]$ *is an eigenvector of the problem* (2.3), (2.4), *and* (3.1) *corresponding to an eigenvalue* $\beta \in \Lambda$. *Then*

$$\mathrm{E}(x) = (\mathcal{B}(\beta)\mathrm{E})\,(x), \quad x \in \mathbb{R}^2, \tag{48}$$

where

$$\begin{aligned}(\mathcal{B}(\beta)\mathrm{E})\,(x) = k^2 \int_\Omega \left(n^2(y) - n_\infty^2\right) \Phi(\beta; x, y)\mathrm{E}(y)dy \\ + \mathrm{Grad}_\beta \int_\Omega (\mathrm{E}, n^{-2}\mathrm{grad}n^2)(y)\Phi(\beta; x, y)dy,\end{aligned} \tag{49}$$

$$\Phi\,(\beta; x, y) = \frac{i}{4} H_0^{(1)}\left(\chi(\beta)\,|x - y|\right). \tag{50}$$

Proof. For all $\beta \in \Lambda$ and $x \in \mathbb{R}^2$ we have

$$\mathrm{E}(x) = \left(k^2 n_\infty^2 + \mathrm{Grad}_\beta \mathrm{Div}_\beta\right) \frac{1}{n_\infty^2} \int_\Omega \left(n^2(y) - n_\infty^2\right) \Phi(\beta; x, y)\mathrm{E}(y)dy. \tag{51}$$

This result is well known for $\beta \in G$ (see, e.g., [40]). The desired assertion for all $\beta \in \Lambda$ is obtained by applying the method of Green functions to the vector Helmholtz equation for the electric field with the use of the relation

$$\int_{\Gamma_R} \left(\frac{\partial \mathrm{E}(y)}{\partial |y|} \Phi(\beta; x, y) - \frac{\partial \Phi(\beta; x, y)}{\partial |y|} \mathrm{E}(y)dy \right) = 0, \quad R \geq R_0, \tag{52}$$

which is valid for any $\beta \in \Lambda$ and an arbitrary function E satisfying the Reichardt condition (24). The validity of relation (52) was proved in [38], [23].

By the supposition of the lemma, $\mathrm{E} \in (C^2(\mathbb{R}^2))^3$. The function n is twice continuously differentiable in $\mathbb{R}^2$ too. Therefore, the following divergence relation is valid:

$$\begin{aligned}\mathrm{Div}_\beta \int_\Omega \left(n^2(y) - n_\infty^2\right) \Phi(\beta; x, y)\mathrm{E}(y)dy \\ = \int_\Omega \mathrm{Div}_\beta \left[\left(n^2(y) - n_\infty^2\right) \mathrm{E}(y)\right] \Phi(\beta; x, y)dy, \quad x \in \mathbb{R}^2.\end{aligned} \tag{53}$$

Taking into account (53) and (20), we obtain the assertion of the lemma. ∎

For all $\beta \in \Lambda$ the operator $\mathcal{B}(\beta)$ determined by (49) will be considered as an operator in the space of complex-valued functions $[L_2(\Omega)]^3$. By definition, set

$$\mathcal{A}(\beta) = I - \mathcal{B}(\beta),$$

where I is the identity operator in $[L_2(\Omega)]^3$. The kernel of the integral operator $\mathcal{B}(\beta)$ is weakly singular for all $\beta \in \Lambda$, and the domain Ω has a Lipschitz boundary. Therefore, the operator $\mathcal{B}(\beta)$ is compact for all $\beta \in \Lambda$ (see, e.g., [41]).

DEFINITION 5.2. *A nonzero vector* $\mathrm{F} \in [L_2(\Omega)]^3$ *is called an eigenvector of an operator-valued function* $\mathcal{A}(\beta)$ *corresponding to an eigenvalue* $\beta \in \Lambda$ *if the relation*

$$\mathcal{A}(\beta)\mathrm{F} = 0 \tag{54}$$

is valid. The set of all $\beta \in \Lambda$ *for which the operator* $\mathcal{A}(\beta)$ *does not have the bounded inverse operator in* $[L_2(\Omega)]^3$ *is called the spectrum of problem* (5.7)

Next we shall prove a theorem on the spectral equivalence of the problem (3), (4), and (24) and the problem (54), but before doing this we consider the following.

DEFINITION 5.3. *A nonzero vector* $u \in C^2(\mathbb{R}^2)$ *is called a* generalized eigenvector *of the problem*

$$[\Delta + (k^2 n^2 - \beta^2)]\, u = 0, \quad x \in \mathbb{R}^2, \tag{55}$$

$$u = \sum_{l=-\infty}^{\infty} a_l H_l^{(1)}(\chi r) \exp(il\varphi) \quad \textit{for all } r \geq R_0 \tag{56}$$

(where the series is supposed to converge uniformly and absolutely), corresponding to an eigenvalue $\beta \in \Lambda$ *if the relations* (5.8) *and* (5.9) *are valid.*

LEMMA 5.4. *The set of all eigenvalues of problem* (5.8) *and* (5.9) *can only be a set of isolated points on* Λ*. The sheet* $\Lambda_0^{(1)}$*, except for the set* G*, is free of the eigenvalues of the problem* (5.8) *and* (5.9)*.*

The proof is found in [42]. Note that the solutions of problem (55) and (56) represent the solutions of the weak-guidance approximation of the original problem (3), (4), and (24).

THEOREM 5.5. *Suppose that* $[\mathrm{E}, \mathrm{H}] \in (C^2(\mathbb{R}^2))^6$ *is an eigenvector of the problem* (2.3)*,* (2.4)*, and* (3.1) *corresponding to an eigenvalue* $\beta_0 \in \Lambda$*. Then* $\mathrm{F} = \mathrm{E} \in [L_2(\Omega)]^3$ *is an eigenvector of the operator-valued function* $\mathcal{A}(\beta)$ *corresponding to the same eigenvalue* β_0*. Suppose that* $\mathrm{F} \in [L_2(\Omega)]^3$ *is an eigenvector of the operator-valued function* $\mathcal{A}(\beta)$ *corresponding to an eigenvalue* $\beta_0 \in \Lambda$*, and also suppose that the same number* β_0 *is not an eigenvalue of the problem* (5.8) *and* (5.9)*. Let* $\mathrm{E} = \mathcal{B}(\beta_0)\mathrm{F}$ *and* $\mathrm{H} = (i\omega\mu_0)^{-1}\mathrm{Rot}_{\beta_0}\mathrm{E}$ *for* $x \in \mathbb{R}^2$*. Then* $[\mathrm{E}, \mathrm{H}] \in (C^2(\mathbb{R}^2))^6$*, and* $[\mathrm{E}, \mathrm{H}]$ *is an eigenvector of the problem* (2.3)*,* (2.4)*, and* (3.1) *corresponding to the same eigenvalue* β_0*.*

Proof. From Lemma 5.1 we obtain the first assertion of the theorem. Now we shall prove the second assertion of the theorem. Suppose that $\mathrm{F} \in [L_2(\Omega)]^3$ is an eigenvector of the operator-valued function $\mathcal{A}(\beta)$ corresponding to an eigenvalue $\beta \in \Lambda$. Assume $\mathrm{E} = \mathcal{B}(\beta)\mathrm{F}$ for $x \in \mathbb{R}^2$. The kernel of the integral operator $\mathcal{B}(\beta)$ is weakly singular for any $\beta \in \Lambda$. By virtue of the well-known property of the integral operator with weakly singular kernel on the domain with a Lipschitz boundary (see, e.g., [41]), we have $\mathrm{E} \in [\mathrm{C}(\overline{\Omega})]^3$. The function n belongs to the space of twice continuously differentiable functions in $\mathbb{R}^2$. By virtue of the well-known properties of the area potential (see, e.g., [41]), we have $\mathrm{E} \in [\mathrm{C}^2(\mathbb{R}^2)]^3$.

Applying the operator Div_β to both sides of (48), and using (10) and (53), we obtain

$$\begin{aligned} \mathrm{Div}_\beta \mathrm{E}(x) = k^2 \int_\Omega \mathrm{Div}_\beta \left[\left(n^2(y) - n_\infty^2\right) \mathrm{E}(y) \right] \Phi(\beta; x, y) dy \\ + \left(\Delta - \beta^2\right) \int_\Omega (\mathrm{E}, n^{-2}\mathrm{grad} n^2)(y) \Phi(\beta; x, y) dy \end{aligned} \tag{57}$$

for all $x \in \mathrm{R}^2$. If we combine this with Poisson's formula

$$\left(\Delta + k^2 n_\infty^2 - \beta^2\right) \int_\Omega \left(n^2(y) - n_\infty^2\right) \Phi(\beta; x, y) \mathrm{E}(y) dy = -\left(n^2(x) - n_\infty^2\right) \mathrm{E}(x), \tag{58}$$

we get

$$\begin{aligned}\mathrm{Div}_\beta \mathrm{E}(x) = k^2 \int_\Omega \mathrm{Div}_\beta \left[\left(n^2(y) - n_\infty^2\right) \mathrm{E}(y)\right] \Phi(\beta; x, y) dy \\ - k^2 n_\infty^2 \int_\Omega (\mathrm{E}, n^{-2} \mathrm{grad} n^2)(y) \Phi(\beta; x, y) dy \\ - (E, n^{-2} \mathrm{grad} n^2)(x)\end{aligned} \tag{59}$$

for all $x \in \mathrm{R}^2$. Using (12), we have

$$\mathrm{Div}_\beta \left[\left(n^2 - n_\infty^2\right) \mathrm{E}\right] = \mathrm{Div}_\beta \left(n^2 \mathrm{E}\right) - n_\infty^2 \mathrm{Div}_\beta \mathrm{E}, \tag{60}$$

$$(\mathrm{E}, n^{-2} \mathrm{grad} n^2) = n^{-2} \mathrm{Div}_\beta \left(n^2 \mathrm{E}\right) - \mathrm{Div}_\beta \mathrm{E}. \tag{61}$$

If we combine this with (59), we see that the function $u = n^{-2} \mathrm{Div}_\beta \left(n^2 \mathrm{E}\right)$ satisfies

$$u = \int_\Omega k^2 \left(n^2(y) - n_\infty^2\right) \Phi(\beta; x, y) u(y) dy, \quad x \in \mathbb{R}^2.$$

If the number β is not an eigenvalue of the problem (55) and (56), then this equation has only the trivial solution (see [42]). Therefore, we have

$$\mathrm{Div}_\beta \left(n^2 \mathrm{E}\right) = 0, \quad x \in \mathbb{R}^2. \tag{62}$$

Using this, (48), and (61), for $x \in \mathbb{R}^2$, we obtain

$$\begin{aligned}\mathrm{E}(x) = k^2 \int_\Omega \left(n^2(y) - n_\infty^2\right) \Phi(\beta; x, y) \mathrm{E}(y) dy \\ - \mathrm{Grad}_\beta \int_\Omega \Phi(\beta; x, y) \mathrm{Div}_\beta \mathrm{E}(y) dy.\end{aligned} \tag{63}$$

Assume $\mathrm{H} = (i\omega\mu_0)^{-1} \mathrm{Rot}_\beta \mathrm{E}$, $x \in \mathbb{R}^2$; i.e., $[\mathrm{E}, \mathrm{H}]$ satisfies (3). Combining (63) and (13), we have

$$\mathrm{H}(x) = -i\omega\varepsilon_0 \mathrm{Rot}_\beta \int_\Omega \left(n^2(y) - n_\infty^2\right) \Phi(\beta; x, y) \mathrm{E}(y) dy, \quad x \in \mathbb{R}^2. \tag{64}$$

Therefore, if $\mathrm{E} \in [\mathrm{C}^2(\mathbb{R}^2)]^3$, then $\mathrm{H} \in [\mathrm{C}^2(\mathbb{R}^2)]^3$.

Now we shall prove that $[\mathrm{E}, \mathrm{H}]$ satisfies (4). Multiplying both sides of (63) by $i\omega\varepsilon_0 n_\infty^2$, applying the operator Rot_β to both sides of (64), and combining the results, we obtain

$$\begin{aligned}\mathrm{Rot}_\beta \mathrm{H} + i\omega\varepsilon_0 n_\infty^2 \mathrm{E} = - i\omega\varepsilon_0 \mathrm{Rot}_\beta \mathrm{Rot}_\beta \int_\Omega \left(n^2(y) - n_\infty^2\right) \Phi(\beta; x, y) \mathrm{E}(y) dy \\ + i\omega\varepsilon_0 n_\infty^2 k^2 \int_\Omega \left(n^2(y) - n_\infty^2\right) \Phi(\beta; x, y) \mathrm{E}(y) dy \\ - i\omega\varepsilon_0 n_\infty^2 \mathrm{Grad}_\beta \int_\Omega \Phi(\beta; x, y) \mathrm{Div}_\beta \mathrm{E}(y) dy\end{aligned} \tag{65}$$

for all $x \in \mathbb{R}^2$. If we combine this with (14) and (53), we obtain

$$\begin{aligned}\mathrm{Rot}_\beta \mathrm{H} + i\omega\varepsilon_0 n_\infty^2 \mathrm{E} = i\omega\varepsilon_0 \left[\Delta + (k^2 n_\infty^2 - \beta^2)\right] \int_\Omega \left(n^2(y) - n_\infty^2\right) \Phi(\beta; x, y) \mathrm{E}(y) dy \\ - i\omega\varepsilon_0 \mathrm{Grad}_\beta \int_\Omega \mathrm{Div}_\beta \left[\left(n^2(y) - n_\infty^2\right) \mathrm{E}(y)\right] \Phi(\beta; x, y) dy \\ - i\omega\varepsilon_0 n_\infty^2 \mathrm{Grad}_\beta \int_\Omega \Phi(\beta; x, y) \mathrm{Div}_\beta \mathrm{E}(y) dy\end{aligned}$$

for all $x \in \mathbb{R}^2$. Using this, (62), and (58), we have

$$\operatorname{Rot}_\beta \mathrm{H} + i\omega\varepsilon_0 n_\infty^2 \mathrm{E} = -i\omega\varepsilon_0 \left(n^2 - n_\infty^2\right) \mathrm{E}, \quad x \in \mathbb{R}^2. \tag{66}$$

Therefore $[\mathrm{E}, \mathrm{H}]$ satisfies (4).

Using the Bessel function addition theorem (see, e.g., [37]), we can readily prove that the number β and the vector $[\mathrm{E}, \mathrm{H}]$ satisfy condition (24). The proof of the theorem is complete. $\square$

THEOREM 5.6. *The set of all eigenvalues of the problem* (2.3)*,* (2.4)*, and* (3.1) *can be only a set of isolated points on* Λ*. Each eigenvalue* β *of the problem* (2.3)*,* (2.4)*, and* (3.1) *depends continuously on* $(\omega, n_\infty) \in \mathbb{R}_+^2$ *and can appear and disappear only at the boundary of* Λ*, i.e., at* $\beta = \pm k n_\infty$ *and at infinity on* Λ*.*

Proof. For any $(x, y) \in \Omega^2$ and any $(\omega, n_\infty) \in \mathbb{R}_+^2$ the kernel of the operator $\mathcal{A}(\beta)$ is analytic in $\beta \in \Lambda$. Hence, the operator-valued function $\mathcal{A}(\beta)$ is holomorphic in $\beta \in \Lambda$ for any $(\omega, n_\infty) \in \mathbb{R}_+^2$. The operator-valued function $\mathcal{A}(\beta; \omega, n_\infty)$ is jointly continuous in $(\beta; \omega, n_\infty) \in \Lambda \times \mathbb{R}_+^2$. For all $(\beta; \omega, n_\infty) \in \Lambda \times \mathbb{R}_+^2$ the operator $\mathcal{B}(\beta; \omega, n_\infty)$ is compact. Therefore, using Theorems 4.1 and 5.5 and Lemma 5.4, we see that the operator $\mathcal{A}(\beta; \omega, n_\infty)$ has a bounded inverse operator in $[L_2(\Omega)]^3$ for all $\beta \in B \bigcup D$ and $(\omega, n_\infty) \in \mathbb{R}_+^2$. Hence, for each $(\omega, n_\infty) \in \mathbb{R}_+^2$ the spectrum of problem (54) can be only a set of isolated points on Λ, which are the eigenvalues of the operator-valued function $\mathcal{A}(\beta)$; each eigenvalue β of the operator-valued function $\mathcal{A}(\beta)$ depends continuously on $(\omega, n_\infty) \in \mathbb{R}_+^2$ and can appear and disappear only at the boundary of Λ, i.e., at $\beta = \pm k n_\infty$ and at infinity on Λ (see [36]). Using Theorem 5.5, we obtain the assertion of the current theorem, which is now complete. $\square$

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