The Riemann boundary value problem on non-rectifiable curves and related questions

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Abstract

We present a brief survey of results on the Riemann boundary value problem for non-rectifiable curves. The solution of the Riemann problem for piecewise smooth curves and arcs was one of well-known achievements of the complex analysis in the first half of the twentieth century. But for non-rectifiable contours the problem was solved only during the latter three decades. In the present paper we consider this solution and certain related questions such as various generalizations of the Cauchy integral over non-rectifiable curves. We describe the technique of investigations, recent results, and open problems.

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1 Introduction.

The Riemann boundary value problem is a classical problem of complex analysis. It has various applications in mechanics of solid media, the theory of elasticity, and other fields.

The main results on this problem for piecewise smooth curves and arcs are presented in well-known monographs of F.D. Gakhov [1] and N.I. Muskhelishvili [2]. These books contain historical review of the problem, its applications and extensive bibliography, which frees the author from the necessity to consider these subjects in detail. But we have to touch the basic points of these researches.

Let Γ be a simple closed curve on the complex plane $\mathbb{C}$ dividing it into domains D^+ and $D^- \ni \infty$. It is required to find a holomorphic in $\overline{\mathbb{C}} \setminus \Gamma$ function $\Phi(z)$ satisfying the equality

$$\Phi^+(t) = G(t)\Phi^-(t) + g(t), \quad t \in \Gamma. \tag{1}$$

Here functions $G(t)$ and $g(t)$ are defined on Γ, and $\Phi^+(t)$ and $\Phi^-(t)$ are limits of $\Phi(z)$ for z tending to a point $t \in \Gamma$ from domains D^+ and D^-, correspondingly. This formulation assumes that the limits exist. Special cases of the Riemann problem are the jump problem

$$\Phi^+(t) - \Phi^-(t) = g(t), \quad t \in \Gamma, \tag{2}$$

and the homogeneous Riemann problem

$$\Phi^+(t) = G(t)\Phi^-(t), \quad t \in \Gamma. \tag{3}$$

The classical solution technique for all these problems is based on the properties of the Cauchy integral

$$\Phi(z) = \frac{1}{2\pi i}\int\limits_{\Gamma} \frac{g(t)dt}{t-z}, \quad z \notin \Gamma. \tag{4}$$

Let g satisfy the Hölder condition

$$\sup\left\{\frac{|g(t') - g(t'')|}{|t' - t''|^\nu} : t', t'' \in \Gamma, t' \neq t''\right\} := h_\nu(g, \Gamma) < \infty \tag{5}$$

with exponent $\nu \in (0, 1]$. We denote by $H_\nu(\Gamma)$ the set of all defined on Γ functions satisfying the Hölder condition (5). If the contour Γ is piecewise smooth and $g \in H_\nu(\Gamma)$, $\nu \in (0, 1]$, then (see, for instance, [1, 2]) function (4) is holomorphic in $\overline{\mathbb{C}} \setminus \Gamma$, has the limits from the left and from the right $\Phi^+(t)$ and $\Phi^-(t)$, correspondingly, and both these functions satisfy the Hölder condition with exponent ν for $\nu \in (0, 1)$ and with any exponent smaller than one for $\nu = 1$. In addition, these limits satisfy formula (2) (it is one of the so called Sokhotskii-Plemelj formulas). Hence, the Cauchy integral with density $g \in H_\nu(\Gamma)$ gives a solution of the jump problem (2) for any $\nu \in (0, 1]$. Problems (3) and (1) are reducible to the jump problem. Thus, the whole theory of the Riemann boundary value problem for piecewise smooth curves is reduced to the application of the cited above result on the boundary behavior of the Cauchy integral.

The boundary behavior of the Cauchy integral over non-smooth rectifiable curves was a subject of investigations for several decades. It was described in 1979 by E.M. Dyn'kin [8, 9] and T. Salimov [10] simultaneously. They proved that the Cauchy integral (4) is continuous in closures of domains D^+ and D^- if $f \in H_\nu(\Gamma)$ for $\nu > \frac{1}{2}$. The boundary values $\Phi^\pm(t)$ satisfy under the latter restriction the Hölder condition with any exponent smaller than $2\nu - 1$. These results are sharp in the following sense:

(i) for any $\nu \in (0, \frac{1}{2})$ there exist a rectifiable curve and a function $f \in H_\nu(\Gamma)$ such that the Cauchy integral (4) loses its continuity at a point $t \in \Gamma$;

(ii) for any $\nu \in (\frac{1}{2}, 1]$ there exist a rectifiable curve and a function $f \in H_\nu(\Gamma)$ such that the boundary values of the Cauchy integral (4) do not satisfy the Hölder condition with exponent $2\nu - 1$ or larger.

Consequently, the jump problem (2) on a non-smooth rectifiable curve Γ with $g \in H_\nu(\Gamma)$ has a solution for $\nu > \frac{1}{2}$, and for $\nu \leq \frac{1}{2}$ it may be unsolvable.

A reader can find a brief bibliography on behavior properties of Cauchy integrals with continuous densities over non-smooth rectifiable curves in the paper [9].

We discern a mismatch between the formulation of the Riemann problem and the classical technique for its solution. Obviously, the boundary condition (1) makes sense for an arbitrary simple curve, but integral (4) is defined only for rectifiable curves. This contradiction was overcame in the early 80's in the papers [3, 4, 5], where a solution of the problem was built without the use of integration over Γ. But recently the authors of works [6, 7] have constructed certain objects (namely, certain distributions with supports on non-rectifiable curves and their Cauchy transforms) which can be considered as extensions of notions of the curvilinear integral and the Cauchy integral on non-rectifiable curves. In the present paper we describe the history of these studies and cite recent results concerning this subject.

In Section 2 we present a method for solving of the Riemann problem on non-rectifiable curves proposed in [3, 4, 5]. As was mentioned above, this method does not use integration over Γ. In Section 3 we consider results concerning integration over non-rectifiable curves including the distributional approach to these questions. The last Section 4 contains a list of open problems.

2 The regularization of quasi-solutions.

Problems (1), (2) and (5) were solved first for non-rectifiable curves and arcs in the following way (see [3, 4, 5]). We build a quasi-solution, i.e., a differentiable in $\mathbb{C} \setminus \Gamma$ function φ satisfying one of these boundary relations and certain additional restrictions, and then regularize it, i.e., turn it by means of special integral-differential operators into a holomorphic in $\overline{\mathbb{C}} \setminus \Gamma$ function $\Phi(z)$ satisfying the corresponding boundary relation. Let us describe this method in detail.

2.1 The jump problem on a closed non-rectifiable curve.

The first solution of problem (2) was obtained by the author of [3, 4, 5]. It is based on the following idea. We assume that the jump $g(t)$ has a differentiable extension $u(z)$ into the domain D^+, i.e., the function $u(z)$ is defined in $\overline{D^+}$, has in D^+ partial derivatives of the first order integrable in D^+ to a power $p > 2$, and $u|_\Gamma = g$. Consider the function

$$\Psi(z) := \frac{1}{2\pi i} \iint\limits_{D^+} \frac{\partial u}{\partial \overline{\zeta}} \frac{d\zeta d\overline{\zeta}}{\zeta - z} \tag{6}$$

The integral operator

$$Tf := \frac{1}{2\pi i} \iint\limits_{\mathbb{C}} \frac{f(\zeta) d\zeta d\overline{\zeta}}{\zeta - z} \tag{7}$$

is well known (see, for instance, [11]). In particular, if f is integrable to a power $p > 2$ and has a compact support, then Tf satisfies the Hölder condition with exponent $1 - \frac{2}{p}$ in the whole complex plane, and $\frac{\partial Tf}{\partial \overline{z}} = f(z), \quad z \in \mathbb{C}$. Hence, the difference $u(z)\chi^+(z) - \Psi(z)$ is a solution of the jump problem (2); here $\chi^+(z)$ is the characteristic function of the domain D^+.

We obtain an analogous result by means of differentiable extension of $g(t)$ into D^-; this extension must have a compact support.

Thus, the solvability of the jump problem is equivalent to the differentiable extendability of the jump into one of domains $D^\pm$ with partial derivatives integrable to a power $p > 2$.

We apply the Whitney extension operator $\mathcal{E}_0$ (see [12], Ch.1, [13], Ch.2) for the set Γ. If $g \in H_\nu(\Gamma)$, then the function $u := \mathcal{E}_0 g$ satisfies the Hölder condition with the same exponent ν in the whole complex plane, $u|_\Gamma = g$, and in $\mathbb{C} \setminus \Gamma$ the function $u(z)$ has partial derivatives satisfying the estimate

$$\left| \frac{\partial^{n+m} \mathcal{E}_0 g(z)}{\partial x^n \partial y^m} \right| \leq \frac{h_\nu(g, \Gamma)}{\mathrm{dist}^{n+m-\nu}(z, \Gamma)}, \quad z = x + iy. \tag{8}$$

In particular, for $\nu = 1$ the first derivatives are bounded, and we obtain the following result on the jump problem for non-rectifiable curves.

Theorem 1 (see [3, 4]). *Let Γ be a simple closed curve of zero area and $g \in H_1(\Gamma)$. Then the function*

$$\Phi(z) := \chi^+(z)\mathcal{E}_0 g(z) - \frac{1}{2\pi i} \iint_{D^+} \frac{\partial \mathcal{E}_0 g}{\partial \overline{\zeta}} \frac{d\zeta d\overline{\zeta}}{\zeta - z} \tag{9}$$

is a solution of the jump problem (2). Its boundary values $\Phi^\pm(t)$ satisfy the Hölder condition on Γ with any exponent smaller than 1.

It remains to find the exponent of integrability of $\frac{\partial u}{\partial \overline{z}}$ for $\nu < 1$. In the works [3, 4, 5] it was estimated in terms of the so called upper metric dimension. This characteristic of sets in a metric space was introduced first by A.N. Kolmogorov and V.M. Tikhomirov [14]. Let S be a compact subset of a metric space X. We consider all coverings of S by balls of diameter $\varepsilon > 0$ and denote by $N(\varepsilon; S)$ the least number of balls in this covering. Then the upper metric dimension of S is

$$\mathrm{dm}\, S := \limsup_{\varepsilon \to 0} \frac{\log N(\varepsilon; S)}{-\log \varepsilon}.$$

In the fractal theory this characteristic is called the Minkowski dimension or the box dimension (see [15]).

If $X = \mathbb{C}$, then the latter definition is equivalent to the following one. Let $\mathcal{Q}_n$ be a partition of $\mathbb{C}$ into non-overlapping dyadic squares with sides 2^{-n}. We denote by $m_n(S)$ the number of all squares from $\mathcal{Q}_n$ having non-empty intersections with S. Then

$$\mathrm{dm}\, S := \limsup_{n \to \infty} \frac{\log_2 m_n(S)}{n}. \tag{10}$$

In the case $X = \mathbb{C}$ we have $\mathrm{dm}\, S \leq 2$. If S is a rectifiable curve, then $\mathrm{dm}\, S = 1$. The upper metric dimension of any continuum on the complex plane is greater than or equal to 1. For $\operatorname{dm} S < 2$ the set S has zero area. Note also that for any set S the upper metric dimension $\operatorname{dm} S$ cannot be lesser than its Hausdorff dimension $\operatorname{dm}_{\mathrm{H}} S$. In the papers [4, 5] the author constructs curves and arcs of the prescribed upper metric dimension.

If a set $S \subset \mathbb{C}$ is compact, then its complement has the so called Whitney partition (see, for instance, [12]). It consists of non-overlapping dyadic squares Q such that

$$C^{-1} \operatorname{diam} Q \leq \operatorname{dist}(Q, S) \leq C \operatorname{diam} Q. \tag{11}$$

Here and in what follows, C stands for various positive constants. This inequality implies that the number $w_n(\Gamma)$ of squares with the side 2^{-n} in the Whitney partition of $\mathbb{C} \setminus \Gamma$ satisfies the estimate $w_n(\Gamma) \leq C m_n(\Gamma)$.

Inequalities (8) and (11) imply that the integral of the function $\left|\frac{\partial \mathcal{E}_0 g}{\partial \bar{\zeta}}\right|^p$ over D^+ does not exceed $C h_\nu(g, \Gamma) \sum\limits_{n=1}^{+\infty} m_n(\Gamma) 2^{-(2-(1-\nu)p)n}$. By the definition of the upper metric dimension we have $m_n \leq 2^{dn}$ for arbitrarily fixed $d > \operatorname{dm} \Gamma$ and sufficiently large n, i.e., the series converges for

$$p < \frac{2 - \operatorname{dm} \Gamma}{1 - \nu}. \tag{12}$$

The right-hand side of the latter bound exceeds 2 for

$$\nu > \frac{1}{2} \operatorname{dm} \Gamma. \tag{13}$$

Thus, it is valid

Theorem 2 (see [3, 4]). *Let Γ be a simple closed curve, $\operatorname{dm} \Gamma < 2$, and $g \in H_\nu(\Gamma)$, $\nu < 1$. If inequality (13) is fulfilled, then function (9) is a solution of the jump problem (2). Its boundary values $\Phi^{\pm}(t)$ satisfy the Hölder condition on Γ with any exponent μ satisfying the inequality*

$$\mu < \frac{2\nu - \operatorname{dm} \Gamma}{2 - \operatorname{dm} \Gamma}. \tag{14}$$

This result is sharp in the following sense.

Theorem 3 (see [3, 4]). *For any values ν and d such that $0 < \nu \leq d/2 < 1$ there exist a simple closed curve and defined on this curve function $g(t)$ such that $\operatorname{dm} \Gamma = d$, $g \in H_\nu(\Gamma)$, and the jump problem (2) has no solution.*

We understand equality (9) as regularization formula for the quasi-solution. The quasi-solution is the function $\phi(z) := \chi^+(z)\mathcal{E}_0 g(z)$. It is continuous and differentiable in $\mathbb{C} \setminus \Gamma$, has the jump $g(t)$ on Γ, and its support is compact. Relation (9) can be rewritten in the form

$$\Phi = (I - T\overline{\partial})\phi,$$

where I is the identity operator, and T is the integral operator (7). Thus, the regularizing operator $I - T\overline{\partial}$ turns the quasi-solution ϕ into the solution Φ. This operator is a projector of the space of continuous in $\overline{D^+}$ and $\overline{D^-}$ functions with integrable Sobolev derivatives onto its subspace consisting of holomorphic in $\overline{\mathbb{C}} \setminus \Gamma$ functions.

Another quasi-solution of the jump problem is the product $-\omega(z)\chi^-(z)\mathcal{E}_0 g(z)$, where $\chi^-(z)$ is the characteristic function of domain D^-, and the smooth function $\omega(z)$ with a compact support equals 1 in a domain containing Γ. Its regularization by the same operator leads to the same solution.

2.2 The uniqueness of the solution.

There is an important difference between the jump problems for rectifiable and non-rectifiable curves. If a curve Γ is rectifiable, then any continuous in a domain $\Delta \supset \Gamma$ and holomorphic in $\Delta \setminus \Gamma$ function is holomorphic in Δ (the Painleve theorem). Consequently, a solution of the jump problem (2) is unique up to an additive constant. But if the Hausdorff dimension of a non-rectifiable curve Γ exceeds 1, then one can find a non-trivial function which is holomorphic in $\overline{\mathbb{C}} \setminus \Gamma$ and continuous in $\overline{\mathbb{C}}$ (see, for instance, [16]). Hence, the difference of two solutions of the jump problem on such a curve may be non-constant.

On the other hand, as was shown by E.P. Dolzhenko [16], if a holomorphic in $\Delta \setminus \Gamma$ function satisfies in a domain $\Delta \supset \Gamma$ the Hölder condition with exponent exceeding $\mathrm{dm_H}\,\Gamma - 1$, then it is holomorphic in Δ. This result allows us to define classes of uniqueness for the jump problem. We denote by $\mathcal{H}_\mu(\Gamma)$ the class of all holomorphic in $\mathbb{C} \setminus \Gamma$ functions $\Phi(z)$ with boundary values Φ^+ and Φ^- belonging to $H_\mu(\Gamma)$. Then restrictions $\Phi|_{D^+}$ and $\Phi|_{D^-}$ satisfy the Hölder condition with exponent μ in the domain D^+ and in any finite part of D^-, correspondingly (see [18]). Hence, a solution of the jump problem in the class $\mathcal{H}_\mu(\Gamma)$ is unique up to an additive constant for $\mu > \mathrm{dm_H}\,\Gamma - 1$. By virtue of theorems 1 and 2 the problem is solvable in this class if either

$$\mathrm{dm_H}\,\Gamma - 1 < \mu < \frac{2\nu - \mathrm{dm}\,\Gamma}{2 - \mathrm{dm}\,\Gamma}, \tag{15}$$

or $\nu = 1$, $\mathrm{dm_H}\,\Gamma < 2$, $\mathrm{dm_H}\,\Gamma - 1 < \mu < 1$.

2.3 The Riemann boundary value problem on a closed non-rectifiable curve.

We consider first the homogeneous problem (3). As usually (see [1, 2]), we assume that the coefficient $G(t)$ does not vanish and belongs to $H_\nu(\Gamma)$. Let $\varkappa := \frac{1}{2\pi}[\arg G]_\Gamma$, where $[\arg G]_\Gamma$ stands for the increment of $\arg G(t)$ at a single circuit of Γ in the positive direction. The value $\varkappa$ is integer. Then we fix a point $z_0 \in D^+$ and represent G as $G(t) = (t - z_0)^\varkappa \exp f(t)$, where $f \in H_\nu(\Gamma)$. The function

$$\Upsilon(z) := \chi^+(z)\mathcal{E}_0 f(z) - \frac{1}{2\pi i}\iint\limits_{D^+} \frac{\partial \mathcal{E}_0 f}{\partial \overline{\zeta}} \frac{d\zeta d\overline{\zeta}}{\zeta - z}$$

is a solution of the jump problem with the jump $f(t)$ satisfying assumptions of theorems 1 or 2. Let $X(z)$ be equal to $\exp \Upsilon(z)$ for $z \in D^+$ and $X(z) = (z - z_0)^{-\varkappa} \exp \Upsilon(z)$ for $z \in D^-$ (in [1, 2] an analogous construction is called canonical function). Then $X^+(t) = G(t)X^-(t), t \in \Gamma$, and $X \in \mathcal{H}_\mu(\Gamma)$, where μ is any value satisfying (14) under assumptions of theorem 2, and any value smaller than 1 under assumptions of theorem 1. If $\Phi(z)$ is a solution of the problem (3) in the class $\mathcal{H}_\mu(\Gamma)$, then the ratio Φ/X is holomorphic in $\mathbb{C}$ under conditions of the previous subsection. Thus, this ratio identically vanishes for negative $\varkappa$, and it is polynomial of degree not greater than $\varkappa$ for $\varkappa \geq 0$.

Now we consider problem (1) with the same coefficient G and $g \in H_\nu(\Gamma)$. It has at least two obvious quasi-solutions: $\phi_1(z) := \chi^+(z)\mathcal{E}_0 g(z)$ and $\phi_2(z) := -\chi^-(z)\omega(z)\mathcal{E}_0(g/G)(z)$, where $\omega(z)$ is the same smooth function as above (see the end of Subsection 1.1). The operator $I - XTX^{-1}\overline{\partial}$ regularizes both these quasi-solutions, and the function

$$\Phi_0 := (I - XTX^{-1}\overline{\partial})\phi$$

is holomorphic in $\mathbb{C} \setminus \Gamma$ and satisfies the boundary condition (1). Here ϕ is one of functions $\phi_{1,2}$. The function Φ_0 is regular at the infinity point for

$\varkappa \geq 0$. Otherwise this point may be a pole. We have

$$\iint\limits_{\mathbb{C}} \frac{\partial \phi}{\partial \overline{\zeta}} \frac{d\zeta d\overline{\zeta}}{X(\zeta)(\zeta - z)} = -\sum_{n=0}^{\infty} \frac{1}{z^{n+1}} \iint\limits_{\mathbb{C}} \frac{\partial \phi}{\partial \overline{\zeta}} \frac{\zeta^n d\zeta d\overline{\zeta}}{X(\zeta)}$$

for sufficiently large $|z|$. Thus, Φ_0 is regular at the infinity point for $\varkappa < 0$ if

$$\iint\limits_{\mathbb{C}} \frac{\partial \phi}{\partial \overline{\zeta}} \frac{\zeta^n d\zeta d\overline{\zeta}}{X(\zeta)} = 0, \quad n = 0, 1, 2, \ldots, -\varkappa - 2. \tag{16}$$

As a result, we obtain

Theorem 4 (see [3, 4]). *Assume that Γ is a simple closed curve, $G, g \in H_\nu(\Gamma)$, $G(t)$ does not vanish on Γ, and we solve problem (1) in the class $\mathcal{H}_\mu(\Gamma)$. Let exponents ν and μ satisfy one of two conditions:*

(a) $\nu = 1$ and $1 > \mu > \mathrm{dm_H}\,\Gamma - 1$;

(b) $\nu \in (0, 1)$ and restrictions (13) and (14) are fulfilled.

Then the following propositions are valid:

(i) for $\varkappa \geq 0$ the problem has a general solution $\Phi = \Phi_0 + XP$, where P is an arbitrary algebraic polynomial of degree no more than $\varkappa$;

(ii) for $\varkappa = -1$ the problem has the unique solution Φ_0;

(iii) for $\varkappa < -1$ the problem is solvable if and only if conditions (16) are fulfilled; in the latter case Φ_0 is its unique solution.

In order words, under assumptions of theorem 4 the solvability of the Riemann boundary value problem for a non-rectifiable closed curve can be described in the same terms as for piecewise smooth curves (see [1, 2]).

The stability of solutions was studied by Liu Hua [17].

2.4 Non-rectifiable arcs.

Let Γ be a simple arc beginning at a point a_1 and ending at a point a_2. In this case we denote by $\Phi^+(t)$ and $\Phi^-(t)$ the limit values of $\Phi(z)$ at a point $t \in \Gamma \setminus \{a_1, a_2\}$ from the left and from the right, correspondingly, and consider boundary conditions (1), (2), and (5) on this arc excluding points a_1 and a_2, where these equalities make no sense. Thus, the Riemann boundary value problem and its special cases for a simple arc turn into the following boundary value problems:

$$\Phi^+(t) = G(t)\Phi^-(t) + g(t), \quad t \in \Gamma \setminus \{a_1, a_2\}, \tag{17}$$

$$\Phi^+(t) - \Phi^-(t) = g(t), \quad t \in \Gamma \setminus \{a_1, a_2\}, \tag{18}$$

$$\Phi^+(t) = G(t)\Phi^-(t), \quad t \in \Gamma \setminus \{a_1, a_2\}. \tag{19}$$

We have to add a certain restriction on the behavior of the desired function at the points a_1 and a_2. The customary restrictions (see [1, 2]) are either boundedness of Φ or its so-called integrability, i.e., validity of the bound

$$|\Phi(z)| \leq C|z - a_j|^{-\gamma}, \quad j = 1, 2, \quad \gamma = \gamma(\Phi) \in (0, 1). \tag{20}$$

Let us construct a quasi-solution for the jump problem (18). We consider a single-valued holomorphic branch of the logarithmic function

$$k_\Gamma(z) := \frac{1}{2\pi i} \ln \frac{z - a_2}{z - a_1}, \tag{21}$$

defined in $\overline{\mathbb{C}} \setminus \Gamma$ by the condition $k_\Gamma(\infty) = 0$. It has the unit jump on the arc Γ, and, consequently, the product $\phi(z) = \omega(z)k_\Gamma(z)\mathcal{E}_0 g(z)$ is a quasi-solution. As above, ω is a smooth function with a compact support such that $\omega|_\Gamma = 1$. The function ϕ has a jump g on Γ, but orders of its singularities at points a_1 and a_2 may be high. As a result, we obtain additional restrictions. For instance, if $k_\Gamma(z)$ is square integrable near points a_j, $j = 1, 2$, and condition (13) is fulfilled, then the function $\Phi(z) = (I - T\overline{\partial})\phi(z)$ satisfies condition (20) by virtue of well-known properties of the operator T. We also have to modify the definition of the uniqueness class $\mathcal{H}_\mu$. If Γ is an arc, then the class $\mathcal{H}_\mu(\Gamma)$ consists of all holomorphic in $\mathbb{C} \setminus \Gamma$ functions $\Phi(z)$ such that limit values $\Phi^+(t)$ and $\Phi^-(t)$ exist for any $t \in \Gamma \setminus \{a_1, a_2\}$ and satisfy the Hölder condition with exponent μ on the set $\Gamma \setminus \bigcup_{j=1}^{2} \{z : |z - a_j| < \epsilon\}$ for any $\epsilon > 0$.

Theorem 5 (cf. [3, 5]). *Let Γ be a simple arc of zero area, and let the function k_Γ be square integrable near the ends of Γ, and $g \in H_\nu(\Gamma)$. If $\nu = 1$ or $\nu > \frac{1}{2}\operatorname{dm}\Gamma$, then the function*

$$\Phi(z) := \omega(z)k_\Gamma(z)\mathcal{E}_0 g(z) - \frac{1}{2\pi i} \iint_{\mathbb{C}} \frac{\partial \omega \mathcal{E}_0 g}{\partial \overline{\zeta}} \frac{k_\Gamma(\zeta) d\zeta d\overline{\zeta}}{\zeta - z} \tag{22}$$

is a solution of the jump problem (18) in the class (20). It is a unique (up to additive constant) solution of the problem in the class $\mathcal{H}_\mu(\Gamma)$ if either $\nu = 1$ and $1 > \mu > \operatorname{dm}_{\mathrm{H}}\Gamma - 1$ or $1 > \nu > \frac{1}{2}\operatorname{dm}\Gamma$ and μ satisfies (15).

Below we solve problems (19) and (17) under restriction

$$k_\Gamma(z) = O(\ln|z-a_j|^{-1}), \quad z \to a_j, \quad j = 1,2. \tag{23}$$

Let $G \in H_\nu(\Gamma)$ and $G(t) \neq 0$ for $t \in \Gamma$. Then $f(t) := \ln G(t) \in H_\nu(\Gamma)$, and under assumptions of the latter theorem and restriction (23) the function

$$\Upsilon(z) := \omega(z)k_\Gamma(z)\mathcal{E}_0 f(z) - \frac{1}{2\pi i}\iint_{\mathbb{C}} \frac{\partial \omega \mathcal{E}_0 f}{\partial \overline{\zeta}} \frac{k_\Gamma(\zeta)d\zeta d\overline{\zeta}}{\zeta - z}$$

satisfies estimates $\Upsilon(z) = f(a_j)k_\Gamma(z) + O(1)$ at points $a_{1,2}$. We put $f(t) = u(t) + iv(t)$. Here $u(t) = \ln|G(t)|$ and $v(t)$ is a fixed single-valued branch of $\arg G(t)$. Then $\operatorname{Re}\Upsilon(z) = (-1)^j(2\pi)^{-1}(u(a_j)\arg(z-a_j) + v(a_j)\ln|z-a_j|) + O(1)$ at a_j, $j = 1,2$, and the function

$$X(z) := (z-a_1)^{-\varkappa_1}(z-a_2)^{-\varkappa_2}\exp\Upsilon(z)$$

satisfies bound (20) for

$$\varkappa_j = 1 + \left]\frac{(-1)^j v(a_j)}{2\pi} + \liminf_{z\to a_j}\frac{(-1)^j u(a_j)\arg(z-a_j)}{2\pi\ln|z-a_j|}\right[, \quad j = 1,2,$$

where $]x[:= \sup\{n \in \mathbb{Z} : n < x\}$, and $\arg(z-a_j)$ is a single-valued branch of the argument defined by means of a cut along Γ. Consequently, the general solution of problem (19), (20) in the class $\mathcal{H}_\mu(\Gamma)$ for $\varkappa := \varkappa_1 + \varkappa_2 \geq 0$ is $\Phi(z) = X(z)P(z)$, where P is an arbitrary algebraic polynomial of degree no more than $\varkappa$. For $\varkappa < 0$ the problem has no non-trivial solutions.

Now let us consider problem (17). Here we are faced with two new obstacles. The first one is related to the construction of quasi-solutions. For instance, we can build a quasi-solution ϕ by putting it continuous on Γ. The equality $\phi^+(t) = \phi^-(t)$ implies $\phi(t) = g(t)/(1-G(t))$, $t \in \Gamma$, i.e., the function $\phi(z) = \omega(z)\mathcal{E}_0(\frac{g}{1-G})(z)$ is a quasi-solution of problem (17) under the additional restriction $G(t) \neq 1$, $t \in \Gamma$. But this restriction is not necessary (see [19]). The second obstacle concerns the regularizing operator $I - XTX^{-1}\overline{\partial}$. Even under restriction (23) orders of singularities of the function X^{-1} at the points $a_{1,2}$ can be high. To exclude this possibility, we assume the existence of finite limits

$$\lim_{z\to a_j}\frac{\arg(z-a_j)}{\ln|z-a_j|}, \quad j = 1,2,$$

for the branch of $\arg(z - a_j)$ defined by means of a cut including the arc Γ. Under this assumption we obtain a full analog of theorem 4.

Note that the index $\varkappa$ for the problem on a non-rectifiable arc depends not only on $\arg G$, but also on the geometry of the arc, too. This fact is also valid for non-smooth rectifiable arcs (see [20]).

2.5 A semi-continuous version of the problem.

In this subsection we seek for solutions of the Riemann boundary value problem on a non-rectifiable curve with discontinuities at several points of the curve. This version of the Riemann problem for a piecewise-smooth curve is called semi-continuous (see [1, 2]).

Let us consider a closed simple curve Γ and a finite set $E = \{a_1, a_2, \dots, a_m\} \subset \Gamma$. We seek for a holomorphic in $\overline{\mathbb{C}} \setminus \Gamma$ function $\Phi(z)$, which has boundary values $\Phi^+(t)$ and $\Phi^-(t)$ from D^+ and D^-, correspondingly, at any point $t \in \Gamma \setminus E$ such that

$$\Phi^+(t) = G(t)\Phi^-(t) + g(t), \quad t \in \Gamma \setminus E \tag{24}$$

and

$$|\Phi(z)| \le C|z - a|^{-\gamma}, \quad a \in E, \quad \gamma = \gamma(\Phi) \in (0, 1). \tag{25}$$

Coefficients G and g also allow singularities at points of the set E. Here we restrict ourselves by situation where G has no discontinuity, and g has simple singularities of the power type. It was investigated first in [29].

Let $w(t) := \prod\limits_{a \in E} |t - a|^{p(a)}$, $0 < p(a) < 1$. We put $H_\nu(\Gamma, w) := \{f : wf \in H_\nu(\Gamma)\}$ and assume that $g \in H_\nu(\Gamma, w)$. Then $\phi(z) := \chi^+(z)w^{-1}(z)\mathcal{E}_0(wg)(z)$ is a quasi-solution for the Riemann boundary value problem (24) and for the corresponding jump problem. As a result, we obtain analogs of all results of items 2.1 and 2.3.

2.6 Certain refinements of characteristics of contours.

J. Harrison and A. Norton [21] introduced the notion of the $d-$summability. Then R. Abreu Blaya, J. Bory Reyes, and J. Marie Vilaire [22] proved that this notion is applicable for the proof of the solvability of the jump problem.

A compact set S is called $d-$summable if

$$\int\limits_0 N(\varepsilon; S)\varepsilon^{d-1} d\varepsilon < \infty.$$

As above, $N(\varepsilon; S)$ stands for the least number of disks of diameter ε covering S. If a curve Γ is $d-$summable then $\operatorname{dm}\Gamma \leq d$, and if $\operatorname{dm}\Gamma < d$, then Γ is $d-$summable (see [21]). As is proved in [22], the jump problem (2) with $g \in H_\nu(\Gamma)$ is solvable if Γ is $d-$summable and $\nu > d/2$. Later R. Abreu Blaya, J. Bory Reyes and T. Moreno Garcia [23] established that the latter condition is sharp in the same sense as condition (13).

Another refinement is connected with the replacement of the upper metric dimension by certain characteristic of partitions of the set $\mathbb{C} \setminus \Gamma$. We use this dimension only for characterizing of the integrability over $\mathbb{C} \setminus \Gamma$ or its parts. Seemingly, the first characterization of this kind was introduced by J. Harrison and A. Norton [21]. They defined the so-called $d-$mass. Let $\mathbf{Q} = \{Q_1, Q_2, \dots\}$ be a partition of the domain D^+ into non-overlapping squares. If $a(Q_j)$ stands for the length of the side of the square Q_j, then the $d-$mass of the partition is the sum $\mathcal{M}_d(\mathbf{Q}) = \sum\limits_{Q_j \in \mathbf{Q}} a^d(Q_j)$ (it may be infinite). The $d-$mass of the domain D^+ is the greatest lower bound for $d-$masses of all square partitions of D^+. If D^+ has a partition of a finite $d-$mass, then the Whitney partition of D^+ also has a finite $d-$mass (see the Peter Jones lemma in [24]). In other words, if the $d-$mass of D^+ is finite, then the series $\sum\limits_{n=0}^{\infty} 2^{-nd} w_n$ converges, and the jump problem has a solution for $\nu > d/2$. In this connection, in [25, 26, 27] the author introduced other metric characteristics of non-rectifiable curves, which were based on partitions of D^+ and D^- into domains with rectifiable boundaries. If B is a domain with a rectifiable boundary, then we denote by $\lambda(B)$ the length of its boundary, and $w(B)$ is the greatest diameter of disks lying inside it. Let $\mathbf{B} = \{B_1, B_2, \dots\}$ be a partition of D^+ into non-overlapping domains with rectifiable boundaries. We put $\mathcal{M}^*_d(\mathbf{B}) := \sum\limits_{B_j \in \mathbf{B}} \lambda(B_j) w^{d-1}(B_j)$ and $\mathcal{M}^*_d(D^+) := \inf\{\mathcal{M}^*_d(\mathbf{B})\}$, where the least upper bound is taken for all rectifiable partitions. Finally, $\operatorname{dma}^+\Gamma := \inf\{d : \mathcal{M}^*_d(D^+) < \infty\}$. The value $\operatorname{dma}^-\Gamma$ is defined analogously, and the approximation dimension $\operatorname{dma}\Gamma$ equals the lesser of values $\operatorname{dma}^+\Gamma$ and $\operatorname{dma}^-\Gamma$. The refined metric dimension $\operatorname{dmr}\Gamma$ is defined in an analogous way, but the rectifiable partitions are replaced here with rectifiable chains, i.e., with sequences of domains with rectifiable boundaries $\mathbf{B} = \{B_1, B_2, \dots, B_n, \dots\}$ such that the symbols $\cup$ and $\setminus$ can be placed between members of the sequence $\mathbf{B}$ so that the limit result of these operations is D^+ or D^-. Theorem 2 keeps validity if we replace $\operatorname{dm}\Gamma$ in condition (13) with one of these dimensions. Additionally, both these dimensions do not exceed $\mathrm{dm}\,\Gamma$, and they are strictly less than $\mathrm{dm}\,\Gamma$ for certain curves. Hence, the replacement of $\mathrm{dm}\,\Gamma$ with $\mathrm{dma}\,\Gamma$ or $\mathrm{dmr}\,\Gamma$ improves theorem 2. But we do not know the exact values of $\mathrm{dma}\,\Gamma$ or $\mathrm{dmr}\,\Gamma$ for any non-trivial curve Γ. In 2013 D.B. Kats [30] introduced a new characteristic of non-rectifiable curves which also sharpens condition (13) and can be calculated for some non-rectifiable curves.

2.7 Other classes of functions.

A similar technique is applicable for solving boundary value problems on non-rectifiable boundaries and for the investigation of related questions for Clifford analysis, multidimensional complex analysis, theories of hyper-analytical and $\beta-$analytical functions, and for other generalizations of the theory of holomorphic functions (see [6, 22, 31, 32, 33, 34, 35, 36, 37] et al).

3 Integration over non-rectifiable curves.

In the preceding section we solved the Riemann boundary value problem on non-rectifiable curves without application of curvilinear integrals. But the representation of its solutions in terms of integrals is of interest. Moreover, in what follows we will see that the jump problem and the problem of generalization of curvilinear integrals for non-rectifiable curves are connected. We consider here certain versions of integration over non-rectifiable curves, which are suitable for representations of solutions.

In addition, integrals over non-rectifiable fractal curves and surfaces are of interest for the theory of elasticity (see, for instance, [38]).

3.1 The Cauchy-Stieltjes integral.

At first glance, the curvilinear integral $\int_\Gamma f(z)dz$ makes no sense for non-rectifiable Γ. But it is not so. Let $z = z(t) : [0,1] \to \Gamma$ be an one-to-one mapping of the segment $[0,1]$ onto the curve Γ. Then $\int_\Gamma f(z)dz = \int_0^1 f(z(t))dz(t)$, and the last term in this equality can be understood as the Stieltjes integral. The best known condition for the existence of this integral includes the boundedness of the variation of function $z(t)$, which implies the rectifiability of Γ. But the Stieltjes integral $\int f(t)dg(t)$ exists not only for functions of bounded variations, but for wider classes, namely, for functions of bounded $\Phi-$variation (see, for instance, [39]). In this connection B.A. Kats ([40, 41, 42]) has introduced the class of $\Phi-$rectifiable curves. Let $\Phi(x)$ be a given increasing function defined for $x \geq 0$, $\Phi(0) = 0$. A curve Γ is called $\Phi-$rectifiable, if

$$\sum_{j=1}^{\infty} \Phi(|t_j - t_{j-1}|) < C$$

for any sequence $\{t_0, t_1, \ldots, t_n\}$ of points of the curve γ enumerated in order of the positive direction of Γ, and the positive constant C is independent of the sequence. If $\Phi(x) = x^q, q > 1$, then the curve is called $q-$rectifiable. The class of $q-$rectifiable curves contains non-rectifiable ones, for instance, it contains the Von Koch snowflake for appropriate q. The author studied the Cauchy-Stieltjes integral

$$\mathcal{CS}_\Gamma f(z) = \frac{1}{2\pi i} \int\limits_\Gamma f(\xi) d\ln(\xi - z), \quad \xi \in \Gamma, z \notin \Gamma,$$

and has proved the following theorems.

Theorem 6 *Let Γ be a closed q-rectifiable curve, $f \in H_\nu(\Gamma)$, $\nu > q-1$ and $\nu > \frac{\mathrm{dm}\,\Gamma}{2}$. Then the Cauchy-Stieltjies integral $\mathcal{CS}_\Gamma f(z)$ exists and has continuous limit values on Γ satisfying the relation*

$$\mathcal{CS}_\Gamma^+ f(t) - \mathcal{CS}_\Gamma^- f(t) = f(t).$$

Theorem 7 *If a curve Γ is Φ-rectifiable for a convex function $\Phi(x)$ and $f \in H_1(\Gamma)$, then the Cauchy-Stieltjes integral exists and has continuous limit values satisfying the same boundary value condition under the restriction*

$$\sum_{n=1}^{\infty} \phi^2(\frac{1}{n}) < \infty,$$

where ϕ is the inverse function for Φ.

Obviously, these results are applicable for solving the Riemann boundary value problem on non-rectifiable curve. We obtain integral representations for its solutions in terms of the Stieltjes integrals.

3.2 Approximative integral.

Let a function $u(z)$ be continuous in the closure of a finite domain D^+ bounded by a rectifiable curve Γ. If u has integrable in D^+ derivatives of the first order, then by virtue of the Green formula we have

$$\int\limits_{\Gamma} u(\zeta)d\zeta = -\iint\limits_{D^+} \frac{\partial u}{\partial \overline{\zeta}} d\zeta d\overline{\zeta}. \tag{26}$$

If Γ is non-rectifiable, then we understand the right-hand side of equality (26) as the definition of the left-hand one, i.e., a defined on the closed non-rectifiable curve Γ continuous function $f(t)$ is integrable over this curve if it has an extension $u(z)$ into the domain D^+ with integrable derivatives, and the integral $\int\limits_{\Gamma} f(t)dt$ equals the right-hand side of equality (26). Seemingly, this approach was formulated first in the note [43]. Then it was investigated in the papers [44, 24, 45, 28], etc.

The first question arising here is the independence of the integral on the choice of extension. The following result is proved in various formulations in all above mentioned papers: if $f \in H_\nu(\Gamma)$ and $\nu > \operatorname{dm}\Gamma - 1$, then f has an extension $u(z) \in H_\nu(\overline{D^+})$ with integrable derivatives in D^+. If u_1 and u_2 are two differentiable extensions of f into D^+ such that $u_{1,2}(z) \in H_\nu(\overline{D^+})$, then

$$\iint\limits_{D^+} \frac{\partial u_1}{\partial \overline{\zeta}} d\zeta d\overline{\zeta} = \iint\limits_{D^+} \frac{\partial u_2}{\partial \overline{\zeta}} d\zeta d\overline{\zeta}.$$

In other words, the value of this integral is independent of the choice of the extension within the class H_ν.

The same integral can be defined in another way. Let $\{\Gamma_1, \Gamma_2, \dots, \Gamma_n, \dots\}$ be a sequence of simple closed polygonal lines bounding polygonal domains D_n^+, $n = 1, 2, \dots\}$ such that $D_1^+ \subset D_2^+ \subset \dots \subset D_n^+ \subset \dots \subset D^+$ and $\bigcup\limits_{n=1}^{+\infty} D_n^+ = D^+$. Then for $f \in H_\nu(\Gamma)$ and $\nu > \operatorname{dm}\Gamma - 1$ we have

$$\lim_{n\to+\infty} \int\limits_{\Gamma_n} u(t)dt = -\lim_{n\to+\infty} \iint\limits_{D_n^+} \frac{\partial u}{\partial \overline{\zeta}} d\zeta d\overline{\zeta} = -\iint\limits_{D^+} \frac{\partial u}{\partial \overline{\zeta}} d\zeta d\overline{\zeta},$$

i.e., the integral under consideration is the limit of integrals of expansion of the integrand over polygonal lines approximating Γ. In this connection, we call it approximative integral. The idea of the approximation by polygonal constructions was essentially developed by J. Harrison [28].

The approximative integral is applicable for the representation of solutions of the Riemann boundary value problem on non-rectifiable curves. In particular, the right-hand side of (9) equals the approximative Cauchy integral $\frac{1}{2\pi i}\int_\Gamma \frac{g(t)dt}{t-z}$.

3.3 The distributional approach.

This approach was proposed recently by the author (see, for instance, [6] and [7]). Let us present its scheme.

We identify any function $F(\zeta)$ on the complex plane with the distribution

$$F : C_0^\infty(\mathbb{C}) \ni \omega \mapsto \iint\limits_{\mathbb{C}} F(\zeta)\omega(\zeta)d\zeta d\overline{\zeta},$$

if the latter integral makes a sense. Let F be holomorphic in $\mathbb{C}\setminus\Gamma$. Then the support of its distributional derivative $\overline{\partial}F$ is compact, because it is a subset of the curve Γ. Let us assume that F has limit values from both sides at any point of Γ. If Γ is rectifiable, then the derivative $\overline{\partial}F$ is representable in the form

$$\langle\overline{\partial}F,\omega\rangle = \int\limits_\Gamma (F^+(\zeta)-F^-(\zeta))\omega(\zeta)d\zeta, \quad \omega\in C^\infty(\mathbb{C}).$$

For non-rectifiable Γ we consider this distribution as a generalized integration over Γ with the weight $F^+(\zeta)-F^-(\zeta)$. The integration with the unit weight corresponds to functions F with unit jump on Γ. For instance, to this end we can use the characteristic function $\chi^+(z)$ of domain D^+ which equals 1 in D^+ and 0 in D^- for a closed curve Γ, or $k_\Gamma(z)$ for an arc Γ.

Then we consider a functional Banach space $\mathfrak{X}$ containing restrictions of functions $\omega\in C^\infty(\mathbb{C})$ onto a neighborhood of Γ. Let $\mathfrak{X}^*$ be the closure of the set of these restrictions in $\mathfrak{X}$. We assume that $\mathfrak{X}^*C^\infty(\mathbb{C})=\mathfrak{X}^*$, i.e., $f\omega\in\mathfrak{X}^*$ for any $\omega\in C^\infty(\mathbb{C}), f\in\mathfrak{X}^*$.

If for any $\omega\in C^\infty(\mathbb{C})$

$$|\langle\overline{\partial}F,\omega\rangle| \le C\|\omega\|_{\mathfrak{X}}, \tag{27}$$

where $\|\cdot\|_{\mathfrak{X}}$ is the norm of $\mathfrak{X}$ and C is a positive constant, then $\overline{\partial}F$ is continuable up to a functional on $\mathfrak{X}^*$ and generates a family of distributions

$$\langle f\overline{\partial}F,\omega\rangle := \overline{\partial}F(f\omega), \tag{28}$$

where f is an arbitrarily fixed function from $\mathfrak{X}^*$. Here we keep the notation $\overline{\partial}F$ for the above mentioned functional. We consider these distributions as integrations over Γ with weights $f(\zeta)(F^+(\zeta) - F^-(\zeta))$. This scheme of integration is dual to the approximative integral from the previous subsection: here we replace the approximation of a curve with the approximation of integrand.

This scheme is realized for the Hölder spaces as $\mathfrak{X}$. As usually, a norm in the Hölder space $H_\nu(B)$ on a compact set $B \subset \mathbb{C}$ equals $\|f\|_\nu := h_\nu(f, B) + \sup\{|f(z)| : z \in B\}$ (see (5)).

Theorem 8 (see [7]). *Let Ω be a finite domain such that $\Gamma \subset \Omega$, and $1 \geq \nu > \operatorname{dm}\Gamma - 1$. If a holomorphic in $\mathbb{C} \setminus \Gamma$ function F is integrable to a power p over Ω for any $p \geq 1$, then inequality (27) is valid for $\mathfrak{X} = H_\nu(\overline{\Omega})$.*

Thus, for $\nu > \operatorname{dm}\Gamma - 1$ the distribution $\overline{\partial}F$ is extendable up to continuous functional on $H_\nu(\overline{\Omega})$, which allows us to define distribution (28) for every $f \in H_\nu(\overline{\Omega})$. As is shown in [6], if $f, g \in H_\nu(\overline{\Omega})$ and $f|_\Gamma = g|_\Gamma$, then $f\overline{\partial}F = g\overline{\partial}F$. Hence, the distribution $f\overline{\partial}F$ is defined, in fact, for any $f \in H_\nu(\Gamma)$.

In order to apply these distributions for solving the Riemann boundary value problem, we study their Cauchy transforms. If ϕ is a distribution with a compact support S on the complex plane, then its Cauchy transform is

$$\mathcal{C}\mathrm{au}\,\phi := \frac{1}{2\pi i}\left\langle \phi, \frac{1}{\zeta - z}\right\rangle,$$

where $z \notin S$, and ϕ is applied to the Cauchy kernel $\frac{1}{2\pi i(\zeta - z)}$ as to a function of variable ζ. In particular, the Cauchy integral with density f over a rectifiable curve Γ is the Cauchy transform of the distribution

$$C^\infty(\mathbb{C}) \ni \omega \mapsto \int_\Gamma f(t)\omega(t)dt.$$

Theorem 9 (see [7]). *Let $F(z)$ be a holomorphic in $\overline{\mathbf{C}} \setminus \Gamma$ and continuous in $\overline{D^+}$ and $\overline{D^-}$ function, and $F(\infty) = 0$. If $g \in H_\nu(\Gamma)$ and $\nu > \frac{1}{2}\operatorname{dm}\Gamma$, then the function $\Phi(z) := \mathcal{C}\mathrm{au}\, g\overline{\partial}F(z)$ is holomorphic in $\overline{\mathbf{C}} \setminus \Gamma$, continuous in $\overline{D^+}$ and $\overline{D^-}$, $\Phi(\infty) = 0$, and*

$$\Phi^+(t) - \Phi^-(t) = g(t)(F^+(t) - F^-(t)), \quad t \in \Gamma.$$

Clearly, this result allows us to solve all versions of the Riemann boundary value problem on non-rectifiable curves in terms of the Cauchy transforms. Let us note the following proposition.

Corollary 1 *Let $J(\Gamma)$ be the set of all continuous functions g such that the jump problem on the curve Γ with a jump g has a solution. If $\nu > \frac{1}{2}\operatorname{dm}\Gamma$, then $fg \in J(\Gamma)$ for any $g \in J(\Gamma)$ and $f \in H_\nu(\Gamma)$, i.e., $J(\Gamma)H_\nu(\Gamma) = J(\Gamma)$.*

The latter two theorems and the corollary can be sharpened by the replacement of the upper metric dimension $\operatorname{dm}\Gamma$ with either the approximation dimension $\operatorname{dma}\Gamma$ or the refined metric dimension $\operatorname{dmr}\Gamma$.

Let us note that the Cauchy transform $\mathcal{C}\mathrm{au}\,\phi$ is the convolution of ϕ and the fundamental solution of the $\overline{\partial}-$equation. Consequently, the jump problem and the problem of integration over non-rectifiable curve are equivalent in a certain sense.

4 Open questions

4.1. In 1958 V.P. Havin [46] obtained the following result. Let μ be an essential measure on a bounded, closed and connected set Γ in the complex plane. A positive measure is called essential on Γ if any its null set $E \subset \Gamma$ satisfies the condition $\overline{\Gamma \setminus E} = \Gamma$. Then any holomorphic in $\overline{\mathbb{C}} \setminus \Gamma$ function is representable as

$$\Phi(z) = const + \sum_{k=0}^{\infty} \int_{\Gamma} \frac{Y_k(t)dt}{(t-z)^{k+1}},$$

where $\int_{\Gamma} |Y_k(t)|^2 d\mu < +\infty$ for any k, and $\lim_{k\to+\infty} \left(\int_{\Gamma} |Y_k(t)|^2 d\mu\right)^{1/k} = 0$. V.P. Havin also obtained a criterion for the finiteness of the sum in this representation.

If Γ is a rectifiable curve, μ is its length, and $\Phi(z)$ is a solution of the jump problem for Γ, then the Havin representation for Φ contains only one term. It is of interest to find this representation for a solution Φ of the jump problem (2) for a given essential measure μ.

4.2 The latter two theorems imply that for $g \in H_\nu(\Gamma)$, $\frac{1}{2}\operatorname{dm}\Gamma > \nu > \operatorname{dm}\Gamma - 1$, the Cauchy transform $\Phi(z) = \mathcal{C}\mathrm{au}\, g\overline{\partial}\chi^+(z)$ exists, but it can be discontinuous on Γ. The problem is to describe its boundary behavior.

4.3 The author assumes that if Γ is a self-similar non-rectifiable curve, $f \in H_\nu(\Gamma)$, and $1 > \nu > \operatorname{dm}\Gamma - 1$, then the Cauchy transform $\mathcal{C}\mathrm{au}\, g\overline{\partial}\chi^+(z)$ is continuous in $\overline{D^+}$ and in $\overline{D^-}$, and its boundary values on Γ from the left and from the right satisfy the Hölder condition with any exponent $\mu < \nu$.

This assumption means that the self-similarity of a non-rectifiable curve improves the boundary properties of the Cauchy transform in almost the same degree as the smoothness of a rectifiable curve improves the boundary behavior of the Cauchy integral.

4.4 We can identify the boundary values $\Phi^+(t)$ and $\Phi^-(t)$ of a holomorphic in $\overline{\mathbb{C}} \setminus \Gamma$ function Φ with distributions $\overline{\partial}(\chi^+\Phi)$ and $-\overline{\partial}(\chi^-\Phi)$, where χ^+ and χ^- are characteristic functions of D^+ and D^-. In this connection, it is of interest to study the distributional Riemann problem

$$\overline{\partial}(\chi^+\Phi) + G\overline{\partial}(\chi^-\Phi) = g,$$

where G and g are given distributions such that the product $G\overline{\partial}(\chi^-\Phi)$ is defined.

A distributional version of the Riemann boundary value problem was investigated earlier (see, for instance, [47]), but only for the real axis and its segments.

4.5 There exists a lot of open questions concerning the Riemann boundary value problem on non-rectifiable arcs. Here we formulate one of them. Let an arc Γ satisfy restriction (23), and

$$\vartheta_j := \limsup_{z \to a_j} \frac{\operatorname{Re} k_\Gamma(z)}{\ln|z - a_j|} - \liminf_{z \to a_j} \frac{\operatorname{Re} k_\Gamma(z)}{\ln|z - a_j|}.$$

Assume that $\ln G(t)$ is a restriction on Γ of a function, which has partial derivative of order $[\vartheta_j] + 1$ at point a_j, and P_j is its Taylor polynomial at this point, $j = 1, 2$. The problem is to describe the solvability of problems (19) and (17) in terms of these polynomials.

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