КАЗАНСКИЙ ФЕДЕРАЛЬНЫЙ УНИВЕРСИТЕТ
ИНСТИТУТ МАТЕМАТИКИ И МЕХАНИКИ
ИМ Н.И. ЛОБАЧЕВСКОГО
Кафедра теории и технологий преподавания
математики и информатики

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НЕОПРЕДЕЛЕННЫЙ ИНТЕГРАЛ

Казань – 2015

УДК 517.9

Принято на заседании кафедры теорий и технологий преподавания математики и информатики
Протокол №2 от 7 октября 2015 года

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Неопределенный интеграл (The indefinite integral)/ М.Ю. Денисова, Л.А. Корнилова – Казань: Казан. фед. ун-т, 2015. – 33 с.

Учебно-методическое пособие предназначено для студентов, изучающих "Математический анализ и охватывает раздел "Неопределенный интеграл". В пособии имеется большое количество примеров, рассмотрение которых помогает усвоению теоретического материала. Весь материал изложен на английском языке.

Educational textbook is intended for students studying the subject "Mathematical analysis"and includes the chapter "The indefinite integral". This manual consists of a sufficient number of examples, examination of which helps to understand the given theoretical material. All the material is presented in English.

1. The primitive and the indefinite integral

The definition 1. The function $F(x)$ is called the primitive of the function $f(x)$ on the interval (a, b), if in all points of this interval equality is executed $F'(x) = f(x)$ (или $dF(x) = f(x)dx$).

The theorem. If the function $F(x)$ is the primitive of the function $f(x)$ on (a, b), the ensemble of all primitives for the function $f(x)$ defines by the formula $F(x) + C$, where C – is the permanent number. The inverse proposition is that any two primitives differ from each other on a constant summand.

The proving. Under the circumstance $F(x)$ – the primitive of the function $f(x)$,i.e. the condition

$$F'(x) = f(x)$$

is carrying out. Then for any constant number C the equality

$$(F(x) + C)' = F'(x) + C' = F'(x) = f(x)$$

takes place i.e the function $F(x) + C$ is the primitive. Let $F_1'(x)$ – to be another, different from отлична $F(x)$, the primitive of the function $f(x)$, i.e. $F_1'(x) = f(x)$. Then for any $x \in (a, b)$ we carry

$$(F_1(x) - F(x))' = F_1'(x) - F'(x) = f(x) - f(x) = 0$$

This means that $F_1(x) - F(x) = C$, где $C = const$. The theorem is proved.

The definition 2. The ensemble of all primitives $F(x) + C$ for the function $f(x)$ is called the indefinite integral from the function $f(x)$ and designates by symbol

$$\int f(x)dx.$$

Thus, by the definition

$$\int f(x)dx = F(x) + C,$$

if $F'(x) = f(x)$.

Accordingly, the indefinite integral itself represents the assemblage of functions $y = F(x) + C$.

The diagram of each curve (primitive) is called the integral curve.

2. Internals of the indefinite integral

1) Differential from the indefinite integral equals to integrand expression, whereas derivative from the indefinite integral equals to integrand function, i.e.

$$d(\int f(x)dx) = f(x)dx,$$

$$(\int f(x)dx)' = f(x).$$

The proving.

$$d(\int f(x)dx = d(F(x) + C = dF(x) + d(C) = F'(x)dx = f(x)$$

$$(\int f(x)dx)' = (F(x) + C)' = F'(x) + C' = f(x).$$

Due to this trait verity of integration is proved by derivation.

2) The indefinite integral from differential of a certain function equals the sum of this function and derivative invariable i.e.

$$\int dF(x) = F(x) + C.$$

The proving.

$$\int dF(x) = \int F'(x)dx = \int f(x)dx = F(x) + C.$$

3) The constant factor can be carried out of the integral sign.

$$\int \alpha f(x)dx = \alpha \int f(x)dx, \tag{1}$$

$$\alpha = const \neq 0.$$

The proving. For proving we take differentials from the left and the right parts of equality (1)

$$d(\int \alpha f(x)dx) = \alpha f(x)dx,$$

$$d(\alpha \int f(x)dx) = \alpha d(\int f(x)dx) = \alpha f(x)dx.$$

Reciprocally the following quality is proved

4) The indefinite integral of the seme (remainder) of two functions equals the sum (remainder) of their integrals, i.e.

$$\int (f(x) \pm g(x))dx = \int f(x)dx \pm \int g(x)dx.$$

The method of mathematical induction easily proves validity of the given quality for the finitesimal number of summands.

3. The table of the basic indefinite integrals

The table of the basic indefinite integrals assembles with the use of the table of derivative elementary functions. For example, it is known that $(x^{n+1})' = (n+1)x^n$, если $n \neq -1$. Herefrom follows that $x^n = (\frac{x^{n+1}}{n+1})'$. Consequently with the definition of the indefinite integral the following equality takes place

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C, (n \neq -1).$$

Analogically the rest formulas of the following table come out.

1. $\int x^n dx = \frac{x^{n+1}}{n+1} + C, (n \neq -1)$
2. $\int \frac{dx}{x} = ln|x| + C,$
3. $\int sinxdx = -cosx + C,$
4. $\int cosxdx = sinx + C,$
5. $\int \frac{dx}{cos^2 x} = tgx + C,$
6. $\int \frac{dx}{sin^2 x} = -ctgx + C,$
7. $\int e^x dx = e^x + C,$
8. $\int \alpha^x dx = \frac{\alpha^x}{ln\alpha} + C,$
9. $\int \frac{dx}{1+x^2} = arcctgx + C,$
10. $\int \frac{dx}{\sqrt{1-x^2}} = arcsinx + C,$

11. $\int shxdx = chx + C$,

12. $\int chxdx = shx + C$,

13. $\int \frac{dx}{ch^2x} = thx + C$,

14. $\int \frac{dx}{sh^2x} = cthx + C$.

4. Integration by the method of variable's replacement (the method of substitution)

Integration by the method of variable's replacement consists in leading-up of a new variable of integration, which helps to depress to the new integral, which is tabular or depresses to it.

Let's compute the integral $\int f(x)dx$. We'll make substitution

$$x = \varphi(t), \tag{1}$$

where$\varphi(t)$ – is a constant function, which has constant derivative $\varphi'(t)$. Then when derivation is completed we receive $(1) dx = \varphi'(t)dt$. Let's prove that the following equality occurs

$$\int f(x)dx = \int f(\varphi(t))\varphi'(t)dt. \tag{2}$$

To prove the ratio (2), it's enough to show that differentials of both parts are equal. Let's differentiate the left part (2)

$$d(\int f(x)dx) = f(x)dx.$$

But since $x = \varphi(t)$, then $dx = \varphi'(t)dt$. It gives

$$d(\int f(x)dx) = f(x)dx = f(\varphi(t))\varphi'(t)dt. \tag{3}$$

Let's differentiate the right part of the ratio (2)

$$d(\int f[\varphi(t)]\varphi'(t)dt) = f[\varphi(t)]\varphi'(t)dt. \tag{4}$$

Ratios (3) and (4) prove validity of formula (2).

After we've found the integral of the right part of the equality we need to proceed from new variable of the integral t back to variable x (they say, that reverse substitution is needed here). Sometimes it's reasonable to use formula (2) from the right to the left.

The 1st example. $\int \frac{dx}{\sqrt{a^2-x^2}}$.

The answer. Let's suppose $x = az$, we find $dx = adz$, using formula (2)

$$\int \frac{dx}{\sqrt{a^2-x^2}} = \int \frac{adz}{\sqrt{a^2-a^2z^2}} = \int \frac{dz}{\sqrt{1-z^2}} = arcsinz + C.$$

Returning again to variable x, we receive $z = \frac{x}{a}$.

$$\int \frac{dx}{\sqrt{a^2-x^2}} = arcsin\frac{x}{a} + C.$$

The 2nd example. $\int x\sqrt{x-3}dx$.

The answer. Let to be the next $\sqrt{x-3} = t$, then comes the following $x = t^2+3$, $dx = 2tdt$. That's why

$$\int x\sqrt{x-3}dx = \int (t^2+3)t \cdot 2tdt = 2\int (t^4+3t^2)dt =$$

$$= 2\int t^4dt + 6\int t^2dt = 2\frac{t^5}{5} + 6\frac{t^3}{3} + C = \frac{2}{5}(x-3)^{\frac{5}{2}} + 2(x-3)^{\frac{3}{2}} + C.$$

The 3rd example.

$$\int cos\, mxdx = \begin{pmatrix} x = \frac{t}{m} \\ dx = \frac{dt}{m} \end{pmatrix} = \int cos\, t\frac{dt}{m} = \frac{1}{m}\int cos\, tdt =$$

$$= \frac{1}{m}sin\, t + C = \frac{1}{m}sin\, mx + C.$$

5. Integration in parts

Let $u = u(x)$ и $v = v(x)$ to be two functions, which have two constant derivatives. It's known that the differential of product uv is computed by formula

$$d(uv) = udv + vdu.$$

Integrating both parts of the last equality

$$\int d(uv) = \int udv + \int vdu$$

or

$$\int udv = \int d(uv) - \int vdu,$$

and because of $\int d(uv) = uv$, (the second quality of the indefinite integral), we receive

$$\int udv = uv - \int vdu. \tag{5}$$

Formula (5) is called formula of integration in parts. This formula gives us an opportunity to depress computation of the integral $\int udv$ to computation of the integral $\int vdu$, which may occur to be easier than the primitive. Sometimes we can use this formula of integration in parts several times.

Let's designate some integral types, which convenient to compute by the method of integration in parts.

1. Integrals of type

   $$\int P(x)e^{kx}\,dx, \ \int P(x)sin\,kx\,dx, \ \int P(x)cos\,kx\,dx,$$

   where $P(x)$ – multinomial, k – number. It's convenient to distribute $u = P(x)$, whereas dv – to denote all other efficients.

2. Integrals of type

   $$\int P(x)arcsin\,x\,dx, \int P(x)arccos\,x\,dx, \int P(x)ln\,x\,dx,$$

   $$\int P(x)arctg\,x\,dx, \quad \int P(x)arcctg\,x\,dx,$$

   is convenient to suppose $P(x)dx = dv$, and for v to mark all other multipliers.

3. Integral of type
$$\int e^{\alpha x} sin\,\beta x\,dx, \quad \int e^{\alpha x} cos\,\beta x\,dx$$
where α, β – numbers.

The 1st example.

$$\int (x+2) cos\,x\,dx = \begin{pmatrix} u = x+2 & dv = cos\,x\,dx \\ du = dx & v = \int cos\,x\,dx = sin\,x \end{pmatrix} =$$

$$= (x+2) sin\,x - \int sin\,x\,dx = (x+2) sin\,x + cos\,x + C.$$

The 2nd example.

$$\int (3x^2+1) ln\,x\,dx = \begin{pmatrix} u = ln\,x & dv = (3x^2+1)dx \\ du = \frac{dx}{x} & v = x^3 + x \end{pmatrix} =$$

$$= (x^3+x) ln\,x - \int (x^3+x)\frac{dx}{x} = (x^3+x) ln\,x - \frac{x^3}{3} - x + C.$$

The 3rd example.

$$\int e^{ax} cos\,bx\,dx = \begin{pmatrix} u = e^{ax} & dv = cos\,bx\,dx \\ du = ae^{ax}dx & v = \frac{1}{b} sin\,bx \end{pmatrix} =$$

$$= \frac{e^{ax}}{b} sin\,bx - \frac{a}{b}\int sin\,bx\,e^{ax}\,dx = \begin{pmatrix} u = e^{ax} & dv = sin\,bx\,dx \\ du = ae^{ax}dx & v = -\frac{1}{b} cos\,bx \end{pmatrix} =$$

$$= \frac{e^{ax}}{b} sin\,bx - \frac{a}{b}(-\frac{1}{b}e^{ax} cos\,bx + \frac{a}{b}\int e^{ax} cos\,bx\,dx) + C =$$

$$= \frac{e^{ax}}{b} sin\,bx + \frac{a}{b^2}e^{ax} cos\,bx - \frac{a^2}{b^2}\int e^{ax} cos\,bx\,dx + C.$$

$$\int e^{ax} cos\,bx\,dx = \frac{e^{ax}}{b} sin\,bx + \frac{a}{b^2}e^{ax} cos\,bx - \frac{a^2}{b^2}\int e^{ax} cos\,bx\,dx + C.$$

$$(1 + \frac{a^2}{b^2})\int e^{ax} cos\,bx\,dx = \frac{e^{ax}}{b^2}(b\,sin\,bx + a\,cos\,bx) + C.$$

$$\int e^{ax} cos\,bx\,dx = \frac{e^{ax}}{a^2+b^2}(b\,sin\,bx + a\,cos\,bx) + C.$$

6. Computation of integrals with the help of recursive formula

Let's derive formula, which allows for any $k \in N$ to express I_k through I_{k-1}.

$$I_k = \int \frac{dx}{(x^2+a^2)^k} = \frac{1}{a^2} \int \frac{a^2+x^2-x^2}{(x^2+a^2)^k} dx =$$

$$= \frac{1}{a^2} \int \frac{dx}{(x^2+a^2)^{k-1}} - \frac{1}{a^2} \int \frac{x^2 dx}{(x^2+a^2)^k} =$$

$$= \frac{1}{a^2} I_{k-1} - \frac{1}{a^2} \int \frac{x^2 dx}{(x^2+a^2)^k}.$$

For the integral $\int \frac{x^2 dx}{(x^2+a^2)^k}$ let's apply the method of integration in parts

$$\int \frac{xxdx}{(x^2+a^2)^k} = \begin{pmatrix} u = x, & du = dx \\ dv = \frac{xdx}{(x^2+a^2)^k}, \; v = & -\frac{1}{2(k-1)(x^2+a^2)^{k-1}} \end{pmatrix} =$$

$$= -\frac{x}{2(k-1)(x^2+a^2)^{k-1}} + \frac{1}{2(k-1)} \int \frac{dx}{(x^2+a^2)^{k-1}} =$$

$$= -\frac{x}{2(k-1)(x^2+a^2)^{k-1}} + \frac{1}{2(k-1)} I_{k-1}.$$

consequently

$$I_k = \frac{1}{a^2} I_{k-1} - \frac{1}{a^2} \left[-\frac{x}{2(k-1)(x^2+a^2)^{k-1}} + \frac{1}{2(k-1)} I_{k-1} \right].$$

Grouping terms, containing I_{k-1}, we'll receive

$$I_k = x2(k-1)(x^2+a^2)^{k-1} + \frac{2k-3}{2a^2(k-1)} I_{k-1}.$$

This is the recursive formula.

The example. It is required to calculate

$$I_3 = \int \frac{dx}{(x^2+1)^3}.$$

Let's express I_3 through I_2:

$$I_3 = -\frac{x}{2(3-1)(x^2+1)^{3-1}} + \frac{2 \cdot 3 - 3}{2(3-1)} I_2 = \frac{x}{4(x^2+1)^2} + \frac{3}{4} I_2.$$

Let's substitute I_2 в I_3, finally we'll receive

$$I_3 = \frac{x}{4(x^2+1)^2} + \frac{3x}{8(x^2+1)} + \frac{3}{8} arctg\, x + C.$$

7. The simplest rational fractions and their integration

Integration of any rational fraction reduces to the integration of regular fractions.
Definition. Types of the regular rational fractions

$I. \frac{A}{x-a}$,

$II. \frac{A}{(x-a)^k} (k = 2, 3, ...)$,

$III. \frac{Ax+B}{x^2+px+q}$ (the roots of the denominator are complex, that is $D < 0$),

$IV. \frac{Ax+B}{(x^2+px+q)^k}$, ($k = 2, 3, ...$, complex denominator's roots),

where A, B, a, p, q – are real numbers, which are called the simplest rational fractions of I, II, III, IV types.

Integration of the fractions of the I and the II types doesn't make any difficulty.

$I. \int \frac{A}{x-a} dx = A \int \frac{d(x-a)}{x-a} = Aln|x-a| + C.$

$II. \int \frac{A}{(x-a)^k} dx = A \int (x-a)^{-k} d(x-a) = -\frac{A}{(k-1)(x-a)^{k-1}} + C.$

Let's proceed to the integration of the rational fractions of the III type.

$III. \int \frac{Ax+B}{x^2+px+q} dx.$

Let's examine the denominator and intercept a complete square

$$x^2 + px + q = \left(x + \frac{p}{2}\right)^2 + \left[q - \frac{p^2}{4}\right].$$

We'll replace the variable $t = x + \frac{p}{2}$, $dx = dt$, $x^2 + px + q = t^2 + a^2$.

$$\int \frac{Ax+b}{x^2+px+q} dx = \int \frac{A\left(t - \frac{p}{2}\right) + B}{t^2+a^2} dt =$$

$$= \left\{ dt^2 = 2tdt \Rightarrow tdt = \frac{1}{2} dt^2 \right\} =$$

$$= \frac{A}{2} \int \frac{d(t^2+a^2)}{t^2+a^2} + \left(B - \frac{Ap}{2}\right) \int \frac{dt}{t^2+a^2} =$$

$$= \frac{A}{2} ln|t^2 + a^2| + \left(B - \frac{Ap}{2}\right) \frac{1}{a} arctg\, \frac{t}{a} + C.$$

Making the reverse substitution

$$\int \frac{Ax + B}{x^2 + px + q} dx = \frac{A}{2} ln(x^2 + px + q) +$$

$$+ \frac{\left(B - \frac{pA}{2}\right)}{\sqrt{q - \frac{p^2}{4}}} arctg\, \frac{x + \frac{p}{2}}{\sqrt{q - \frac{p^2}{4}}} + C.$$

Let's examine the integration of the fractions of the IVth type.

$IV. \int \frac{Ax+B}{(x^2+px+q)^k} dx$. Let's bring in,as in the III case, a new variable to

$$t = x + \frac{p}{2},$$

$$dx = dt,$$

$$x^2 + px + q = t^2 + a^2, \qquad \text{где } a^2 = q - \frac{p^2}{4}$$

Consequently

$$\int \frac{Ax + B}{(x^2 + px + q)^k} = A \int \frac{tdt}{(t^2 + a^2)^k} + \left(B - \frac{pA}{2}\right) \int \frac{dt}{(t^2 + a^2)^k}. \qquad (1)$$

The first of the integrals' ratio (1) is easily compute

$$\int \frac{tdt}{(t^2 + a^2)^k} = \frac{1}{2} \int \frac{d(t^2 + a^2)}{(t^2 + a^2)^k} = \frac{1}{2} \int (t^2 + a^2)^{-k} d(t^2 + a^2) =$$

$$= -\frac{1}{2(k-1)(t^2 + a^2)^{k-1}} + C.$$

For the integration of the second fraction we are going to apply the recursion formula we've inducted earlier.

$$\int \frac{dx}{(x^2 + a^2)^k} = \frac{1}{2a^2(k-1)} \cdot \frac{x}{(x^2 + a^2)^{k-1}} + \frac{2k-3}{2a^2(k-1)} \int \frac{dx}{(x^2 + a^2)^{k-1}}.$$

The example.

$$\int \frac{x - 1}{(x^2 + 2x + 3)^2} dx = \begin{pmatrix} x^2 + 2x + 3 = (x+1)^2 + 2 \\ t = x + 1 \quad \Rightarrow \quad dx = dt \end{pmatrix} =$$

$$= \int \frac{t - 2}{(t^2 + 2)^2} dt = \int \frac{d(t^2 + 2)}{(t^2 + 2)^2} - 2 \int \frac{dt}{(t^2 + 2)^2} =$$

$$= -\frac{1}{2}\cdot\frac{1}{t^2+2} - 2\left[\frac{1}{2\cdot 2}\cdot\frac{t}{t^2+2} + \frac{1}{2\cdot 2}\int\frac{dt}{t^2+2}\right] =$$

$$= -\frac{1}{2}\cdot\frac{1}{t^2+2} - \frac{1}{2}\cdot\frac{1}{t^2+2} - \frac{1}{2}arctg\,\frac{t}{\sqrt{2}} + c = -\frac{1}{2}\cdot\frac{t+1}{t^2+2} - \frac{1}{2\sqrt{2}}arctg\,\frac{t}{\sqrt{2}} + c =$$

$$= -\frac{1}{2}\cdot\frac{x+2}{x^2+2x+1} - \frac{1}{2\sqrt{2}}arctg\,\frac{x+1}{\sqrt{2}} + C.$$

8. Transformation of rational fractal into the simplest fractions

Let's show, that we can transform any regular rational number into the sum of the simplest fractions.

Here the regular rational fractal $\frac{P(x)}{Q(x)}$ is given. We'll suppose, that indexes of multinomial included in this fractal are real numbers, and that the given fractal is the irreducible (i.e. the numerator and the denominator don't have common roots).

The theorem 1. If the of the regular rational fractal looks like $(x-\alpha)^k Q_1(x)$, where $Q_1(x)$ -is an indivisible for $x-\alpha$ multinomial then we can represent this fractal as the sum of two right fractals

$$\frac{P(x)}{(x-\alpha)^k Q_1(x)} = \frac{A}{(x-\alpha)^k} + \frac{P_1(x)}{(x-\alpha)^{k-1}Q_1(x)}, \tag{2}$$

where A – is a certain number.

The proving. Let's examine the difference

$$\frac{P(x)}{(x-\alpha)^k Q_1(x)} - \frac{A}{(x-\alpha)^k} = \frac{P(x)-AQ_1(x)}{(x-\alpha)^k Q_1(x)}. \tag{3}$$

The fraction in the right part (3) is correct, because it is the difference of two regular fractals. We have to prove, that the difference (3)equals to the second fractal, which we see in the right part (2).

Let's define A so as the multinomial $P(x)-AQ_1(x)$ will be divided for $x-\alpha$ without the residual. It's possible when α is multinomial's root, i.e.

$$P(\alpha) - AQ_1(\alpha) = 0 \quad\Rightarrow\quad A = \frac{P(\alpha)}{Q_1(\alpha)}$$

$Q_1(\alpha) \neq 0$ (by theorem's condition). This means, that A is determined ambiguously, multinomial $P(x) - AQ_1(x)$ divides without the remainder for $x - \alpha$. Designating quotient from division through $P_1(x)$, we'll derive

$$P(x) - AQ_1(x) = (x - \alpha)P_1(x).$$

Consequently

$$\frac{P(x)}{(x-\alpha)^k Q_1(x)} - \frac{A}{(x-\alpha)^k} = \frac{P(x) - AQ_1(x)}{(x-\alpha)^k Q_1(x)} = \frac{P_1(x)}{(x-\alpha)^{k-1} Q_1(x)}.$$

Thus, the theorem is proved.

Corollary. To the regular rational fractal

$$\frac{P_1(x)}{(x-\alpha)^{k-1} Q_1(x)}$$

we can apply the same reasoning. That's why, if the denominator has the root $x = \alpha$, which is multiple to k, then we can write

$$\frac{P(x)}{Q(x)} = \frac{A}{(x-\alpha)^k} + \frac{A_1}{(x-\alpha)^{k-1}} + \cdots + \frac{A_{k-1}}{x-\alpha} + \frac{P_k(x)}{Q_1(x)},$$

where $\frac{P_k(x)}{Q_1(x)}$ – is a regular irreducible fractal.

The theorem 2. Let square trinomial $x^2 + px + q$, which doesn't have real roots, is a part of denominator's expansion $Q(x)$ of the regular rational fractal $\frac{P(x)}{Q(x)}$ in the power m.

$$Q(x) = (x^2 + px + q)^m Q_2(x).$$

Then we can find such real numbers as A и B, that has identity

$$\frac{P(x)}{Q(x)} = \frac{Ax + B}{(x^2 + px + q)^m} + \frac{F_1(x)}{(x^2 + px + q)^{m-1} Q_2(x)},$$

where $\frac{F_1(x)}{(x^2+px+q)^{m-1}Q_2(x)}$ – is the regular rational fractal.

The proving. For proving let's examine the difference

$$\frac{P(x)}{Q(x)} - \frac{Ax + B}{(x^2 + px + q)^m} = \frac{P(x)}{(x^2 + px + q)^m Q_2(x)} - \frac{Ax + B}{(x^2 + px + q)^m} =$$

$$= \frac{P(x) - (Ax + B)Q_2(x)}{(x^2 + px + q)^m Q_2(x)}$$

and we'll prove, that the difference $\frac{P(x)-(Ax+B)Q_2(x)}{(x^2+px+q)^m Q_2(x)}$ equals to $\frac{F_1(x)}{(x^2+px+q)^{m-1}Q_2(x)}$. assemble numbers A и B so as the multinomial

$$P(x) - (Ax + B)Q_2(x) \tag{1}$$

will be divided without the remainder into $x^2 + px + q$. For this case it's required and sufficient for

$$P(x) - (Ax + B)Q_2(x) = 0$$

to have the same roots $\alpha \pm i\beta$, as the multinomial $x^2 + px + q$. Consequently

$$P(\alpha + i\beta) - [A(\alpha + i\beta) + B]Q_2(\alpha + i\beta) = 0$$

or

$$A(\alpha + i\beta) + B = \frac{P(\alpha + i\beta)}{Q_2(\alpha + i\beta)},$$

where$\frac{P(\alpha+i\beta)}{Q_2(\alpha+i\beta)}$ – is the definite finite number, which can be written as $k + iL$, where k и L – are real numbers. That is

$$A(\alpha + i\beta) + B = k + iL,$$

then

$$A \cdot \alpha + B = k, \ \ A\beta = L.$$

Consequently

$$B = \frac{k\beta - L\alpha}{\beta}, \ \ A = \frac{L}{\beta}.$$

In this values A and B multinomial (1) has as the root the number $\alpha + i\beta$, and consequently it has and the adjoint number $\alpha - i\beta$. But in this case the multinomial (1) divides without the remainder by differences $x - (\alpha + i\beta)$ и $x - (\alpha - i\beta)$, and consequently on their product $x^2 + px + q$. Denoting the quotient of division through $F(x)$, we'll receive

$$P(x) - (Ax + B)Q_2(x) = (x^2 + px + q)F(x),$$

then

$$\frac{P(x) - (Ax + B)Q_2(x)}{(x^2 + px + q)^m Q_2(x)} = \frac{F(x)}{(x^2 + px + q)^{m-1} Q_2(x)}$$

that's what we need to prove.

Consequence. For the fraction

$$\frac{F(x)}{(x^2+px+q)^{m-1}Q_2(x)}$$

the results of the theorem can be applied 2. we can also consequently find out all the simplest fractions, corresponding to all denominator's roots.

Thus analyzing theorems 1 and 2 we come to the following result. If the denominator of the fraction $Q(x)$ can be presented as

$$Q(x) = (x-\alpha)^a(x-\beta)^b \dots (x^2+px+q)^\mu \dots (x^2+lx+s)^\lambda,$$

then the fraction $\frac{P(x)}{Q(x)}$ can be produced as

$$\frac{P(x)}{Q(x)} = \frac{A}{(x-\alpha)^\alpha} + \frac{A_1}{(x-\alpha)^{\alpha-1}} + \cdots \frac{A_{\alpha-1}}{x-\alpha} +$$

$$+\frac{B}{(x-\beta)^b} + \frac{B_1}{(x-\beta)^{b-1}} + \cdots + \frac{B_{b-1}}{x-\beta} +$$

$$+\frac{Mx+N}{(x^2+px+q)^\mu} + \frac{M_1x+N_1}{(x^2+px+q)^{\mu-1}} + \cdots + \frac{M_{\mu-1}x+N_{\mu-1}}{x^2+px+q} +$$

$$+\frac{Px+Q}{(x^2+lx+s)^\lambda} + \cdots + \frac{P_{\lambda-1}x+Q_{\lambda-1}}{x^2+lx+s}.$$

9. The method of indefinite indexes

The method of indefinite indexes is one of the simplest methods of finding indexes in transformation of regular fraction into simple fractions. Let's clarify appliance of this methods in the following examples.

The 1st example. To transform into the simplest fractions

$$\frac{x^2+3x-1}{(x-1)^2(x^2+x+1)}.$$

The answer.

$$\frac{x^2+3x-1}{(x-1)^2(x^2+x+1)} = \frac{A}{(x-1)^2} + \frac{B}{x-1} + \frac{Cx+d}{x^2+x+1}$$

$$\begin{cases} B+C &= 0, \\ A-2C+D &= 1, \\ A+C-2D &= 3, \\ A-B+D &= -1, \end{cases}$$

$$\begin{cases} A &= 1, \\ B &= \frac{2}{3}, \\ C &= -\frac{2}{3}, \\ D &= -\frac{4}{3}. \end{cases}$$

$$\frac{x^2+3x-1}{(x-1)^2(x^2+x+1)} = \frac{1}{x-1} + \frac{2}{3}\cdot\frac{1}{(x-1)^2} - \frac{1}{3}\cdot\frac{2x+4}{x^2+x+2}.$$

The 2nd example. To transform into the simplest fractions

$$\frac{x^2+x+1}{(x^2+1)^2(x^2+x+2)}.$$

The answer.

$$\frac{x^2+x+1}{(x^2+1)^2(x^2+x+2)} = \frac{Ax+B}{(x^2+1)^2} + \frac{Cx+D}{x^2+1} + \frac{Ex+F}{x^2+x+2}$$

$$\begin{cases} C+E &= 0, \\ C+D+F &= 0, \\ A+3C+D+2E &= 0, \\ A+B+C+3D+2F &= 1, \\ 2A+B+2C+D+E &= 1, \\ 2B+2D+F &= 1, \end{cases}$$

$$\begin{cases} A &= \frac{1}{2}, \\ B &= \frac{1}{2}, \\ C &= -\frac{1}{4}, \\ D &= -\frac{1}{4}, \\ E &= \frac{1}{4}, \\ F &= \frac{1}{2}. \end{cases}$$

10. Integration of the rational fractions by the method of indefinite indexes

Considered material allows us to state *the general rule of rational functions' integration.*

1. If the fraction is indefinite, then it is presented as a sum of a multinomial and a regular fraction.
2. Regular fraction's denominator is expansioned into factors.
3. Regular rational fraction is presented as a sum of the simplest fractions.
4. To integrate the multinomial and the obtained sum of the simplest fractions.

The example. To find the integral

$$I = \int \frac{x^5 + 2x^3 + 4x + 4}{x^4 + 2x^3 + 2x^2} dx.$$

The answer. Under the integral's sign is the indefinite fraction. Let's intercept its integer. We derive

$$\frac{x^5 + 2x^3 + 4x + 4}{x^4 + 2x^3 + 2x^2} = x - 2 + \frac{4x^3 + 4x^2 + 4x + 4}{x^4 + 2x^3 + 2x^2}.$$

Let's expansion the real rational fraction into the simplest fractions

$$\frac{4x^3 + 4x^2 + 4x + 4}{x^4 + 2x^3 + 2x^2} = \frac{4x^3 + 4x^2 + 4x + 4}{x^2(x^2 + 2x + 2)} = \frac{A}{x^2} + \frac{B}{x} + \frac{Cx + D}{x^2 + 2x + 2},$$

$$4x^3 + 4x^2 + 4x + 4 = A(x^2 + 2x + 2) + Bx(x^2 + 2x + 2) + (Cx + D)x^2,$$

$$4x^3 + 4x^2 + 4x + 4 = (B + C)x^3 + (A + 2B + D)x^2 + (2A + 2B)x + 2A.$$

$$\left\{ \begin{array}{rcl} B + C &=& 4, \\ A + 2B + D &=& 4, \\ 2A + 2B &=& 4, \\ 2A &=& 4, \end{array} \right. \quad \left\{ \begin{array}{rcl} A &=& 2, \\ B &=& 0, \\ C &=& 4, \\ D &=& 2. \end{array} \right.$$

Thus

$$\frac{4x^3 + 4x^2 + 4x + 4}{x^4 + 2x^3 + 2x^2} = \frac{2}{x^2} + \frac{4x + 2}{x^2 + 2x + 2}.$$

$$\frac{x^5 + 2x^3 + 4x + 4}{x^4 + 2x^3 + 2x^2} = x - 2 + \frac{2}{x^2} + \frac{4x + 2}{x^2 + 2x + 2}.$$

We are integrating the obtained equality

$$\int \left(x - 2 + \frac{2}{x^2} + \frac{4x + 2}{x^2 + 2x + 2} \right) dx = \frac{x^2}{2} - 2x - \frac{2}{x} + \int \frac{4x + 2}{x^2 + 2x + 2} dx,$$

$$\int \frac{4x+2}{x^2+2x+2}dx = 2\,ln\,(x^2+2x+2) - 2arctg\,(x+1) + C.$$

Consequently,

$$I = \frac{x^2}{2} - 2x - \frac{2}{x} + 2\,ln\,(x^2+2x+2) - 2arctg\,(x+1) + C.$$

11. Integrals coming out of irrational functions

Integral is evaluated by elementary functions not from each irrational function. Let's examine those irrational functions, integrals out of which with the help of method of substitution depress to integrals of rational functions, and consequently they integrate till the end.

I. Let's examine the integral

$$\int R(x,\, x^{\frac{m}{n}}, \cdots,\, x^{\frac{r}{s}})dx,$$

where R – is a rational function of its arguments.

Let k – to be the common denominator of fractions $\frac{m}{n}, \cdots, \frac{r}{s}$. Let's carry out the substitution

$$x = t^k, \qquad dx = kt^{k-1}dt$$

Then eachfractional power x is expressed throughwhole power,and consequently, the integrand function transforms into the rational function.

Example. We must compute

$$\int \frac{\sqrt{x}dx}{\sqrt[4]{x^3}+1}.$$

The answer. The common denominator of fractions $\frac{1}{2}$ and $\frac{3}{4}$ is 4, therefore let's perform the substitution $x = t^4$, $dx = 4t^3dt$, then

$$\int \frac{\sqrt{x}dx}{\sqrt[4]{x^3}+1} = \int \frac{t^2 4t^3 dt}{t^3+1} = 4\int \frac{t^5 dt}{t^3+1} = 4\int \left(t^2 - \frac{t^2}{t^3+1}\right)dt =$$

$$= 4\frac{t^3}{3} - \frac{4}{3}ln\,|t^3+1| + C = \frac{4}{3}\left[\sqrt[4]{x^3} - ln\,|\sqrt[4]{x^3}+1|\right] + C.$$

II. Let's examine the integral of the following type

$$\int R\left[x,\left(\frac{ax+b}{cx+d}\right)^{\frac{m}{n}},\cdots,\left(\frac{ax+b}{cx+d}\right)^{\frac{r}{s}}\right]dx,$$

where R – is a rational function.

This integral comes to the integral of a rational function by the method of substitution.

$$\frac{ax+b}{cx+d}=t^k$$

where k – is the common denominator of fractions $\frac{m}{n},\cdots,\frac{r}{s}$.

Example. We need to compute

$$\int\frac{1+\sqrt{\frac{1-x}{1+x}}}{1+x}dx.$$

The answer. We'll substitute $\frac{1-x}{1+x}=t^2$, $dx=\frac{-4t}{(1+t^2)^2}dt$. Consequently,

$$\int\frac{1+\sqrt{\frac{1-x}{1+x}}}{1+x}dx=\int\frac{1+t}{1+\frac{1-t^2}{1+t^2}}\cdot\frac{-4t}{(1+t^2)^2}dt=$$

$$=-2\int\frac{t^2+t}{1+t^2}dt=-2\int\left(1+\frac{t-1}{t^2+1}\right)dt=$$

$$=-2t-ln\,|t^2+1|+2\,arctg\,t+C=$$

$$=-2\sqrt{\frac{1-x}{1+x}}-ln\,|1+x|+2\,arctg\sqrt{\frac{1-x}{1+x}}+C.$$

12. The differential binomials' integration

The expression of type

$$x^m(a+bx^n)^p dx \tag{1}$$

where a, b, m, n, p – are constant numbers, is called a differential binomial.

The theorem. The integral from the differential binomial

$$\int x^m(a+bx^n)^p dx, \tag{2}$$

if m, n, p – are rational numbers, comes to the integral out of the rational function, and consequently, it's expressed through the elementary functions in three following cases

1) p – is the integer;

2) $\frac{m+1}{n}$ – is the integer;

3) $\frac{m+1}{n}+p$ – is the integer.

The proving: Let's make a replacement of the variable in the integral (2)

$$x^n = z$$

$$x = \sqrt[n]{z}$$

$$dx = \frac{1}{n} z^{\frac{1}{n}-1} dz$$

Consequently,

$$\int x^m (a+bx^n)^p dx = \int z^{\frac{m}{n}} (a+bz)^p \frac{1}{n} z^{\frac{1}{n}-1} dz =$$

$$= \frac{1}{n} \int z^{\frac{m+1}{n}-1} (a+bz)^p dz = \frac{1}{n} \int z^q (a+bz)^p dz \qquad (3)$$

where $q = \frac{m+1}{n} - 1$. Now let's examine the different cases. 1) Let p – to be the integer, q – is the rational fraction ($q = \frac{r}{s}$). Thus, the following replacement of the variable takes place

$$z^{\frac{1}{s}} = t$$

$$z = t^s$$

$$dz = st^{s-1} dt$$

$$\frac{1}{n} \int z^{\frac{r}{s}} (a+bz)^p dz = \frac{s}{n} \int t^{r+s-1} (a+bt^s)^p dt$$

2) Let $\frac{m+1}{n}$ – to be the integer, then $\frac{m+1}{n} - 1$ – is also the integer, p – is the rational fraction ($p = \frac{r}{s}$). In this case by the variable's replacement

$$(a+bz)^{\frac{1}{s}} = t$$

$$a + bz = t^s$$

$$z = \frac{t^s - a}{b}$$

$$dz = \frac{st^{s-1}}{b} dt$$

the integral (3) comes to the integral from the rational function

$$\frac{1}{n}\int z^q(a+bz)^{\frac{r}{s}}dz = \frac{s}{nb}\int \left(\frac{t^s-a}{b}\right)^q t^{r+s-1}dt$$

3) Let $\frac{m+1}{n}+p$ – to be the integer. But then $\frac{m+1}{n}-1+p=q+p$ – is the integer. We have to transform the integral (3)

$$\int z^q(a+bz)^p dz = \int z^{q+p}\left(\frac{a+bz}{z}\right)^p dz$$

where $p+q=l$ – is the integer, $p=\frac{r}{s}$ – is the rational number. Then

$$\frac{1}{n}\int z^q(a+bz)^p dz = \frac{1}{n}\int z^l\left(\frac{a+bz}{z}\right)^{\frac{r}{s}} dz$$

By making the replacement

$$\frac{a+bz}{z}=t^s$$

the given integral comes to the integral from the rational function.

The great Russian mathematician P.L. Tchebishev have proved, that if m, n, p – are the rational numbers, which don't obey to any of the stated above three cases, then the integral from the differential binomial doesn't express in the elementary functions.

The 1st example.

$$\int \frac{dx}{\sqrt[3]{x^2}(1+\sqrt[3]{x^2})} = \int x^{-\frac{2}{3}}\left(1+x^{\frac{2}{3}}\right)^{-1}dx.$$

Here $p=-1$ – is the integer. We'll replace

$$x^{\frac{2}{3}}=z,\ dx=\frac{3}{2}z^{\frac{1}{2}}dz,$$

$$\int x^{-\frac{2}{3}}\left(1+x^{\frac{2}{3}}\right)^{-1}dx = \int z^{-1}(1+z)\frac{3}{2}z^{\frac{1}{2}}dz = \frac{3}{2}\int z^{-\frac{1}{2}}(1+z)^{-1}dz.$$

Now we'll make the replacement

$$z^{\frac{1}{2}}=t,\ dz=2tdt,$$

$$\frac{3}{2}\int z^{-\frac{1}{2}}(1+z)^{-1}dz = \frac{3}{2}\int t^{-1}(1+t^2)^{-1}2tdt = 3\int\frac{dt}{1+t^2} =$$

$$= 3\,arctg\,t + C = 3\,arctg\,\sqrt{z}+C = 3\,arctg\,\sqrt[3]{x}+C.$$

The 2nd example.

$$\int \frac{x^3}{\sqrt{1-x^2}}dx = \int x^3(1-x^2)^{-\frac{1}{2}}dx$$

$$m = 3, \quad n = 2, \quad p = -\frac{1}{2} \qquad \frac{m+1}{n} = 2 \text{ – is the integer}$$

$$x^2 = z, \; 1 - z = t^2.$$

The answer: $\frac{\sqrt{1-x^2}}{3}(-x^2-2)+C$.

The 3rd example.

$$\int \frac{dx}{x^2\sqrt{(1+x^3)^2}} = \int x^{-2}(1+x^2)^{-\frac{3}{2}}dx$$

$$m = -2, \quad n = 2, \quad p = -\frac{3}{2}, \qquad \frac{m+1}{n} + p = -2 \text{ – is the integer}$$

$$x^2 = z, \; \left(\frac{1+z}{z}\right)^{\frac{1}{2}} = t.$$

The answer: $-\frac{\sqrt{1+x^2}}{x} - \frac{x}{\sqrt{1+x^2}} + C$.

13. Integrals of type $\int R(x, \sqrt{ax^2+bx+c})dx$. Eiler's substitutions

Let's examine the integrals of type

$$\int R(x, \sqrt{ax^2+bx+c})dx \tag{1}$$

where R – is the rational expression from x и $\sqrt{ax^2+bx+c}$.

Such integrals come to the integral from the rational function with the help of one of Eiler's substitution.

1. The first Eiler's substitution. It's applicable if $a > 0$. In this case we suppose

$$\sqrt{ax^2+bx+c} = \pm\sqrt{a}\cdot x + t.$$

For determinacy we 'll take before $\sqrt{a}$ the plus sing. Then

$$ax^2+bx+c = ax^2 + 2\sqrt{a}xt + t^2$$

wherefrom

$$x = \frac{t^2 - c}{b - 2\sqrt{a}t}$$

(it means that dx is rationally expressed through t). Consequently,

$$\sqrt{ax^2 + bx + c} = \sqrt{a}x + t = \sqrt{a}\frac{t^2 - c}{b - 2\sqrt{a}t} + t,$$

i.e. $\sqrt{ax^2 + bx + c}$ – is also the rational function from также t. Consequently, the integral (1) comes to the integral from the rational function from t.

2. The second Eiler's substitution. If $c > 0$, then we consider

$$\sqrt{ax^2 + bx + c} = xt \pm \sqrt{c}.$$

Then

$$ax^2 + bx + c = x^2t^2 + 2xt\sqrt{c} + c$$

(For determinacy we take the plus sign before the root). Herefrom

$$x = \frac{2\sqrt{c}t - b}{a - t^2}$$

i.e. x – is the rational function from e, and this means the same for dx,and consequently the integral (1) comes to the integral from the rational function from t.

3. The third Eiler's substitution. It's used only if the trinomial ax^2+bx+c has the real roots. Let α and β – to be the roots of the trinomial $ax^2 + bx + c$. We suppose

$$\sqrt{ax^2 + bx + c} = (x - 2)t.$$

Since $ax^2 + bx + c = a(x - \alpha)(x - \beta)$, then

$$\sqrt{a(x - \alpha)(x - \beta)} = (x - \alpha)t,$$

$$a(x - \alpha)(x - \beta) = (x - \alpha^2t^2,$$

$$a(x - \beta) = (x - \alpha)t^2.$$

Herefrom we find out x as the rational function from t

$$x = \frac{a\beta - \alpha t^2}{a - t^2}.$$

It means that the integral (1) comes to the integral from the rational function from t.

Observation 1. The third Eiler's substitution is used when $a > 0$ and when $a < 0$.

Observation 2. Integrals of type

$$\int \frac{Mx+N}{\sqrt{a\,x^2+bx+c}}dx$$

can be found with the help of the first or the third Eiler's substitution. However computation of these integrals easier to do with the following substitution:

$$z = ax + \frac{b}{2}$$

($ax + \frac{b}{2}$ – is $\frac{1}{2}$ the derivative from ax^2+bx+c).

$$\int \frac{Mx+N}{\sqrt{ax^2+bx+c}}dx = M_1 \int \frac{zdz}{\sqrt{az^2+m}} + N_1 \int \frac{dz}{\sqrt{az^2+m}}.$$

The first integral comes to the integral from the power function, and the second under the circumstances $a < 0$ comes to arc sine. With $a > 0$ we'll examine the answer in the next example.

The example. We should compute the integral

$$\int \frac{dx}{\sqrt{x^2+c}}.$$

The answer. Since here we have $a = 1 > 0$, then the first Eiler's substitution is used

$$\sqrt{x^2+c} = -x + t,$$

$$x^2 + c = x^2 - 2xt + t^2,$$

$$x = \frac{t^2-c}{2t},\ dx = \frac{t^2+c}{2t^2}dt,$$

$$\sqrt{x^2+c} = \frac{t^2+c}{2t}.$$

We receive

$$\int \frac{dx}{\sqrt{x^2+c}} = \int \frac{\frac{t^2+c}{2t^2}dt}{\frac{t^2+c}{2t}} = \int \frac{dt}{t} = ln\,|x + \sqrt{x^2+c}| + C_1.$$

The example. We need to compute the integral

$$\int \frac{dx}{\sqrt{x^2+3x-4}}$$

The answer. Since $x^2+3x-4=(x+4)(x-1)$, then we suppose

$$\sqrt{(x+4)(x-1)}=(x+4)t$$

Then

$$(x+4)(x-1)=(x+4)^2t^2,$$

$$x-1=(x+4)t^2,$$

$$x=\frac{1+4t^2}{1-t^2},\ dx=\frac{10t}{(1-t^2)^2}dt,$$

$$\sqrt{(x+4)(x-1)}=(x+4)t=\left(\frac{1+4t^2}{1-t^2}+4\right)t=\frac{5t}{1-t^2}.$$

We receive,

$$\int \frac{dx}{\sqrt{x^2+3x-4}}=\int \frac{10t(1-t^2)}{(1-t^2)^2 5t}dt=\int \frac{2}{1-t^2}dt=$$

$$=ln\left|\frac{1+t}{1-t}\right|+C=ln\left|\frac{\sqrt{x+4}+\sqrt{x-1}}{\sqrt{x+4}-\sqrt{x-1}}\right|+C.$$

14. Integrals of the type $\int \frac{P_n(x)dx}{\sqrt{ax^2+bx+c}}$

Integrals of the type

$$\int \frac{P_n(x)dx}{\sqrt{ax^2+bx+c}},$$

where $P_n(x)$ – is the multinomial of the power n, using the formula the following formula we can compute

$$\int \frac{P_n(x)dx}{\sqrt{ax^2+bx+c}}=Q_{n-1}(x)\sqrt{ax^2+bx+c}+\lambda\int \frac{dx}{\sqrt{ax^2+bx+c}}, \qquad (2)$$

where $Q_{n-1}(x)$ – is the multinomial with the indefinite indexes of the power $n-1$, and λ – is also the indefinite index.

All the indefinite indexes are got from identity, which is received by differentiating of the both parts of the equality (2)

$$\frac{P_n(x)}{\sqrt{ax^2+bx+c}} = \left(Q_{n-1}(x)\sqrt{ax^2+bx+c}\right)' + \frac{\lambda}{\sqrt{ax^2+bx+c}},$$

after what we must equate the indexes under the equal powers of the unknown x.

The example. It is required to to calculate the integral

$$I = \int \frac{x^2}{\sqrt{1-2x-x^2}}dx$$

The answer. Using the formula (2), we get

$$I = \int \frac{x^2}{\sqrt{1-2x-x^2}}dx = (Ax+B)\sqrt{1-2x-x^2} + \lambda \int \frac{dx}{\sqrt{1-2x-x^2}}$$

Differentiating this equality, we get

$$\frac{x^2}{\sqrt{1-2x-x^2}} = A\sqrt{1-2x-x^2}+$$

$$+(Ax+B)\frac{-2-2x}{2\sqrt{1-2x-x^2}} + \frac{\lambda}{\sqrt{1-2x-x^2}}$$

$$x^2 = A(1-2x-x^2) + (Ax+B)(-1-x) + \lambda.$$

Comparing the indexes under the equal powers x

$$\begin{cases} -2A &= 1, \\ -3A - B &= 0, \\ A - B + \lambda &= 0. \end{cases} \quad \begin{cases} A &= -\frac{1}{2}, \\ B &= \frac{3}{2}, \\ \lambda &= 2. \end{cases}$$

Consequently,

$$I = \left(-\frac{1}{2}x + \frac{3}{2}\right)\sqrt{1-2x-x^2} + 2\int \frac{dx}{\sqrt{2-(x+1)^2}} =$$

$$= \left(-\frac{1}{2}x + \frac{3}{2}\right)\sqrt{1-2x-x^2} + 2\,arcsin\,\frac{x+1}{\sqrt{2}} + C.$$

15. Integration of some trigonometrical functions

The universal trigonometrical substitution. Let's examine the integral of the type

$$\int R(\sin x,\, \cos x)dx, \tag{1}$$

where$R$ – is its arguments' rational function. We'll show that this integral cones to the integral from the rational function with the help of the method of substitution

$$tg\,\frac{x}{2} = t. \tag{2}$$

We'll express $\sin x$ and $\cos x$ through $tg\,\frac{x}{2}$, and consequently through t.

$$\sin x = \frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{1} = \frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{\sin^2\frac{x}{2}+\cos^2\frac{x}{2}} = \frac{2tg\,\frac{x}{2}}{1+tg^2\,\frac{x}{2}} = \frac{2t}{1+t^2},$$

$$\cos x = \frac{\cos^2\frac{x}{2}-\sin^2\frac{x}{2}}{1} = \frac{\cos^2\frac{x}{2}-\sin^2\frac{x}{2}}{\sin^2\frac{x}{2}+\cos^2\frac{x}{2}} = \frac{1-tg^2\,\frac{x}{2}}{1+tg^2\,\frac{x}{2}} = \frac{1-t^2}{1+t^2}.$$

Considering (2), we'll find

$$x = 2arctg\,t,\ dx = \frac{2dt}{1+t^2}.$$

Thus $\sin x,\ \cos x$ and dx were rationally expressed through t, that's why substituting the received expressions in (1), we'll have the integral from the rational function

$$\int R(\sin x,\, \cos x)dx = \int R\left(\frac{2t}{1+t^2}, \frac{1-t^2}{1+t^2}\right)\frac{2dt}{1+t^2}.$$

The example.

$$\int \frac{dx}{\sin x} = \int \frac{\frac{2dt}{1+t^2}}{\frac{2t}{1+t^2}} = \int \frac{dt}{t} = \ln|t| + C = \ln|tg\,\frac{x}{2}| + C.$$

With the help of the universal trigonometrical substitution the integral (1) always comes to the integral from the rational function. However practically it often leads to too cumbrous calculations That's why it's useful to remember another substitutions, which make these calculations much easier.

1) The integral of the type

$$\int R(\sin x)\cos x\,dx$$

with the method of substitution $\sin x = t, \cos x\, dx = dt$ comes to the integral

$$\int R(t)dt.$$

2) The integral of the type

$$\int R(\cos x)\sin x\, dx$$

comes to the integral from the rational function with the method of substitution $\cos x = t,\ , \sin x\, dx = -dt.$

3) If the integrand function depends only on $tg\, x$, then the replacement $tg\, x = t,\ \ dx = \frac{dt}{1+t^2}$, depresses the integral to the integral from the rational function.

The example. It's required to find the integral

$$\int \frac{\sin^3 x}{2+\cos x} dx.$$

The answer. This integral depresses to the form $\int R(\cos x)\sin x\, dx$

$$\int \frac{\sin^3 x}{2+\cos x} dx = \int \frac{\sin^2 x\ \sin x}{2+\cos x} dx = \int \frac{1-\cos^2 x}{2+\cos x} \sin x\, dx$$

We'll substitute $\cos x = t$, then $\sin x\, dx = -dt$

$$\int \frac{\sin^3 x}{2+\cos x} dx = \int \frac{t^2-1}{t+2} dt = \int (t-2+\frac{3}{t+2})dt =$$

$$= \frac{t^2}{2} - 2t + 3\ln|t+2| + C = \frac{\cos^2 x}{2} - 2\cos x + 3\ln(\cos x + 2) + C.$$

4) The integrals of type

$$\int \sin^m x\ \cos^n x\, dx,$$

where m и n – the integers. Here three variants are possible

a) one of the numbers m и n – is uneven. Let, for example, n – is uneven, then $n = 2p+1$.

$$\int \sin^m x\ \cos^{2p+1} x\, dx = \int \sin^m x\,(1-\sin^2 x)^p \cos x\, dx.$$

Let's make the substitution $\sin x = t, \cos x\, dx = dt$. We get

$$\int \sin^m x\ \cos^n x\, dx = \int t^m(1-t^2)^p dt.$$

The example.

$$\int \sin^2 x \cos^3 x\, dx = \int \sin^2 x \cos^2 x \cos x\, dx =$$

$$= \int \sin^2 x (1 - \sin^2 x) \cos x\, dx = \begin{pmatrix} \sin x = t \\ \cos x\, dx = dt \end{pmatrix} =$$

$$= \int t^2 (1 - t^2) dt = \int (t^2 - t^4) dt = \frac{t^3}{3} - \frac{t^5}{5} + C =$$

$$= \frac{\sin^3 x}{3} - \frac{\sin^5 x}{5} + C.$$

б) m and n – are the even nonnegative numbers. We'll use the formulas of the power's decrease.

$$\sin^2 x = \frac{1 - \cos 2x}{2}, \qquad \cos^2 x = \frac{1 + \cos 2x}{2}.$$

$$\int \sin^4 x\, dx = \frac{1}{2^2} \int (1 - \cos 2x)^2 dx = \frac{1}{4} \int (1 - 2\cos 2x + \cos^2 2x) dx =$$

$$= \frac{1}{4} \left[x - \sin 2x + \frac{1}{2} \int (1 + \cos 4x) dx \right] = \frac{1}{4} \left[\frac{3}{2} x - \sin 2x + \frac{\sin 4x}{8} \right] + C.$$

в) m и n – are even, although even one of them is negative. We must make the substitution $tg\, x = t$ (or $ctg\, x = t$).

The example.

$$\int \frac{\sin^2 x}{\cos^6 x} dx = \int \frac{\sin^2 x (\sin^2 x + \cos^2 x)^2}{\cos^6 x} dx = \int tg^2 x (1 + tg^2 x)^2 dx =$$

$$= \begin{pmatrix} tg\, x = t \quad x = arctg\, t \\ dx = \frac{dt}{1+t^2} \end{pmatrix} = \int t^2 (1 + t^2)^2 \frac{dt}{1 + t^2} = \int t^2 (1 + t^2) dt =$$

$$= \frac{t^3}{3} + \frac{t^5}{5} + C = \frac{tg^3 x}{3} + \frac{tg^5 x}{5} + C.$$

5) The integrals of the type

$$\int \cos mx \cos nx\, dx, \int \sin mx \cos nx\, dx, \int \sin mx \sin nx\, dx.$$

are calculated with formulas

$$\cos mx \cos nx = \frac{1}{2} [\cos(m + n)x + \cos(m - n)x],$$

$$\sin mx\ \cos nx = \frac{1}{2}[\sin(m+n)x + \sin(m-n)x],$$

$$\sin mx\ \sin nx = \frac{1}{2}[-\cos(m+n)x + \cos(m-n)x].$$

The example.

$$\int \sin 5x\ \sin 3x\, dx = \frac{1}{2}\int(-\cos 8x + \cos 2x)dx = -\frac{\sin 8x}{16} + \frac{\sin 2x}{4} + C.$$

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