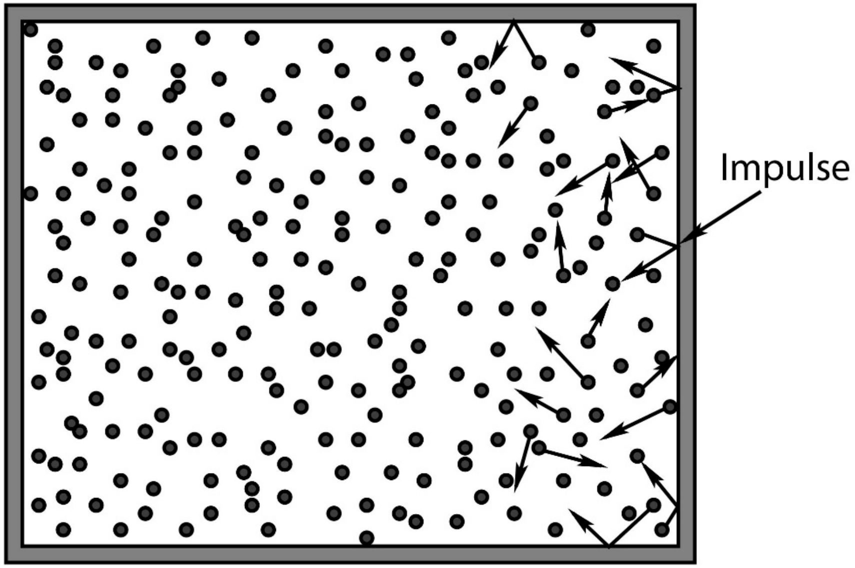
Lecture 5. Fluids

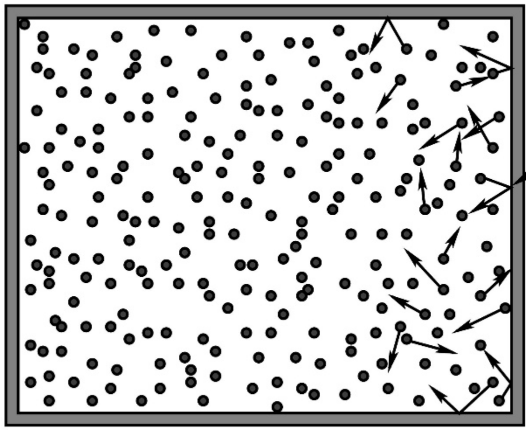
General Fluid Properties

Fluids are the generic name given to two states of matter, liquids and gases characterized by a lack of long range order and a high degree of mobility at the molecular scale.



A large number of atoms or molecules are confined within in a “box”, where they bounce around off of each other and the walls. They exert a *force* on the walls equal and opposite the force the walls exert on them as the collisions more or less elastically reverse the particles’ momenta perpendicular to the walls.

General Fluid Properties



Many particles all of mass *m* are *constantly moving* in random, constantly changing directions (as the particles collide with each other and the walls) with an average kinetic energy related to the temperature of the fluid. Some of the particles (which might be atoms such as helium or neon or molecules such as H_2 or O_2) happen to be close to the walls of the container and moving in the right direction to bounce (elastically) off of those walls.

When they do, their *momentum* perpendicular to those walls is *reversed*. Since *many, many* of these collisions occur each second, there is a nearly continuous momentum transfer between the walls and the gas and the gas and the walls. This transfer, per unit time, becomes the *average force* exerted by the walls on the gas and the gas on the walls

Eventually, we will transform this simple picture into the ***Kinetic Theory of Gases*** and use it to derive the venerable ***Ideal Gas Law***

$$PV = Nk_bT$$

Pressure

To describe the forces that confine and act on the fluids in terms of ***pressure***, defined to be the ***force per unit area*** with which a fluid pushes on a confining wall or the confining wall pushes on the fluid:

$$P = \frac{F}{A}$$

Pressure gets its own SI units, which clearly must be Newtons per square meter. We give these units their own name, ***Pascals***:

$$1\ \text{Pascal} = \frac{\text{Newton}}{\text{meter}^2}$$

A Pascal is a tiny unit of pressure – a Newton isn't very big, recall (one kilogram weighs roughly ten Newtons) so a Pascal is the weight of a quarter pound spread out over a square meter.

A more convenient measure of pressure in our everyday world is a form of the unit called a ***bar***:

$$1\text{bar} = 10^5\ \text{Pa} = 100\ \text{kPa}$$

Pressure

The average air pressure at sea level is very nearly 1 bar.

The symbol ***atm*** stands for ***one standard atmosphere***. The connection between atmospheres, bars, and pascals is:

$$1 \text{ standard atmosphere} = 101.325 \text{ kPa} = 1013.25 \text{ mbar}$$

The extra significant digits therefore refer only to a fairly arbitrary value (in pascals) historically related to the original definition of a standard atmosphere in terms of “millimeters of mercury” or *torr* :

$$1 \text{ standard atmosphere} = 760.00 \text{ mmHg} = 760.00 \text{ torr}$$

In this class we will use the simple rule 1 bar ≈ 1 atm

Note well: in the field of medicine blood pressures are given in mm of mercury (or torr) by long standing tradition (largely because for at least a century blood pressure was measured with a mercury-based sphygmomanometer). These can be converted into atmospheres by dividing by 760, remembering that one is measuring the difference between these pressures and the standard atmosphere (so the actual blood pressure is always greater than one atmosphere).

Density

Even a very tiny volume of fluid has many, many atoms or molecules in it.

We can work to create a *vacuum* – a volume that has relatively *few* molecules in it per unit volume, but it is almost impossible to make that number zero – even the hard vacuum of outer space has on average one molecule per cubic meter or thereabouts. We live at the bottom of a gravity well that confines our *atmosphere* – the air that we breathe – so that it forms a relatively thick soup that we move through and breathe with order of *Avogadro's Number* (6×10^{23}) molecules per *liter* – hundreds of billions of billions per cubic centimeter.

At this point we cannot possibly track the motion and interactions of all of the individual molecules, so we *coarse grain* and *average*.

The properties of *oxygen* molecules and *helium* molecules might well be very different, so the molecular count alone may not be the most useful quantity. Since we are interested in how forces might act on these small volumes, we need to know their mass, and thus we define the *density* of a fluid to be:

$$\rho = \frac{dm}{dV}$$

Compressibility

A major difference between fluids and solids, and liquids and gases within the fluids, is the ***compressibility*** of these materials. Compressibility describes how a material responds to changes in *pressure*.

This can be expressed as a simple linear relationship:

$$\Delta P = -B\frac{\Delta V}{V}$$

Pressure up, volume down and vice versa. The constant of proportionality B is called the ***bulk modulus*** of the material.

Note well that we haven't really specified *yet* whether the "material" is solid, liquid or gas. All three of them have densities, all three of them have bulk moduli. Where they differ is in the *qualitative* properties of their compressibility.

Lecture 5. Fluids

Compressibility

- **Solids** are typically *relatively* incompressible (large B), although there are certainly exceptions. They have long range order – all of the molecules are packed and tightly bonded together in structures and there is usually very little free volume.
- **Liquids** are also relatively incompressible (large B). They differ from solids in that they lack long range order. All of the molecules are constantly moving around and any small “structures” that appear due to local interaction are short-lived. The molecules of a liquid are close enough together that there is often significant physical and chemical interaction, giving rise to surface tension and wetting properties – especially in water, which is an amazing fluid!
- **Gases** are in contrast quite *compressible* (small B). One can usually squeeze gases smoothly into smaller and smaller volumes, until they reach the point where the molecules are basically all touching and the gas converts to a liquid! Gases per se (especially hot gases) usually remain “weakly interacting” right up to where they become a liquid, although the correct (non-ideal) equation of state for a real gas often displays features that are the results of moderate interaction, depending on the pressure and temperature.

Compressibility

Water is, as noted, a remarkable liquid. H_2O is a polar molecules with a permanent dipole moment, so water molecules are very strongly interacting, both with each other and with other materials. It organizes itself quickly into a state of relative order that is very ***incompressible***.

The bulk modulus of water is 2.2×10^9 Pa, which means that even deep in the ocean where pressures can be measured in the tens of millions of Pascals (or hundreds of atmospheres) the density of water only varies by a few percent from that on the surface. Its density varies much more rapidly with *temperature* than with pressure.

We will idealize water by considering it to be *perfectly incompressible* in this course, which is close enough to true for nearly any mundane application of hydraulics that you are most unlikely to ever observe an exception that matters.

Lecture 5. Fluids

Viscosity and fluid flow

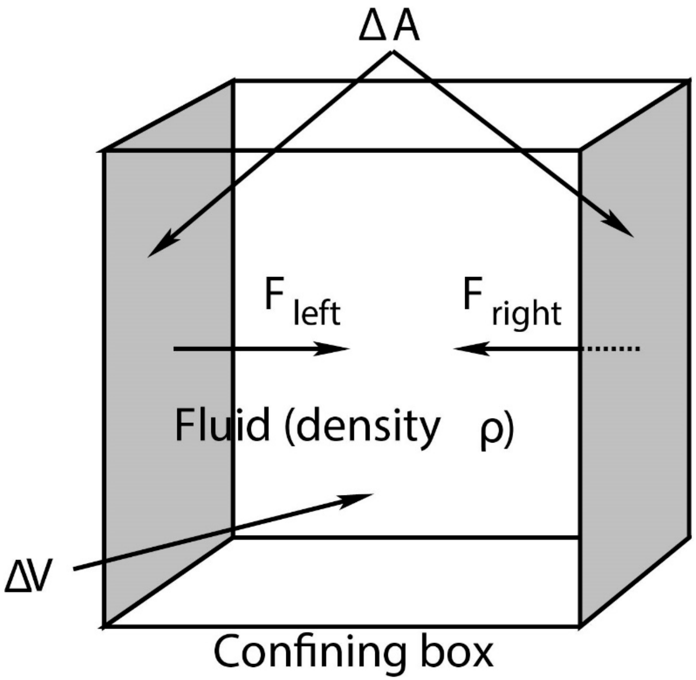
Fluids, whether liquid or gas, have some internal “stickiness” that resists the relative motion of one part of the fluid compared to another, a kind of internal “friction” that tries to equilibrate an entire body of fluid to move together. They also interact with the walls of any container in which they are confined.

The **viscosity** of a fluid (symbol μ) is a measure of this internal friction or stickiness. Thin fluids have a low viscosity and flow easily with minimum resistance; thick sticky fluids have a high viscosity and resist flow.

Fluid, when flowing through (say) a cylindrical pipe tends to organize itself in one of two very different ways – a state of *laminar flow* where the fluid at the very edge of the flowing volume is at rest where it is in contact with the pipe and the speed concentrically and symmetrically increases to a maximum in the center of the pipe, and *turbulent flow* where the fluid tumbles and rolls and forms eddies as it flows through the pipe. Turbulence and flow and viscosity are properties that will be discussed in more detail below.

Lecture 5. Fluids

Static Fluids. Pressure and Confinement of Static Fluids

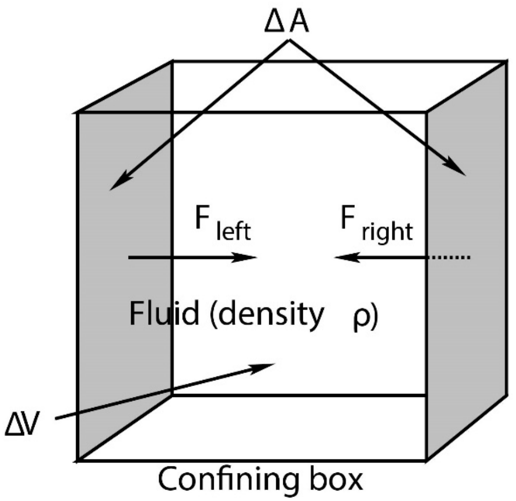


In figure we see a box of a fluid that is confined within the box by the rigid walls of the box.

We will imagine that this particular box is in “free space” far from any gravitational attractor and is therefore at rest with no external forces acting on it. We know from our intuition based on things like cups of coffee that no matter how this fluid is initially stirred up and moving within the container, after a very long time the fluid will damp down any initial motion by interacting with the walls of the container and arrive at *static equilibrium*.

Lecture 5. Fluids

Static Fluids. Pressure and Confinement of Static Fluids



A fluid in static equilibrium has the property that every single tiny chunk of volume in the fluid has to *independently* be in force *equilibrium* – the total force acting on the differential volume chunk must be *zero*.

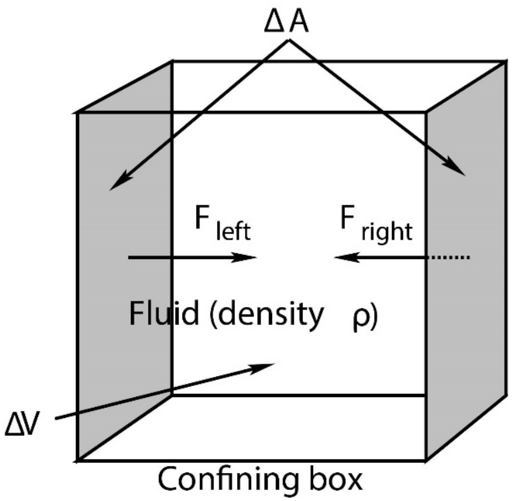
In addition the net torques acting on all of these differential subvolumes must be zero, and the fluid must be at rest, neither translating nor rotating.

Fluid rotation is more complex than the rotation of a static object because a fluid can be *internally* rotating even if all of the fluid in the outermost layer is in contact with a contain and is *stationary*. It can also be *turbulent* – there can be lots of internal eddies and swirls of motion, including some that can exist at very small length scales and persist for fair amounts of time.

We will idealize all of this – when we discuss static properties of fluids we will assume that all of this sort of internal motion has disappeared.

Lecture 5. Fluids

Static Fluids. Pressure and Confinement of Static Fluids



We can now make a few very simple observations about the forces exerted by the walls of the container on the fluid within. First of all the mass of the fluid in the box above is clearly:

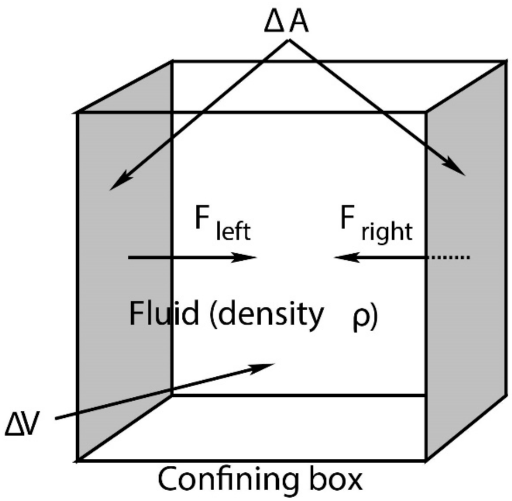
$$\Delta M = \rho \Delta V$$

We drew a symmetric box to make it easy to see that the magnitudes of the forces exerted by opposing walls are equal $F_{left} = F_{right}$ (for example). Similarly the forces exerted by the top and bottom surfaces, and the front and back surfaces, must cancel.

Suppose (as shown) the cross-sectional area of the left and right walls are ΔA originally. Consider now what we expect if we *double* the size of the box and at the same time add enough additional fluid for the fluid density to remain the same, making the side walls have the area $2\,\Delta A$. With twice the area (and twice the volume and twice as much fluid), we have twice as many molecular collisions per unit time on the doubled wall areas (with the same average impulse per collision). The average force exerted by the doubled wall areas therefore *also doubles*.

Lecture 5. Fluids

Static Fluids. Pressure and Confinement of Static Fluids



From this simple argument we can conclude that the average force exerted by any wall is proportional to the area of the wall. This force is therefore most naturally expressible in terms of pressure:

$$F_{\text{left}} = P_{\text{left}}\Delta A = P_{\text{right}}\Delta A = F_{\text{right}}$$

which implies that the pressure at the left and right confining walls is the same:

$$P_{\text{left}} = P_{\text{right}} = P$$

An important property of fluids is that *one part of a fluid can move independent of another* so the fluid in at least *some* layer with a finite thickness near the wall would therefore experience a *net* force and would *accelerate*. But this violates our assumption of static equilibrium, so a fluid in *static equilibrium* exerts no tangential force on the walls of a confining container and vice versa.

We therefore conclude that the direction of the force exerted by a confining surface with an area ΔA on the fluid that is in contact with it is: $\vec{F} = P\Delta A\hat{n}$. Where $\hat{n}$ is an inward-directed unit vector *perpendicular* to (normal to) the surface.

Pressure and Confinement of Static Fluids in Gravity

The principle change brought about by setting our box of fluid down on the ground in a gravitational field is that an additional external force comes into play: The weight of the fluid. A static fluid, confined in some way in a gravitational field, must ***support the weight of its many component parts*** internally, and of course the box itself must support the weight of the entire mass ΔM of the fluid.

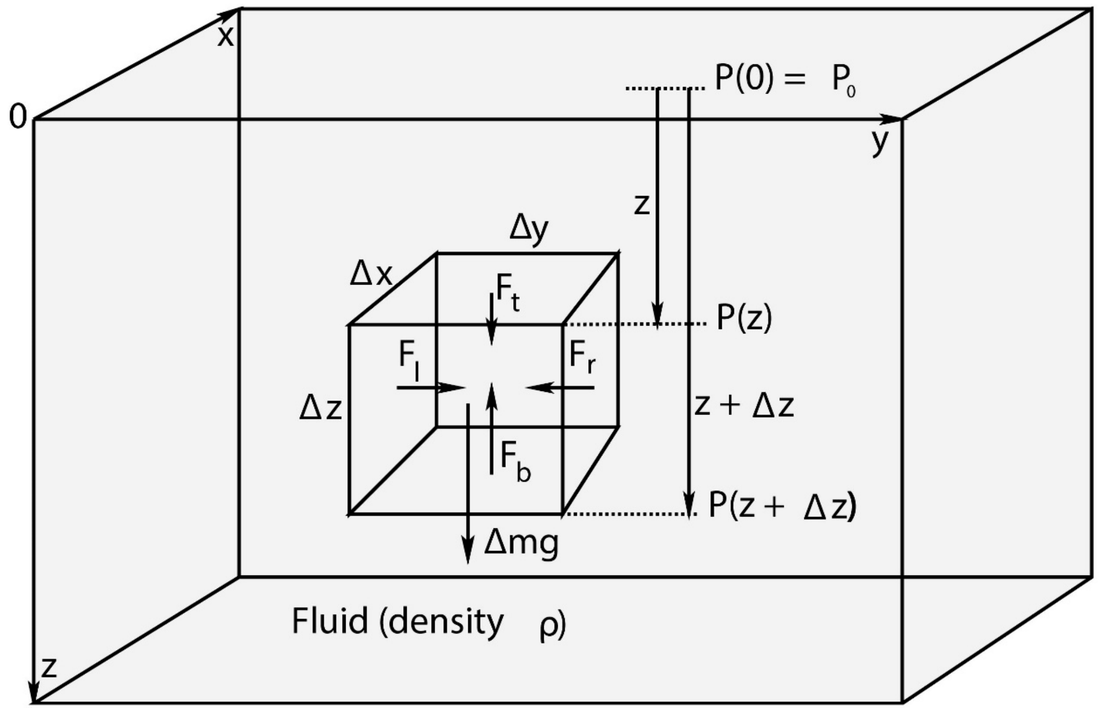
As hopefully you can see if you carefully read the previous section. The only force available to provide the necessary internal support or confinement force is the ***variation of pressure within the fluid***. We would like to know how the pressure varies as we move up or down in a static fluid so that it supports its own weight.

If we consider a tiny (eventually differentially small) chunk of fluid in force equilibrium, gravity will pull it down and the only thing that can push it up is a pressure difference between the top and the bottom of the chunk.

Lecture 5. Fluids

Pressure and Confinement of Static Fluids in Gravity

A fluid in static equilibrium confined to a sealed rectilinear box in a near-Earth gravitational field $\vec{g}$. Note well the small chunk of fluid with dimensions Δx, Δy, Δz in the middle of the fluid. Also note that the coordinate system selected has z increasing from the top of the box down, so that z can be thought of as the depth of the fluid.



Pressure and Confinement of Static Fluids in Gravity

In figure a (portion of) a fluid confined to a box is illustrated. The box could be a completely sealed one with rigid walls on all sides, or it could be something like a cup or bucket that is open on the top but where the fluid is still confined there by e.g. atmospheric pressure.

Let us consider a small (eventually infinitesimal) chunk of fluid somewhere in the *middle* of the container. As shown, it has physical dimensions Δx, Δy, Δz; its upper surface is a distance z below the origin (where z increases down and hence can represent “depth”) and its lower surface is at depth $z + \Delta z$. The areas of the top and bottom surfaces of this small chunk are e.g. $\Delta A_{tb} = \Delta x\Delta y$, the areas of the sides are $\Delta x\Delta z$ and $\Delta y\Delta z$ respectively, and the volume of this small chunk is $\Delta V = \Delta x\, \Delta y\Delta z$.

This small chunk is itself in static equilibrium – therefore the forces between any pair of its horizontal sides (in the x or y direction) must cancel. As before (for the box in space) $F_l = F_r$ in magnitude (and opposite in their y-direction) and similarly for the force on the front and back faces in the x-direction, which will always be true if the pressure does not vary horizontally with variations in x or y. In the z-direction, however, force equilibrium requires that:

$$F_t + \Delta mg - F_b = 0$$

Pressure and Confinement of Static Fluids in Gravity

The only possible source of F_t and F_b are the pressure in the fluid itself which will vary with the depth z: $F_t = P(z)\Delta A_{tb}$ and $F_b = P(z + \Delta z)\Delta A_{tb}$. Also, the mass of fluid in the (small) box is $\Delta m = \rho \Delta V$ (using our ritual incantation "the mass of the chunks is..."). We can thus write:

$$P(z)\Delta x\Delta y + \rho(\Delta x\Delta y\Delta z)g - P(z+\Delta z)\Delta x\Delta y = 0$$

$$\frac{\Delta P}{\Delta z} = \frac{P(z+\Delta z) - P(z)}{\Delta z} = \rho g$$

Finally, we take the limit $\Delta z \to 0$ and identify the ***definition of the derivative*** to get:

$$\frac{dP}{dz} = \rho g$$

Identical arguments but without any horizontal external force followed by $\Delta x \to 0$ and $\Delta y \to 0$ lead to:

$$\frac{dP}{dx} = \frac{dP}{dy} = 0$$

as well – P does not vary with x or y as already noted

Pressure and Confinement of Static Fluids in Gravity

$$\frac{dP}{dz} = \rho g$$

In order to find $P(z)$ from this differential expression (which applies, recall, to *any* confined fluid in static equilibrium in a gravitational field) we have to *integrate* it. This integral is very simple if the fluid is incompressible because in that case ***ρ is a constant***. The integral isn't *that* difficult if ρ is *not* a constant as implied by the equation we wrote above for the bulk compressibility.

We will therefore first do incompressible fluids, then compressible ones.

Lecture 5. Fluids

Variation of Pressure in Incompressible Fluids

In the case of incompressible fluids, ρ is a constant and does not vary with pressure and/or depth. Therefore we can easily multiple $dP/dz = \rho g$ above by dz on both sides and integrate to find P:

$$dP = \rho g \, dz$$

$$\int dP = \int \rho g \, dz$$

$$P(z) = \rho g z + P_0$$

where P_0 is the *constant of integration* for both integrals, and practically speaking is the pressure in the fluid at zero depth (wherever that might be in the coordinate system chosen).

Lecture 5. Fluids

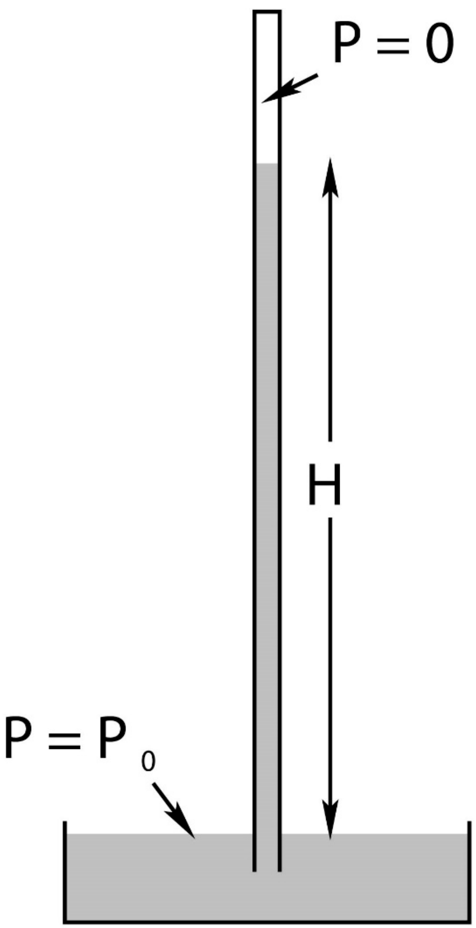
Barometers

Mercury barometers were originally invented by ***Evangelista Torricelli*** a natural philosopher who acted as Galileo's secretary for the last three months of Galileo's life under house arrest.

Torricelli demonstrated that a shorter glass tube filled with mercury, when inverted into a dish of mercury, would fall back into a column with a height of roughly 0.76 meters with a vacuum on top, and soon thereafter discovered that the height of the column fluctuated with the pressure of the outside air pressing down on the mercury in the dish, correctly concluding that water would behave exactly the same way.

Lecture 5. Fluids

Barometers



A simple mercury barometer is shown in figure. It consists of a tube that is completely filled with mercury. Mercury has a specific gravity of 13.534 at a typical room temperature, hence a density of 13534 kg/m^3). The filled tube is then inverted into a small reservoir of mercury. The mercury falls (pulled down by gravity) out of the tube, leaving behind a vacuum at the top. We can easily compute the expected height of the mercury column if P_0 is the pressure on the exposed surface of the mercury in the reservoir. In that case:

$$P = P_0 + \rho g z$$

as usual for an incompressible fluid. Applying this formula to both the top and the bottom, $P(0) = P_0$ and

$$P(H) = P_0 - \rho g H$$

$$P_0 = \rho g H$$

and one can easily convert the measured height H of mercury above the top surface of mercury in the reservoir into P_0, the air pressure on the top of the reservoir.

Lecture 5. Fluids

Barometers

At one standard atmosphere, we can easily determine what a mercury barometer at room temperature will read (the height H of its column of mercury above the level of mercury in the reservoir):

$$P_0 = 13534\frac{\text{kg}}{\text{m}^3} \times 9.80665\frac{\text{m}}{\text{sec}^3} \times H = 101325\ \text{Pa}$$

Dividing we find the value of H expected at one standard atmosphere:

$$H_{\text{atm}} = 0.76000 = 760.00\ \text{millimeters}$$

$$P(H) = P_0 - \rho g H$$

$$P_0 = \rho g H$$

and one can easily convert the measured height H of mercury above the top surface of mercury in the reservoir into P_0, the air pressure on the top of the reservoir.

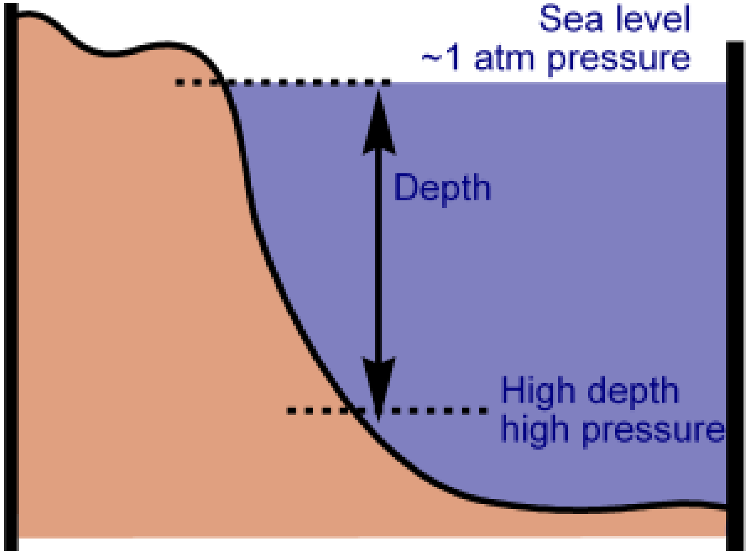
Variation of Oceanic Pressure with Depth

The pressure on the surface of the ocean is, approximately, by definition, one atmosphere. Water is a highly incompressible fluid with ρ_w = 1000 kilograms per cubic meter. $g \approx 10$ meters/second^2. Thus:

$$P(z) = P_0 + \rho_w g z = (10^5 + 10^4 z) \ \text{Pa}$$

$$\text{or} \quad P(z) = (1.0 + 0.1z) \ \text{bar} = (1000 + 100z) \ \text{mbar}$$

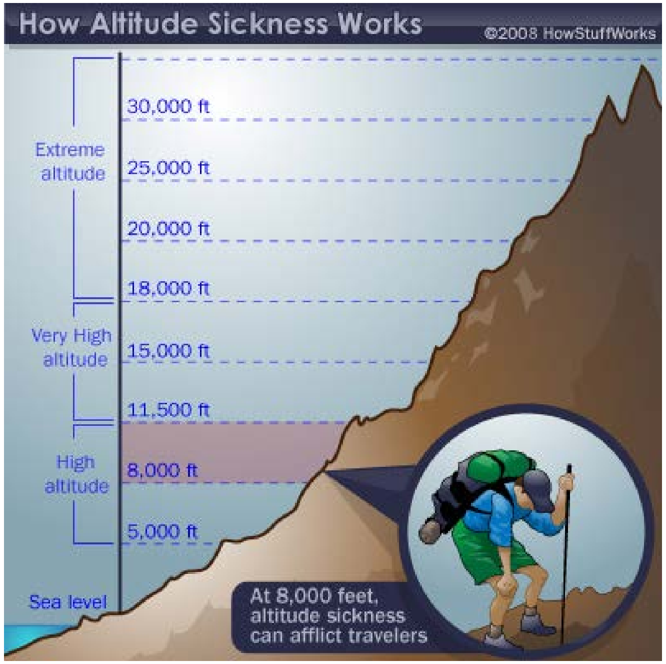
Every ten meters of depth (either way) increases water pressure by (approximately) one atmosphere!



http://www.calctool.org/CALC/other/games/depth_press

Lecture 5. Fluids

Variation of Atmospheric Pressure with Height



http://adventure.howstuffworks.com/outdoor-activities/climbing/altitude-sickness1.htm

Using z to describe depth is moderately inconvenient, so let us define the height h above sea level to be −z. In that case P_0 is 1 Atmosphere. The molar mass of dry air is M = 0.029 kilograms per mole. R = 8.31 Joules/(mole-K°). Hence a bit of multiplication at T = 300°:

$$\frac{M\,g}{RT} = \frac{0.029 \times 10}{8.31 \times 300} = 1.12 \times 10^{-4}\ \text{meters}^{-1}$$

$$P(h) = 10^5 \exp(-0.00012\,h)\ \text{Pa}$$
$$= 1000\ \exp(-0.00012\,h)\ \text{mbar}$$

This equation predicts that air pressure should drop to 1/e of its sea-level value of 1000 mbar at a height of around 8000 meters, the height of the so-called ***death zone***. We can compare the actual (average) pressure at 8000 meters, 356 mbar, to 1000 × e^{-1} = 368 mbar.

Lecture 5. Fluids

Pascal's Principle and Hydraulics

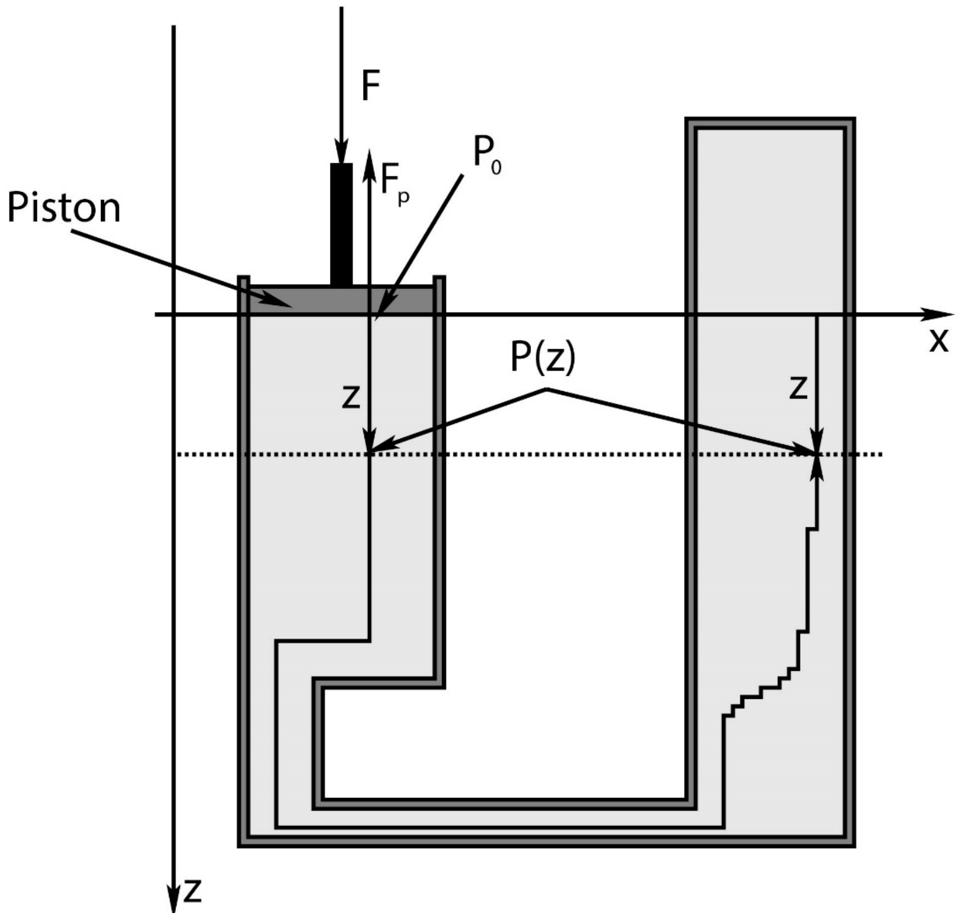
We note that (from the above) the general form of P of a fluid confined to a sealed container has the most general form:

$$P(z) = P_0 + \int_0^z \rho g dz$$

where P_0 is the constant of integration or value of the pressure at the reference ***depth*** $z = 0$. This has an important consequence that forms the basis of *hydraulics*.

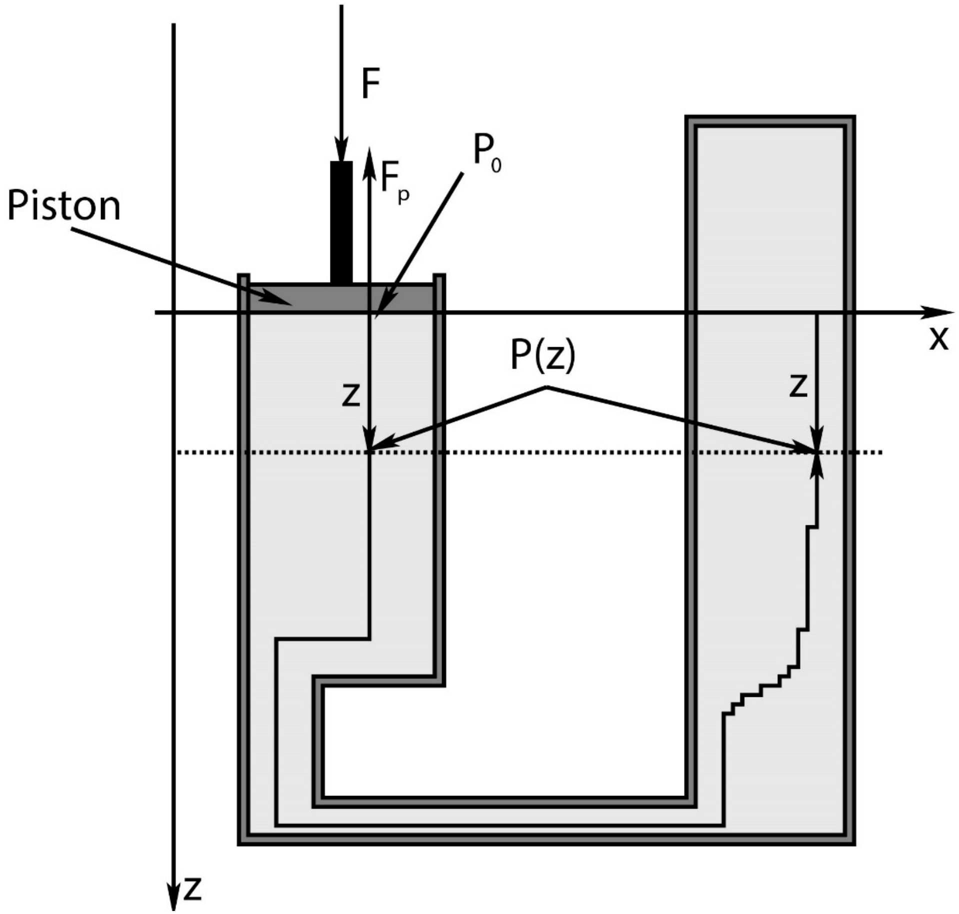
Lecture 5. Fluids

Pascal's Principle and Hydraulics



Suppose, that we have an ***incompressible fluid*** e.g. water confined within a sealed container by e.g. a *piston* that can be pushed or pulled on to *increase or decrease* the *confinement pressure* on the surface of the piston.

Pascal's Principle and Hydraulics



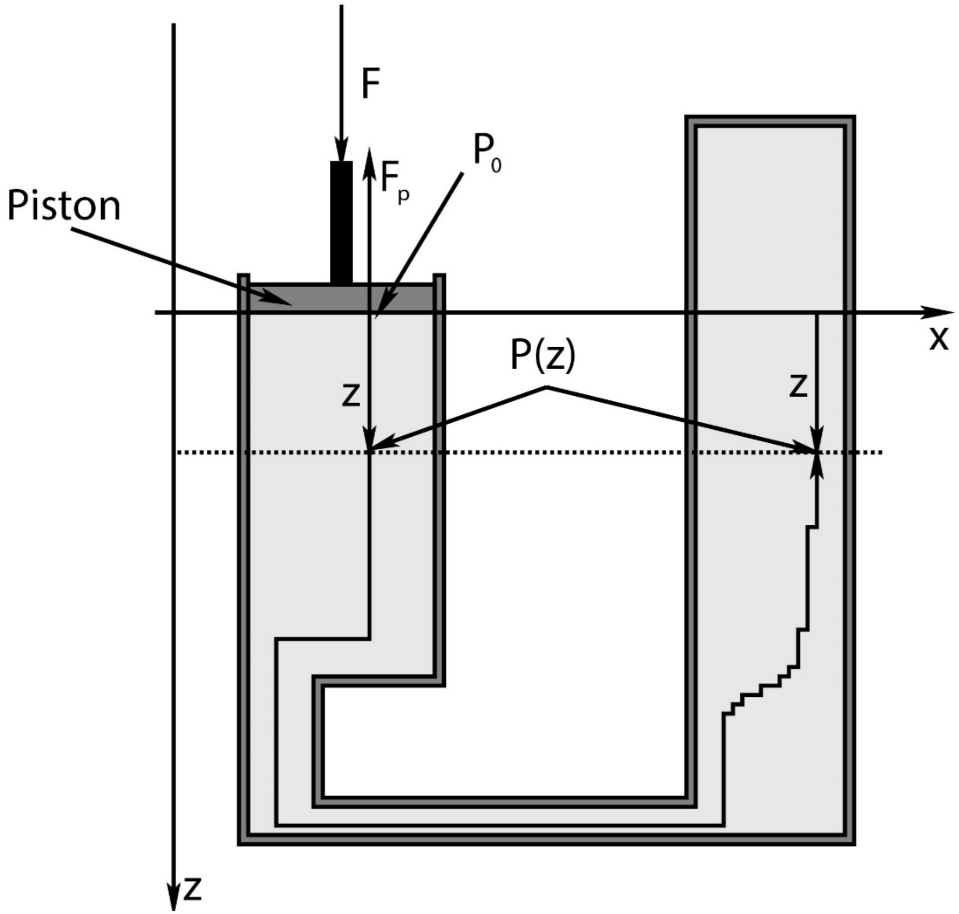
We can push down (or pull back) on the piston with any total downward force F that we like that leaves the system in equilibrium. Since the piston itself is in static equilibrium, the force we push with must be opposed by the pressure in the fluid, which exerts an equal and opposite upwards force:

$$F = F_p = P_0 A$$

where A is the cross sectional area of the piston and where we've put the cylinder face at $z = 0$, which we are obviously free to do.

Lecture 5. Fluids

Pascal's Principle and Hydraulics



The pressure at a depth z in the container is then

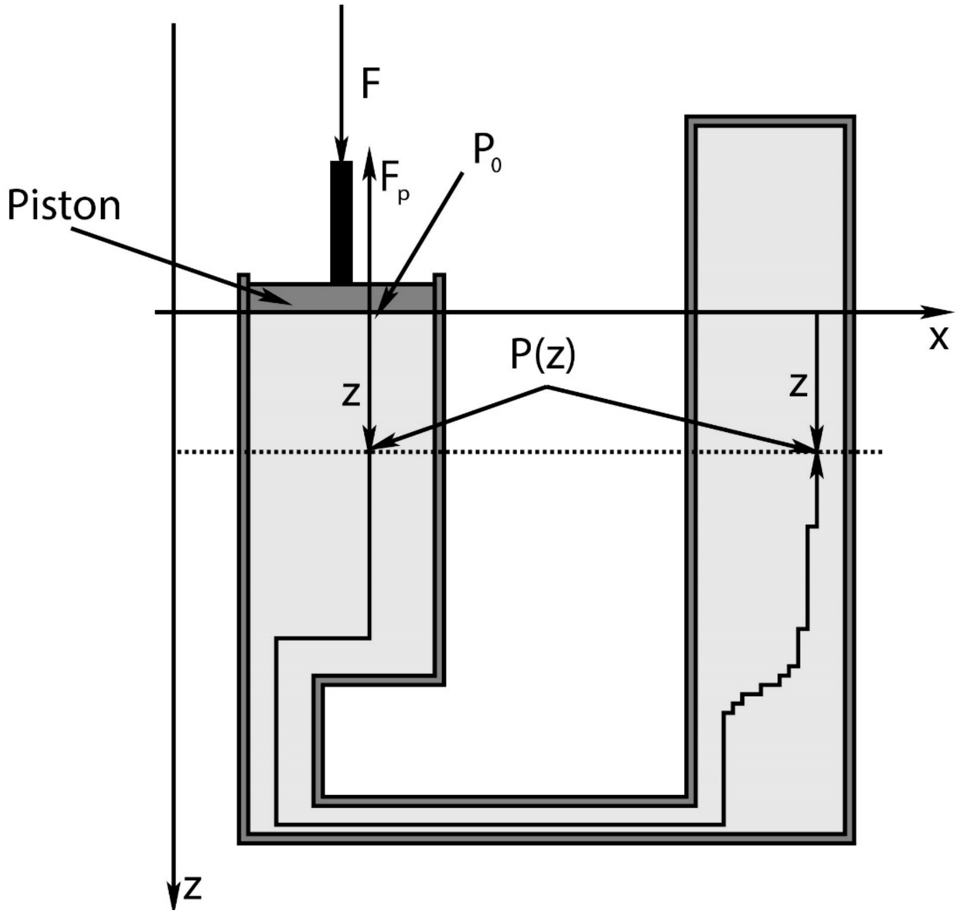
$$P(z) = P_0 + \rho g z$$

where A is the cross sectional area of the piston and where we've put the cylinder face at $z = 0$, which we are obviously free to do.

where $\rho = \rho_w$ if the cylinder is indeed filled with water, but the cylinder could equally well be filled with hydraulic fluid

Lecture 5. Fluids

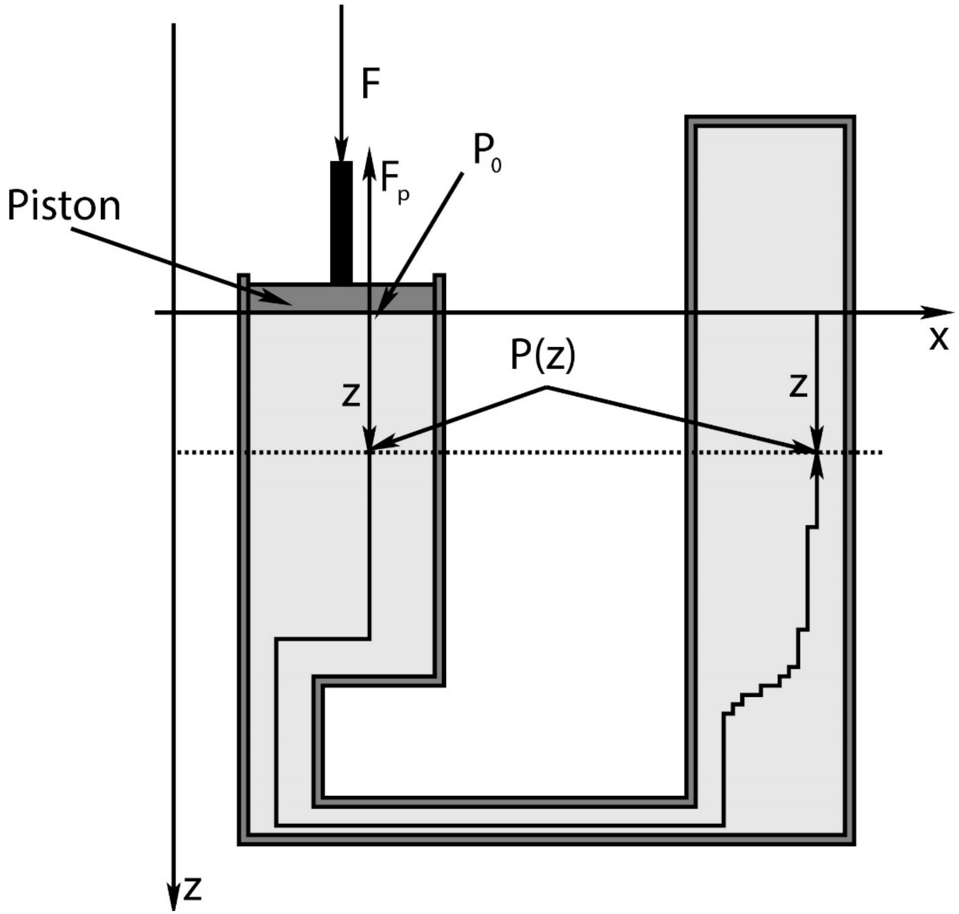
Pascal's Principle and Hydraulics



We recall that the pressure changes only when we change our depth. Moving laterally does not change the pressure, because e.g. $dP/dx = dP/dy = 0$. We can always find a path consisting of vertical and lateral displacements from $z = 0$ to any other point in the container – two such points at the same depth z are shown in figure, along with a vertical/horizontal path connecting them. Clearly these two points ***must have the same pressure P(z)!***

Lecture 5. Fluids

Pascal's Principle and Hydraulics



Now consider the following. Suppose we start with pressure P_0 (so that the pressure at these two points is $P(z)$, but then change F to make the pressure P'_0 and the pressure at the two points $P'(z)$. Then:

$$P(z) = P_0 + \rho g z$$

$$P'(z) = P'_0 + \rho g z$$

$$\Delta P(z) = P'(z) - P(z) = P'_0 - P_0 = \Delta P_0$$

That is, the pressure change at depth z does not depend on z *at any point in the fluid*! It depends *only on the change in the pressure exerted by the piston*!

Pascal’s Principle and Hydraulics

This result is known as ***Pascal’s Principle*** and it holds (more or less) for any compressible fluid, not just incompressible ones, but in the case of compressible fluids the piston will move up or down or in or out and the density of the fluid will change and hence the treatment of the integral will be too complicated to cope with. Pascal’s Principle is more commonly given in *English words* as:

Any change in the pressure exerted at a given point on a confined fluid is transmitted, undiminished, throughout the fluid.

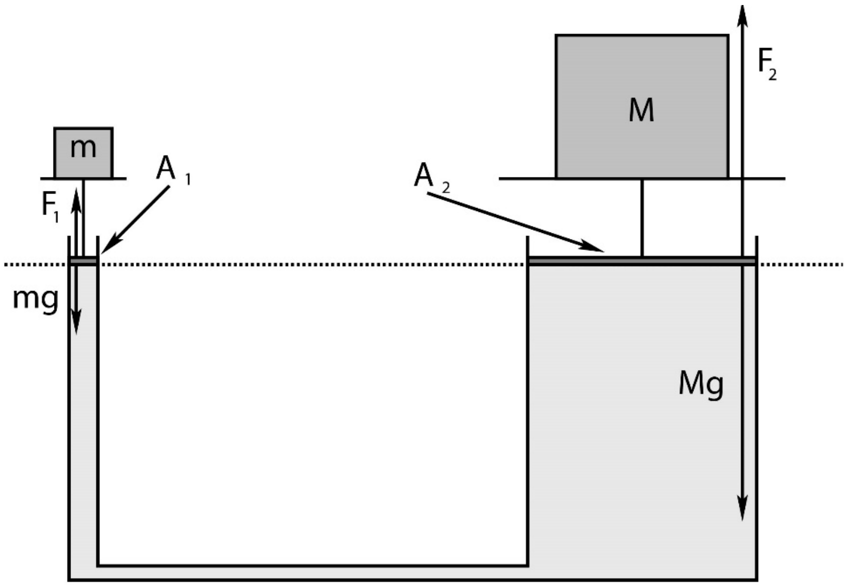
Pascal’s principle is the basis of ***hydraulics***. Hydraulics are a kind of fluid-based simple machine that can be used to greatly amplify an applied force.

A Hydraulic Lift

Figure illustrates the way we can multiply forces using Pascal's Principle.

Two pistons seal off a pair of cylinders connected by a closed tube that contains an incompressible fluid. The two pistons are deliberately given the same height (which might as well be $z = 0$), then, in the figure, although we could easily deal with the variation of pressure associated with them being at different heights since we know $P(z) = P_0 + \rho g z$.

The two pistons have cross sectional areas A_1 and A_2 respectively, and support a small mass m on the left and large mass M on the right in static equilibrium.



A Hydraulic Lift

For them to be in equilibrium, clearly:

$$F_1 - mg = 0$$
$$F_2 - Mg = 0$$

We also/therefore have:

$$F_1 = P_0 A_1 = mg$$
$$F_2 = P_0 A_2 = Mg$$

Thus

$$\frac{F_1}{A_1} = P_0 = \frac{F_2}{A_2}$$

or (substituting and cancelling g):

$$M = \frac{A_2}{A_1} m$$

A small mass on a small-area piston can easily balance a much larger mass on an equally larger area piston!

A Hydraulic Lift

$$M = \frac{A_2}{A_1} m$$

If we try to lift (say) a car with a hydraulic lift, we have to move the same volume $\Delta V = A\Delta z$ from under the small piston (as it descends) to under the large one (as it ascends). If the small one goes down a distance z_1 and the large one goes up a distance z_2, then:

$$\frac{z_1}{z_2} = \frac{A_2}{A_1}$$

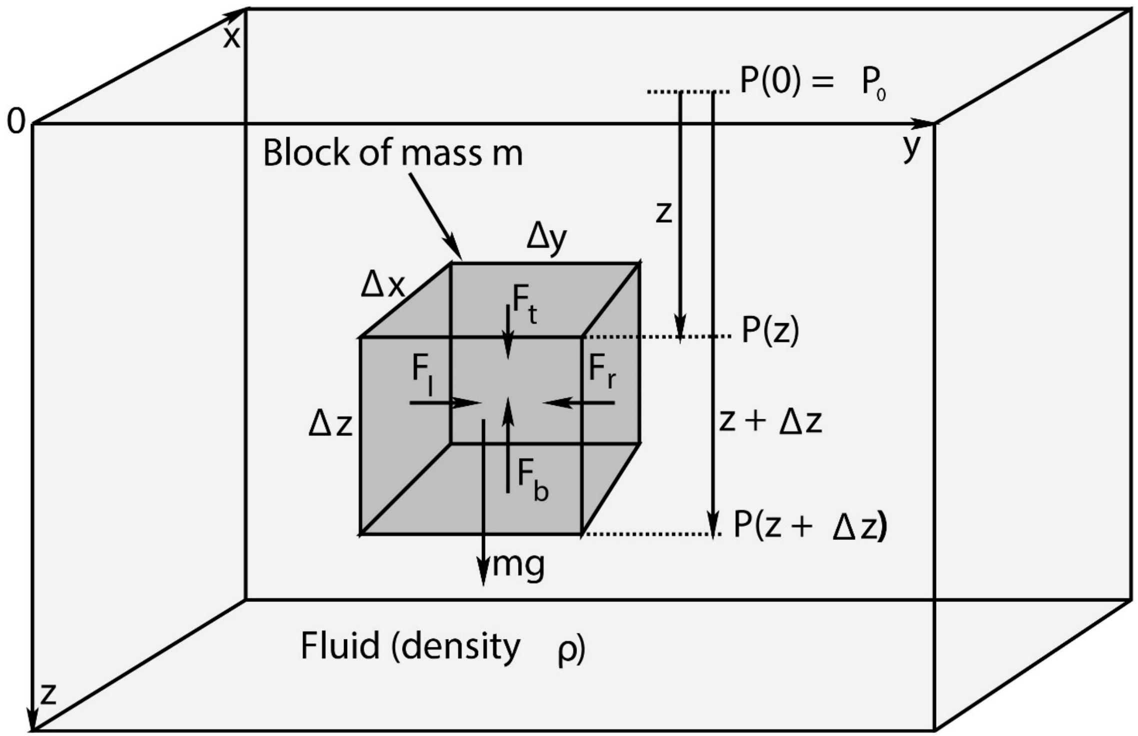
The work done by the two cylinders thus precisely balances:

$$W_2 = F_2 z_2 = F_1 \frac{A_2}{A_1} z_2 = F_1 \frac{A_2}{A_1} z_1 \frac{A_1}{A_2} = F_1 z_1 = W_1$$

The hydraulic arrangement thus transforms pushing a small force through a large distance into a large force moved through a small distance so that the work done ***on*** piston 1 matches the work done ***by*** piston 2.

Lecture 5. Fluids

Archimedes' Principle



A solid chunk of "stuff" of mass *m* and the dimensions shown is immersed in a fluid of density ρ at a depth *z*. The vertical pressure difference in the fluid (that arises as the fluid itself becomes static static) exerts a vertical force on the cube.

Archimedes' Principle

The net upward force exerted by the fluid is called the ***buoyant force*** F_b and is equal to:

$$F_b = P(z + \Delta z)\Delta x \Delta y - P(z)\Delta x \Delta y =$$
$$= \Big(\big(P_0 + \rho g (z + \Delta z)\big) - (P_0 + \rho g z)\Big)\Delta x \Delta y =$$
$$= \rho g \Delta z \Delta x \Delta y =$$
$$= \rho g \Delta V$$

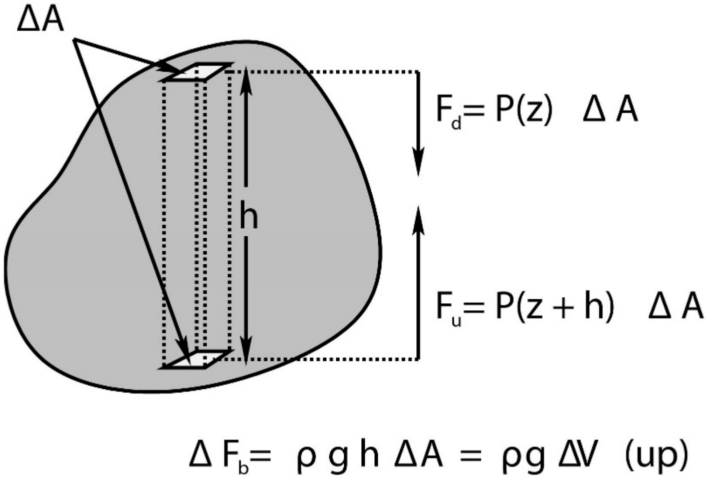
where ΔV is the volume of the small block.

The buoyant force is thus the ***weight of the fluid displaced*** by this single tiny block. This is all we need to show that the same thing is true for an *arbitrary* immersed shape of object.

Lecture 5. Fluids

Archimedes' Principle

In figure, an arbitrary blob-shape is immersed in a fluid of density ρ. Imagine that we've taken a french-fry cutter and cuts the whole blob into nice rectangular segments, one of which (of length h and cross-sectional area ΔA) is shown. We can trim or average the end caps so that they are all perfectly horizontal by making all of the rectangles arbitrarily small (in fact, differentially small in a moment). In that case the *vertical* force exerted by the fluid on just the two lightly shaded surfaces shown would be:



$$F_d = P(z)\Delta A$$

$$F_u = P(z+h)\Delta A$$

where we assume the upper surface is at depth z. Since $P(z+h) = P(z)+\rho g h$, we can find the net upward buoyant force exerted on this little cross-section by subtracting the first from the second:

$$\Delta F_b = F_u - F_d = \rho g h \Delta A = \rho g \Delta V$$ where the volume of this piece is $\Delta V = h\,\Delta A$.

Archimedes' Principle

$\Delta F_b = F_u - F_d = \rho g h \Delta A = \rho g \Delta V$ where the volume of this piece is $\Delta V = h\,\Delta\,A$.

We can now let $\Delta A \to dA$, so that $\Delta V \to dV$, and write

$$F_b = \int F_b = \int_{V \text{ of blob}} \rho g dV = \rho g V = m_f g$$

where $m_f = \rho V$ is the mass of the fluid displaced, so that $m_f g$ is its weight.

That is:

The total buoyant force on the immersed object is the weight of the fluid displaced by the object.

This statement – in the English or algebraic statement as you prefer – is known as ***Archimedes' Principle,***