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COMPUTING SOLUTION OPERATORS OF BOUNDARY-VALUE PROBLEMS FOR SOME LINEAR HYPERBOLIC SYSTEMS OF PDES

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Abstract. We discuss possibilities of application of Numerical Analysis methods to proving computability, in the sense of the TTE approach, of solution operators of boundary-value problems for systems of PDEs. We prove computability of the solution operator for a symmetric hyperbolic system with computable real coefficients and dissipative boundary conditions, and of the Cauchy problem for the same system (we also prove computable dependence on the coefficients) in a cube $Q \subseteq \mathbb{R}^m$. Such systems describe a wide variety of physical processes (e.g. elasticity, acoustics, Maxwell equations). Moreover, many boundary-value problems for the wave equation also can be reduced to this case, thus we partially answer a question raised in [WZ02]. Compared with most of other existing methods of proving computability for PDEs, this method does not require existence of explicit solution formulas and is thus applicable to a broader class of (systems of) equations.

1. Introduction

We consider boundary-value problems for systems of PDEs of the form

$$\begin{cases} L\mathbf{u}(y) = f(y) \in C^p(\Omega, \mathbb{R}^n), \quad y \in \Omega \subset \mathbb{R}^k \\ \mathcal{L}\mathbf{u}(y)|_\Gamma = \varphi(y\,|_\Gamma) \in C^q(\Gamma, \mathbb{R}^n),\ \Gamma \subseteq \partial\Omega, \end{cases} \tag{1.1}$$

where L and $\mathcal{L}$ are differential operators (the differential order of $\mathcal{L}$ is less than the one of L), Γ is a part of the boundary $\partial\Omega$ of some area Ω. In particular, if $\Gamma = \{t = 0\}$ and t is among the variables $y_1, y_2, \ldots, y_k$, then (1.1) is a Cauchy (or initial-value) problem. Assuming existence and uniqueness of the solution $\mathbf{u}$ in Ω, we study computability properties of the solution operator $R : (L, \mathcal{L}, f, \varphi) \mapsto \mathbf{u}$. Note that in (1.1) the number k of "space" variables $y_1, y_2, \ldots, y_k$ is not necessarily equal to the number n of the unknown functions $u_1, u_2, \ldots, u_n$, e.g. for the linear elasticity equations (2.4) we have $n = 9, k = 4$.

Key words and phrases: Systems of PDEs, boundary-value problem, Cauchy problem, computability, solution operator, symmetric hyperbolic system, wave equation, difference scheme, stability, finite-dimensional approximation, constructive field, algebraic real.

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Computability will be understood in the sense of Weihrauch's TTE approach [We00]. Recently the following main achievements in the study of computability properties of PDEs were made. Computability of solution operators of initial-value problems for the wave equation [WZ02], Korteveg de Vries equation [GZZ01, WZ05], linear and nonlinear Schrödinger equations [WZ06] was established; also computability of fundamental solutions of PDEs with constant coefficients $Pu = \sum\limits_{|\alpha|\leq M} c_\alpha D^\alpha u = f$ was proved in [WZ06-2]. Most of the methods of the mentioned papers are based on a close examination of explicit solution formulas and the Fourier transformation method, except for the paper [WZ05] where a method based on fixed point iterations is introduced. In these papers, the initial data and solutions are mainly assumed to belong to some Sobolev classes of generalized functions.

As is well-known, explicit solution formulas for boundary-value problems (even for the Cauchy initial-value problems) exist rarely. Even for the simplest example of the wave equation the computability of the solution operator for boundary-value problem was formulated in [WZ02] as an open question, and we have not seen any paper where this question would be answered. Results of our paper provide, in particular, a positive answer to this question for the case of computable real coefficients and dissipative boundary conditions, for classes of continuously differentiable functions with uniformly bounded derivatives.

In [SS09] we propounded an approach to study the computability of PDEs based on finite-dimensional approximations (difference schemes widely used in numerical analysis) and established computability, in the sense of the TTE approach, of the solution operator $\varphi \mapsto \mathbf{u}$ of the Cauchy problem for a symmetric hyperbolic system, with a zero right-hand part, of the form

$$\begin{cases} A\frac{\partial \mathbf{u}}{\partial t} + \sum\limits_{i=1}^{m} B_i \frac{\partial \mathbf{u}}{\partial x_i} = 0, \ t \geq 0, \\ \mathbf{u}|_{t=0} = \varphi(x_1, \dots, x_m). \end{cases} \tag{1.2}$$

Here $A = A^* > 0$ and $B_i = B_i^*$ are constant symmetric computable $n \times n$-matrices, $t \geq 0$, $x = (x_1, \dots, x_m) \in Q = [0,1]^m$, $\varphi : Q \to \mathbb{R}^n$ and $\mathbf{u} : Q \times [0, +\infty) \rightharpoonup \mathbb{R}^n$ is a partial function acting on the domain H of existence and uniqueness of the Cauchy problem (1.2). In [SS09] the computability of the domain H (which is a convex polyhedron depending only on A, B_i) was also proved. The operator R mapping a C^{p+1} function φ to the unique C^p solution ($p \geq 2$) is computable, if the norms of the first and second partial derivatives of φ are uniformly bounded.

Such systems can be used to describe a wide variety of physical processes like those considered in the theories of elasticity, acoustics, electromagnetism etc., see e.g. [Fr54, God71, God76, LL86, LL04, KPS01, GM98]. They were first considered in 1954 by K.O. Friedrichs [Fr54]. He proved the existence theorem based on finite difference approximations, in contrast with the Schauder-Cauchy-Kovalevskaya method based on approximations by analytic functions and a careful study of infinite series. The notion of a hyperbolic system (applicable also to broader classes of systems) is due to I.G. Petrovskii [Pe37], see also the very interesting discussion on different notions of hyperbolicity and their motivations in [Fr54].

Recall that a linear first-order differential operator $E = \sum\limits_{\mu=1}^{m} A_\mu \frac{\partial}{\partial x_\mu} + B$, where A_μ, B are real $n \times n$ matrices, $\mu = 1, 2, \dots, m$, is called *hyperbolic* in the sense of Petrovskii, if there is a $\xi^0 \in \mathbb{R}^m$ such that, for all $\xi \in \mathbb{R}^m$, the matrix pencil

$$\sum_{\mu=1}^{m} \xi_\mu A_\mu - \lambda \sum_{\mu=1}^{m} \xi_\mu^0 A_\mu$$

has real eigenvalues λ. In particular, if all the matrices A_μ, $\mu = 1, 2, \ldots, m$ are symmetric and one of them is positive-definite, as in (1.2), then the operator E is obviously hyperbolic in this sense.

The Friedrichs' method has turned out to be interesting from the computational point of view because it yields algorithms for solving PDEs's in the exact sense of Computable Analysis which are based on methods really used in Numerical Analysis.

In this paper we prove computability for a broad class of boundary-value problems for (1.2), by using the difference approximations approach stemming from the work [Fr54] and developed in [GR62, God71, God76, KPS01] and others. Many details of our proofs are similar to those of the proof of the existence theorem for the linear hyperbolic systems in [God71, God76, Fr54] but, since we refer to more rigorous approach of computable analysis we are forced to establish several additional estimates. Actually, these proofs are based on careful estimates of the difference approximations of the considered differential operators, ideas of which can be also found e.g. in [Fr54, St04, GR62, GV96].

Our study intensively uses the well-known classical theorem of the theory of difference schemes stating that the approximation and stability properties of a difference scheme imply its convergence to the solution of the correspondent differential equation in a suitable grid norm uniformly on steps.

The proofs of this paper rely also on the well-known fact that the ordered field of algebraic real numbers and some extensions of this field are strongly constructivizable (this is closely related to the Tarski's quantifier elimination for real closed fields, see e.g. [Ta51, BPR06]) which implies computability of necessary spectral characteristics of symmetric matrices with algebraic real coefficients. This makes obvious computability of all steps in the iterative process induced by the difference scheme used in this paper. This trick also leads to an improvement of the main result in [SS09] to the result that the solution operator for the Cauchy problem (1.2) is computable not only on φ but also on the coefficients A, B_i (under some additional assumptions, see Theorem 5.3).

Our proofs here give some additional details compared to those in [SS09]. They make use of results in several fields: PDEs, difference schemes, computable analysis, computable fields. The results and proofs establish a close connection between computable number fields and computable reals and apply this connection to proving computability of solutions of some PDEs. Unfortunately, they do not yield reasonable upper bounds for the complexity of solving the initial and boundary value problems for PDEs because the corresponding results on computable fields are established here with the use of unbounded search algorithms. Search for more feasible algorithms is a natural further step in the study of computability properties of PDEs.

In Section 2 we describe the considered problems and assumptions we need to prove the computability of solution operators. Some necessary notions and facts are recalled in Section 3. Section 4 is devoted to the construction of a difference operator approximating the differential problem and its basic properties. In Section 5 we formulate precisely the main results of the paper and describe the proof schemes, without technical details of the corresponding estimates. The technical details are proved in Section 6. We conclude in Section 7 by a short discussion on more general systems (1.1).

2. Statement of the boundary-value problem and examples

Along with the Cauchy problem (1.2) we now consider the following boundary-value problem:

$$\begin{cases} A\frac{\partial \mathbf{u}}{\partial t} + \sum\limits_{i=1}^{m} B_i \frac{\partial \mathbf{u}}{\partial x_i} = f, \\ \mathbf{u}|_{t=0} = \varphi(x_1, \ldots, x_m), \\ \Phi_i^{(1)}\mathbf{u}(x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_m, t) = 0, \\ \Phi_i^{(2)}\mathbf{u}(x_1, \ldots, x_{i-1}, 1, x_{i+1}, \ldots, x_m, t) = 0, \\ i = 1, 2, \ldots, m, \end{cases} \tag{2.1}$$

where

- $A = A^* > 0$ is positively definite and $B_i = B_i^*$ are fixed computable symmetric $n \times n$-matrices;
- $0 \leq t \leq T$ for a computable real T;
- $x = (x_1, \ldots, x_m) \in Q = [0, 1]^m$;
- $\varphi \in C^{p+1}(Q, \mathbb{R}^n)$, $f \in C^p(Q \times [0, T], \mathbb{R}^n)$, $p \geq 2$ (in this paper we let $f = 0$ for simplicity);
- the boundary coefficients $\Phi_i^{(1)}$, $\Phi_i^{(2)}$ are fixed computable matrices meeting the following conditions:
  1) The number of rows of $\Phi_i^{(1)}$ (respectively, $\Phi_i^{(2)}$) is equal to the number of positive (respectively, negative) eigenvalues of the matrices $A^{-1}B_i$, and the boundary values of $\mathbf{u}$ are consistent with the initial conditions φ;
  2) The boundary conditions are assumed to be dissipative which means that

$$\langle B_i\mathbf{u}, \mathbf{u}\rangle \leq 0 \text{ for } x_i = 0, \quad \langle B_i\mathbf{u}, \mathbf{u}\rangle \geq 0 \text{ for } x_i = 1, \quad i = 1, 2, \ldots, m. \tag{2.2}$$

Remark 2.1. Note that the assumptions 1) regarding the dimensions of the matrices Φ_i and consistency of the boundary conditions with the initial ones are needed for proving existence of a solution $\mathbf{u} \in C^p(Q \times [0, T], \mathbb{R}^n)$ of (2.1), while the assumption (2.2) provides uniqueness of the solution [Fr54, God71, Ev98, Jo66].

Moreover, these assumptions are needed [God76, GR62] for proving stability of the difference scheme constructed below in Section 4, which is one of the main ingredients in the proof of computability results.

The smoothness assumptions $\varphi \in C^{p+1}(Q, \mathbb{R}^n)$, $f \in C^p(Q \times [0, T], \mathbb{R}^n)$, $p \geq 2$, are needed to provide at least C^2 smoothness of the solution, which is essential to establish estimates of the convergence constant in what follows.

The consistency conditions have to be found for each particular case of the boundary conditions and matrix coefficients and are usually nontrivial. We don't go into details since the algorithms below do not depend on their concrete expressions. They matter only for proofs of the existence and uniqueness theorems.

An example of dissipative boundary conditions for the system (2.1) are conservative boundary conditions, stating that the energy flow through the boundary is constant:

$$\oint\limits_S \langle [\tau A + \sum_{i=1}^{m} \xi_i B_i]\mathbf{u}, \mathbf{u}\rangle dS = \int\limits_{Q \times [0,T]} 2\langle f, \mathbf{u}\rangle d\Omega, \tag{2.3}$$

where $(\tau, \xi_1, \ldots, \xi_m)$ is the extrinsic normal vector for the surface $S = \partial(Q \times [0, T])$.

E.g. for the linear elasticity equations (which constitute a symmetric hyperbolic system with 9×9-matrices)

$$\begin{cases} \frac{1}{2\mu}\frac{\partial \sigma_{ij}}{\partial t} - \frac{\lambda}{2\mu(3\lambda+2\mu)}\delta_{ij}\frac{\partial(\sigma_{11}+\sigma_{22}+\sigma_{33})}{\partial t} - \frac{1}{2}(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}) = 0, \quad i,j = 1,2,3, \\ \rho\frac{\partial u_i}{\partial t} - \frac{\partial \sigma_{ij}}{\partial x_j} = 0, \quad i = 1,2,3, \end{cases} \tag{2.4}$$

in particular, the following boundary equations are conservative

$$\begin{cases} x_1 = 0,\ x_2 = 1: \ \sigma_{11} = \sigma_{12} = \sigma_{13} = 0, \\ y_1 = 0,\ y_2 = 1: \ \sigma_{12} = \sigma_{22} = \sigma_{23} = 0, \\ z_1 = 0,\ z_2 = 1: \ \sigma_{13} = \sigma_{23} = \sigma_{33} = 0, \end{cases}$$

which means that the tangent stresses at any boundary are zero. Indeed, the energy conservation law (4.12) takes the form

$$\oint_S [-2\xi(u_1\sigma_{11}+u_2\sigma_{12}+u_3\sigma_{13}) - 2\eta(u_1\sigma_{12}+u_2\sigma_{22}+u_3\sigma_{23}) - 2\zeta(u_1\sigma_{13}+u_2\sigma_{23}+u_3\sigma_{33})]dS = 0.$$

Here u_i are the velocities, σ_{ij} is the stresses tensor (a symmetric 3×3–matrix with 6 independent variables), ρ is the density and λ, μ are the Lame coefficients.

Interestingly, the boundary-value problem for the wave equation

$$\begin{cases} p_{tt} - c_0^2(p_{xx} + p_{yy} + p_{zz}) = 0, \quad (x,y,z) \in \Omega = [0,1]^3, \\ p|_{t=0} = \varphi(x,y,z), \\ p_t|_{t=0} = \psi(x,y,z), \\ p|_{\partial\Omega} = 0. \end{cases} \tag{2.5}$$

can be reduced, in several ways, to a symmetric hyperbolic system (see e.g. [Gor79, Ev98, Jo66]), in particular to the three-dimensional acoustics equations

$$\begin{cases} \rho_0\frac{\partial u}{\partial t} + \frac{\partial p}{\partial x} = 0, \\ \rho_0\frac{\partial v}{\partial t} + \frac{\partial p}{\partial y} = 0, \\ \rho_0\frac{\partial w}{\partial t} + \frac{\partial p}{\partial z} = 0 \\ \frac{\partial p}{\partial t} + \rho_0 c_0^2(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}) = 0, \\ p|_{t=0} = \varphi(x,y,z), \\ u|_{t=0} = -\frac{1}{\rho_0 c_0^2}\int\limits_0^x \psi(\xi,y,z)d\xi, \\ v|_{t=0} = 0, \\ w|_{t=0} = 0, \\ p|_{\partial\Omega} = 0, \end{cases} \tag{2.6}$$

where u, v, w are the velocities, p is the pressure, ρ_0 is the density and c_0 is the speed constant.

Obviously, such a reduction can be done effectively, since integration is a computable operation. Thus the methods of proving computability for symmetric hyperbolic systems can be also applied to prove computability for the wave equation. This gives a partial answer to an open question raised in [WZ02]: a boundary-value problem for the wave equation is computable (in classes of functions with uniformly bounded derivatives, see the exact formulation below) provided that it is dissipative (i.e., the corresponding boundary-value problem for a symmetric hyperbolic system to which the wave equation is reduced, is dissipative) and c_0 is a computable real.

We prove computability of the solution operator $\varphi \mapsto \mathbf{u}$ of the boundary-value problem (2.1) under the following additional assumptions/modifications.

- First of all, note, that the cube $Q = [0,1]^m$ can easily be replaced by a computable parallelepiped
$$[x_1^{(1)}, x_2^{(1)}] \times [x_1^{(2)}, x_2^{(2)}] \times \ldots \times [x_1^{(m)}, x_2^{(m)}],$$
and, in place of $t \geq 0$, we can assume that $t \geq t_0$, where t_0 is a computable real.
- The first and second partial derivatives of the initial function φ are bounded by a uniform constant.
- The considered spaces C^p are equipped with the *sup*-norm on Q or the sL_2-norm on $Q \times [0,T]$, which is an L_2-norm over Q and a *sup*-norm over $[0,T]$, see the more precise definitions in the next section.
- For the Cauchy problem (1.2), we also prove computability of the solution on the matrices A, B_i assuming them to belong to the set of symmetric matrices with $A > 0$, the norms
$$||A||_2 = \lambda_{max}(A), ||A^{(-1)}||_2 = \frac{1}{\lambda_{min}(A)}, \; ||B_1||_2, ||B_2||_2, \ldots, ||B_m||_2$$
to be bounded by a uniform constant, the matrix pencils $\lambda A - B_i$ to have no zero eigenvalues, and to have the cardinalities of spectra (as well as the cardinality of spectrum of the matrix A) given as inputs. Here $\lambda_{max}(A)$ and $\lambda_{min}(A)$ are the maximal and minimal eigenvalues of A, respectively. This result improves the main result in [SS09].

3. Preliminaries

In this section we briefly summarize some relevant notions and facts. In Subsection 3.1 we briefly recall some basic notions and facts about constructive structures, with an emphasis on computable fields. Subsection 3.2 contains some relevant information on computable metric spaces. Concrete metric spaces relevant to this paper are carefully described in Subsection 3.3.

3.1. **Computability on countable structures.**

We briefly recall some relevant notions and facts from computable model theory.

Recall that a *numbering* of a set B is a surjection β from $\mathbb{N}$ onto B. For numberings β, γ of B, β is *reducible* to γ (in symbols $\beta \leq \gamma$) iff $\beta = \gamma \circ f$ for some computable function f on $\mathbb{N}$, and β is *equivalent* to γ (in symbols $\beta \equiv \gamma$) iff $\beta \leq \gamma$ and $\gamma \leq \beta$. These notions (introduced by A.N. Kolmogorov) enable to transfer the computability theory over $\mathbb{N}$ to computability theory over many other countable structures. The notions apply to arbitrary functions $\beta, \gamma : \mathbb{N} \to B$ (not only surjections). Natural extensions of these notions to partial numberings (i.e., functions defined on a subset of $\mathbb{N}$) are also of use. In this case, $\beta \leq \gamma$ means the existence of a computable partial function ψ on $\mathbb{N}$ such that $\beta(n) = \gamma\psi(n)$ for each $n \in dom(\beta)$ (of course, the equality assumes that $n \in dom(\psi)$ and $\psi(n) \in dom(\gamma)$).

In the context of algebra and model theory, the transfer of Computability Theory was initiated in the 1950-s by A. Mostowski [Mo52, Mo53], A.V. Kuznetsov [Ku56, Ku58], and A. Fröhlich and J.C. Shepherdson [FS56]. The subject was strongly influenced by the work of M.O. Rabin [Ra60] and A.I. Mal'cev [Ma61]. The seminal paper of A.I. Mal'cev was fundamental for the extensive subsequent work in computable algebra by the Siberian school of algebra and logic. In parallel, active research in this area was conducted in the West. The resulting rich theory was summarized in the monographs [Er80, EG99] and the handbook [EGNR98].

Definition 3.1. A structure $\mathbb{B} = (B; \sigma)$ of a finite signature σ is called constructivizable iff there is a numbering β of B such that all signature predicates and functions, and also the equality predicate, are β-computable. Such a numbering β is called a constructivization of $\mathbb{B}$, and the pair $(\mathbb{B}, \beta)$ is called a constructive structure.

Recall, in particular, that a binary relation P on B is β-computable (resp. β-computably enumerable) if the corresponding binary relation $P(\beta(m), \beta(n))$ on $\mathbb{N}$ is computable (resp. computably enumerable). In the case when β is a partial numbering, P is called β-computably enumerable if there is a computably enumerable binary relation $\widehat{P}$ on $\mathbb{N}$ such that $\{(m,n) \mid \ P(\beta(m), \beta(n))\} = \widehat{P} \cap (D \times D)$ where $D = dom(\beta)$.

The notion of a constructivizable structure is equivalent to the notion of a computably presentable structure popular in the western literature. Obviously, $(\mathbb{B}, \beta)$ is a constructive structure iff given a quantifier-free σ-formula $\phi(v_1, \ldots, v_k)$ with free variables among $v_1, \ldots, v_k$ and given $n_1, \ldots, n_k \in \mathbb{N}$, one can compute the truth-value $\phi^{\mathbb{B}}(\beta(n_1), \ldots, \beta(n_k))$ of ϕ in $\mathbb{B}$ on the elements $\beta(n_1), \ldots, \beta(n_k) \in B$.

Definition 3.2. A structure $\mathbb{B} = (B; \sigma)$ of a finite signature σ is called strongly constructivizable iff there is a numbering β of B such that, given a first-order σ-formula $\phi(v_1, \ldots, v_k)$ with free variables among $v_1, \ldots, v_k$ and given $n_1, \ldots, n_k \in \mathbb{N}$, one can compute the truth-value $\phi^{\mathbb{B}}(\beta(n_1), \ldots, \beta(n_k))$ of ϕ in $\mathbb{B}$ on the elements $\beta(n_1), \ldots, \beta(n_k) \in B$. Such a numbering β is called a strong constructivization of $\mathbb{B}$, and the pair $(\mathbb{B}, \beta)$ is called a strongly constructive structure.

By the definitions above, any strongly constructivizible structure is constructivizible and has a decidable first-order theory. Note that the notion of a strongly constructive structure is equivalent to the notion of a decidable structure popular in the western literature.

We illustrate the introduced notions by some number structures. Let $\mathbb{N} = (N; <, +, \cdot, 0, 1)$ be the ordered semiring of naturals, $\mathbb{Z} = (Z; <, +, \cdot, 0, 1)$ the ordered ring of integers, $\mathbb{Q} = (Q; <, +, \cdot, 0, 1)$ the ordered field of rationals, $\mathbb{R} = (R; <, +, \cdot, 0, 1)$ the ordered field of reals, $\mathbb{R}_c = (R_c; <, +, \cdot, 0, 1)$ the ordered field of computable reals [We00], and $\mathbb{A} = (A; <, +, \cdot, 0, 1)$ the ordered field of algebraic reals [vdW67] (by definition, the algebraic reals are the real roots of polynomials with rational coefficients). As is well known [We00], any algebraic real is computable, so $\mathbb{A}$ is a substructure of $\mathbb{R}_c$.

As is well-known, the fields $\mathbb{A}$, $\mathbb{R}_c$ and $\mathbb{R}$ are real closed (we use some standard algebraic notions which may be found e.g. in [vdW67].) The following properties of the mentioned number structures are well-known. Details and additional references may be found in the vast literature on computable rings and fields (see e.g. [Mo66, Er68, MN79, Er74, ST95, ST99]).

Example 3.3.

(1) The structures $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$ are constructivizable but not strongly constructivizable.
(2) The structure $\mathbb{A}$ is strongly constructivizable.
(3) The structure $\mathbb{R}_c$ is not constructivizable, but there is a partial numbering ρ of $\mathbb{R}_c$ such that the field operations are ρ-computable and the relation $<$ is ρ-computably enumerable.

For this paper, Example 3.3 (2) is of a special interest. We also need some extensions of this assertion which may be deduced from some known facts about computable fields (and from classical algebraic facts in [vdW67]).

First we recall definition of the partial numbering ρ of $\mathbb{R}_c$ mentioned in the Example 3. Let $\varkappa$ be a constructivization of $\mathbb{Q}$ and $\{\varphi_n\}$ be a standard numbering of the computable partial functions on $\mathbb{N}$. A sequence $\{x_n\}$ in $\mathbb{R}$ is called *fast Cauchy* iff $\forall n \forall i > n(|x_i - x_n| < 2^{-n})$. Now, define ρ as follows: $\rho(n) = x$ iff φ_n is total, $\{\varkappa\varphi_n(i)\}_i$ is fast Cauchy, and converges to x. Let us collect some facts relating the introduced notions.

Lemma 3.4.
(1) *Let $\mathbb{B}$ be an ordered subfield of $\mathbb{R}$ and β a constructivization of $\mathbb{B}$. Then $\beta \leq \rho$, in particular $\mathbb{B} \subseteq \mathbb{R}_c$.*
(2) *Let $\mathbb{B}$ be a subfield of $(\mathbb{R}; +, \cdot, 0, 1)$ and β a constructivization of $\mathbb{B}$ such that $\beta \leq \rho$. Then β is a constructivization of the ordered field $(\mathbb{B}; <)$.*
(3) *Let $\mathbb{B}$ be a real closed subfield of $(\mathbb{R}; +, \cdot, 0, 1)$ and β a constructivization of $\mathbb{B}$. Then β is a strong constructivization of the ordered field $(\mathbb{B}; <)$.*

Proof.
(1) Since β is a constructivization, $\varkappa \leq \beta$. Hence, for some computable functions f, g we have $\varkappa f(n,i) < \beta(n) < \varkappa g(n,i)$ and $\varkappa g(n,i) - \varkappa f(n,i) < 2^{-i}$. Let h be a computable function satisfying $\varkappa h(n,i) = (\varkappa g(n,i) - \varkappa f(n,i))/2$. Then $\{\varkappa h(n,i)\}_i$ is a fast Cauchy sequence converging to $\beta(n)$. Choosing a computable function u with $h(n,i) = \varphi_{u(n)}(i)$ we see that $\beta \leq \rho$ via u.
(2) It suffices to show that $<$ is β-computable. Since $\beta \leq \rho$ and $<$ is ρ-computably enumerable, $<$ is also β-computably enumerable. Hence, given n one can compute which of the alternatives $\beta(n) < 0, \beta(n) = 0, \beta(n) > 0$ holds. Thus, $<$ is β-computable.
(3) Since $\mathbb{B}$ is real closed, $0 \leq \beta(n)$ is equivalent to $\exists m(\beta(n) = \beta(m)^2)$. Then $\leq$ and $<$ are β-computably enumerable. As in the previous paragraph, $<$ is β-computable, hence β is a constructivization of $(\mathbb{B}; <)$. By the Tarski quantifier elimination for real closed fields, given any first order σ-formula ϕ, $\sigma = \{<, +, \cdot, 0, 1\}$, one can compute a quantifier-free σ-formula equivalent to ϕ in $(\mathbb{B}; <)$. Thus, β is a strong constructivization of $(\mathbb{B}; <)$. □

Lemma 3.5. *For any finite set $F \subseteq \mathbb{R}_c$ there is a strongly constructive real closed ordered subfield $(\mathbb{B}, \beta)$ of $\mathbb{R}_c$ with $F \subseteq B$.*

Proof. If $F \subseteq \mathbb{A}$ we can take $\mathbb{B} = \mathbb{A}$ and $\beta = \alpha$, where α is a strong constructivization of $\mathbb{A}$. Otherwise, let x be the least element of $F \setminus \mathbb{A}$, so in particular x is a computable transcendental real number. Let $\mathbb{D} = \mathbb{A}(x)$ be the subfield of $\mathbb{R}_c$ generated by $\mathbb{A} \cup \{x\}$ and δ be the numbering of $\mathbb{D}$ induced by the strong constructivization α of $\mathbb{A}$ and the Gödel numbering of σ-terms with the variable x. Since $\alpha \leq \rho$ and $x \in \mathbb{R}_c$, $\delta \leq \rho$. Moreover, from the well-known structure of $\mathbb{D}$ it follows that δ is a constructivization of $(\mathbb{D}; <)$.

Let now $\mathbb{A}_1$ be the real algebraic closure of $\mathbb{D}$ in $\mathbb{R}_c$. As is well known (see e.g. [Er74, Theorem 3, p. 101]), $\mathbb{A}_1$ is constructivizable, even strongly constructivizable by item (3) of Lemma 3.4. If $F \subseteq \mathbb{A}_1$ we can take $\mathbb{B} = \mathbb{A}_1$ and $\beta = \alpha_1$, where α_1 is a strong constructivization of $\mathbb{A}_1$. Otherwise, iterate the construction $\mathbb{A} \mapsto \mathbb{A}_1$ sufficiently many times in order to get the desired $\mathbb{B}$. □

Remark 3.6. The proof of the last lemma is non-constructive (i.e., from ρ-indices of elements of F one cannot compute a constructivisation β). This lemma and Theorem 5.1 which is based on it are "pure existence theorems".

Let $(\mathbb{B}, \beta)$ be a strongly constructive real closed ordered subfield of $\mathbb{R}_c$. Then one can compute, given a polynomial $p(x) = a_0 + a_1x^1 \cdots + a_kx^k$ with coefficients in $\mathbb{B}$ (i.e., given a string $n_0, \ldots, n_k$ of naturals with $\beta(n_0) = a_0, \ldots, \beta(n_k) = a_k$) the string $r_1 < \cdots < r_m$, $m \geq 0$, of all distinct real roots of $p(x)$ (i.e., a string $l_1, \ldots, l_m$ of naturals with $\beta(l_1) = r_1, \ldots, \beta(l_m) = r_m$), as well as the multiplicity of any root r_j. This fact immediately implies

Lemma 3.7. *Let $(\mathbb{B}, \beta)$ be a strongly constructive real closed ordered subfield of $\mathbb{R}_c$. Given a symmetric $n \times n$-matrix M with coefficients in $\mathbb{B}$, one can compute (w.r.t. β) an orthonormal basis $(\mathbf{v}_1, \ldots, \mathbf{v}_n) \in \mathbb{B}^n$ of eigenvectors of M.*

Proof. Let $(\lambda_1, \ldots, \lambda_n)$ be a string of all complex roots of the characteristic polynomial $\det(\lambda I_n - M)$ taken with their multiplicities. Since M is symmetric, $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$. Since $\mathbb{B}$ is real closed and the coefficients of the characteristic polynomial are in $\mathbb{B}$, $\lambda_1, \ldots, \lambda_n \in \mathbb{B}$. Since $(\mathbb{B}, \beta)$ is strongly constructive, given β-names for the coefficients of M one can compute β-names for $\lambda_1, \ldots, \lambda_n$ (without loss of generality we may even assume that $\lambda_1 \leq \cdots \leq \lambda_n$). Repeating well-known computations from linear algebra (cf. e.g. the proof of [ZB01, Theorem 13]) one can compute the desired eigenvectors $\mathbf{v}_1, \ldots, \mathbf{v}_n \in \mathbb{B}^n$ such that $M \cdot \mathbf{v}_i = \lambda_i \cdot \mathbf{v}_i$ for each $i = 1, \ldots, n$. □

Remark 3.8. Of course, the orthonormal basis of eigenvectors is not unique. It is only important that some such basis is computable (w.r.t. β).

3.2. **Computability on metric spaces.** We use the TTE-approach to computability over metric spaces developed in the K. Weihrauch's school (for more details see e.g. [We00, WZ02, Br03, BHW03] and references therein). Let (M, d) be a metric space. A sequence $\{x_n\}$ in M is called *fast Cauchy* iff $d(x_i, x_n) < 2^{-n}$ for all n and $i > n$. The following lemma is straightforward.

Lemma 3.9. *Let (M, d) be a metric space and let $x, x_n, x_{n,m} \in M$ for all $m, n \in \mathbb{N}$.*
(1) *If $\{x_n\}$ is fast Cauchy and converges to x then $\forall n (d(x, x_n) \leq 2^{-n})$.*
(2) *If $\forall n (d(x, x_n) \leq 2^{-n})$ then $\{x_n\}$ converges to x and $\{x_{n+1}\}$ is fast Cauchy.*
(3) *Let for any n $\{x_{n,m}\}_m$ is fast Cauchy and converges to x_n, and let $\{x_n\}$ is fast Cauchy and converges to x. Then $\{x_{n+2,n+2}\}$ is fast Cauchy and converges to x.*

Let $\mathcal{N} = \omega^\omega$ be the Baire space (instead of the Baire space people often use in this context the Cantor space Σ^ω of infinite words over a finite alphabet Σ containing at least two symbols.). Relate to any function $\nu : \mathbb{N} \to M$ the partial function ν^* from $\mathcal{N}$ to M as follows: $\nu^*(p) = x$ iff the sequence $\{\nu_{p(n)}\}$ is fast Cauchy and converges to x.

Lemma 3.10. *Let (M, d) be a metric space and $\mu, \nu : \mathbb{N} \to M$ be such that $\mu \leq \nu^*$ (i.e., $\mu = \nu^* \circ f$ for a computable function $f : \mathbb{N} \to \mathcal{N}$). Then $\mu^* \leq \nu^*$ (i.e., $\mu^* = \nu^* \circ g$ for a computable partial function g on $\mathcal{N}$).*

Proof. For any $p \in dom(\mu^*)$, $\mu^*(p) = lim_n \mu_{p(n)}$ and $\{\mu_{p(n)}\}$ is fast Cauchy. For each n, $\mu(p(n)) = \mathrm{lim_m} \nu_{\mathrm{f(p(n))(m)}}$ and $\{\nu_{f(p(n))(m)}\}_m$ is fast Cauchy. By item (3) of Lemma 3.9, $\{\nu_{f(p(n+2))(n+2)}\}_n$ is fast Cauchy and converges to $\mu^*(p)$. Let g be the computable function on $\mathcal{N}$ defined by $g(p)(n) = f(p(n+2))(n+2)$. Then g has the desired property. □

Definition 3.11. A computable metric space is a triple (M, d, μ) where (M, d) is a metric space and $\mu : \omega \to M$ is a numbering of a dense subset $\mathrm{rng}(\mu)$ of M such that $\{d(\mu_m, \mu_n)\}$ is a computable double sequence of reals. The partial surjection μ^* from $\mathcal{N}$ onto M is called the Cauchy representation of (M, d, μ).

Note that the computability of the double sequence $\{d(\mu_m, \mu_n)\}$ is equivalent to the computable enumerability of the set

$$\{(i, j, q, r) \mid i, j \in \omega,\ q, r \in \mathbb{Q},\ q < d(\nu_i, \nu_j) < r\}.$$

A partial function $f : M \rightharpoonup M_1$ on the elements of computable metric spaces (M, d, ν) and (M_1, d_1, ν_1) is *computable* if there is a computable partial function $\hat{f}$ on $\mathcal{N}$ which realizes f w.r.t. the Cauchy representations of M and M_1, i.e., $\nu_1^*(\hat{f}(p)) = f(\nu^*(p))$ for each $p \in \mathrm{dom}(\nu^*)$ (in other words, if $\{\nu(p(n))\}$ is a fast Cauchy sequence converging to $x \in M$ then $\{\nu_1(\hat{f}(p)(n))\}$ is a fast Cauchy sequence converging to $f(x) \in M_1$).

A standard example of a computable metric space is $(\mathbb{R}, d, \varkappa)$ where $d(x, y) = |x - y|$ is the standard metric on $\mathbb{R}$ and $\varkappa$ is a constructivization of $\mathbb{Q}$ (see the previous subsection). A less standard example is $(\mathbb{R}, d, \beta)$ where β is a strong constructivization of a real closed ordered subfield $\mathbb{B}$ of $\mathbb{R}_c$. Though formally different, these two computable metric spaces are equivalent in the following sense:

Lemma 3.12. *The Cauchy representations $\varkappa^*, \beta^*$ of $\mathbb{R}$ induced by the numberings $\varkappa, \beta$ respectively, are equivalent, i.e. $\varkappa^* \leq \beta^*$ and $\beta^* \leq \varkappa^*$.*

Proof. Since $\varkappa \leq \beta$ and $\beta \leq \beta^*$, we have $\varkappa \leq \beta^*$, hence $\varkappa^* \leq \beta^*$ by Lemma 3.10. For the converse reduction, by Lemma 3.10 it suffices to show that $\beta \leq \varkappa^*$. This follows from item (1) of Lemma 3.4. □

3.3. **Spaces under consideration.** For any $n \geq 1$, the vector space $\mathbb{R}^n$ carries the sup-norm $||x||_\infty = \max\{|x_i|\}$ and the Euclidean norm $||x||_2 = \sqrt{\sum x_i^2}$; we denote the corresponding metrics by d_∞ and d_2, respectively.

Define the function $\varkappa^n : \mathbb{N} \to \mathbb{R}^n$ by $\varkappa^n\langle k_1, \ldots, k_n\rangle = (\varkappa(k_1), \cdots, \varkappa(k_n))$ where $\langle\cdot\rangle$ is a computable coding function of n-tuples of naturals and $\varkappa$ is a constructivization of $\mathbb{Q}$ (see Section 3.1). Let $(\mathbb{B}, \beta)$ be a strongly constructive real closed ordered subfield of $\mathbb{R}_c$. Define the function $\beta^n : \mathbb{N} \to \mathbb{R}^n$ in the same way as $\varkappa^n$, with $\varkappa$ replaced by β.

Lemma 3.13. *For any $n \geq 1$ and $d \in \{d_\infty, d_2\}$, $(\mathbb{R}^n, d, \varkappa^n)$ and $(\mathbb{R}^n, d, \beta^n)$ are equivalent computable metric spaces.*

Proof. For $n = 1$ this follows from Lemma 3.12 because $d_\infty = d_2$. For $n \geq 2$, computability of the spaces and the reducibility $\varkappa^n \leq \beta^n$ are obvious. Since $\beta \leq \rho$ by item (1) of Lemma 3.4, there is a computable function f on $\mathbb{N}$ such that $d_\infty(\beta(k), \varkappa f\langle k, l\rangle) \leq 2^{-l}$ for all k, l. By Lemma 3.10 and the argument of Lemma 3.12, for the metric $d = d_\infty$ it suffices to find a computable function g on $\mathbb{N}$ such that $d_\infty(\beta^n(k), \varkappa^n g\langle k, l\rangle) \leq 2^{-l}$ for all k, l. Define g by $g\langle k, l\rangle = \langle f\langle k_1, l\rangle, \ldots, f\langle k_n, l\rangle\rangle$ where $k = \langle k_1, \ldots, k_n\rangle$. Then we have

$$d_\infty(\beta^n(k), \varkappa^n g\langle k, l\rangle) = \sup\{d_\infty(\beta(k_1), \varkappa f\langle k_1, l\rangle), \ldots, d_\infty(\beta(k_n), \varkappa f\langle k, l\rangle)\} \leq 2^{-l}$$

which completes the case $d = d_\infty$.

For $d = d_2$, the assertion follows from the obvious estimate $d_2(x, y) \leq \sqrt{n} d_\infty(x, y)$. □

We will consider some subspaces of the introduced metric spaces, in particular the space $S \subseteq \mathbb{R}^{n\times n}$ of symmetric real matrices, the space S_+ of symmetric real positively definite matrices, and the m-dimensional unitary cube $Q = [0,1]^m$. For these subspaces the analog of Lemma 3.13 clearly holds.

In the study of difference equations, some norms on the spaces of grid functions are quite useful. For any $N \geq 0$, let $G = G_N$ be the *uniform grid on* Q formed by the binary-rational vectors $(x_1, \ldots, x_m)$ where $x_i = \frac{y_i}{2^N}$ and $y_i \in \{0, 1, \ldots, 2^N\}$. Note that the number of such vectors is $(2^N+1)^m$, so the set $\mathbb{R}^G$ of grid functions $f : G_N \to \mathbb{R}$ may be identified with $\mathbb{R}^{(2^N+1)^m}$ while the set $(\mathbb{R}^n)^G$ of grid functions $f : G_N \to \mathbb{R}^n$ may be identified with $\mathbb{R}^{n\cdot(2^N+1)^m}$. In the last case, we obtain the following norms

$$||\varphi||_s = \max_{x\in G_N} ||\varphi(x)||,\ ||\varphi||^2_{L_2} = h^m \sum_{x\in G_N} \langle \varphi(x), \varphi(x)\rangle.$$

Note that d_s coincides with d_∞ for the corresponding metric spaces, and Lemma 3.13 applies to these spaces.

For all rational $\tau > 0$ and integers $N \geq 0$ and $L \geq 1$, let G^τ_N be the grid in $Q \times [0, T]$, $T = L\tau$, with step $h = \frac{1}{2^N}$ on the space coordinates x_i and step τ on the time coordinate t. Just as above, we can define the sup- and L_2-norms on the vector space $M = \mathbb{R}^{G^\tau_N}$ of grid functions on such grids and Lemma 3.13 applies to the corresponding metric spaces.

The vector space M carries also another natural norm (called the sL_2-norm.) The sL_2-norm of a grid function $f : G^\tau_N \to \mathbb{R}$ is defined by

$$||f||^2_{sL_2} = \max_{0\leq l\tau\leq T} \left(h^m \sum_{x\in G_N} f^2(x, l\tau) \right).$$

Let d_{sL_2} be the corresponding metric on M. Let $(\mathbb{B}, \beta)$ be a strongly constructive real closed ordered subfield of $\mathbb{R}_c$. Let μ (resp. ν) be the numbering of the set $\{f \mid f : G^\tau_N \to \mathbb{Q}\}$ (resp. $\{f \mid f : G^\tau_N \to \mathbb{B}\}$) induced by the natural numbering $G^\tau_0, G^\tau_1, \cdots$ of all such grids and the constructivization $\varkappa$ of $\mathbb{Q}$ (resp. the strong constructivization β of $\mathbb{B}$). The following analog of Lemma 3.13 is an easy corollary of the estimate $d_{sL_2}(f, g) \leq d_s(f, g)$ which follows from the definition of sL_2-norm.

Lemma 3.14. *In the notation of the previous paragraph,* (M, d_{sL_2}, μ) *and* (M, d_{sL_2}, ν) *are equivalent computable metric spaces. This extends in the obvious way to the space* $(\mathbb{R}^n)^{G^\tau_N}$ *of grid functions* $f : G^\tau_N \to \mathbb{R}^n$*.*

We will work with several functional spaces most of which are subsets of the set $C(\mathbb{R}^m, \mathbb{R}^n) \simeq C(\mathbb{R}^m, \mathbb{R})^n$ of integrable continuous functions $\varphi : \mathbb{R}^m \to \mathbb{R}^n$ equipped with the L_2-norm. In particular, we deal with the space $C(Q, \mathbb{R}^n) \simeq C(Q, \mathbb{R})^n$ (resp. $C^k(Q, \mathbb{R}^n)$) of continuous (resp. k-time continuously differentiable) functions $\varphi : Q \to \mathbb{R}^n$ equipped with the L_2-norm

$$||\varphi||_{L_2} = \left(\int_Q |\varphi(x)|^2 dx \right)^{\frac{1}{2}},\ |\varphi(x)|^2 = \langle \varphi, \varphi\rangle = \sum_{i=1}^n \varphi^2_i(x).$$

We will also use the sup-norm

$$||\varphi||_s = \sup_{x\in Q} |\varphi(x)|,\ ||f||_s = \sup_{(x,t)\in Q\times[0,T]} |f(x, t)|$$

on $C(Q,\mathbb{R}^n)$ or $C(Q\times[0,T],\mathbb{R}^n)$ and the sL_2-norm

$$||u||_{sL_2} = \sup_{0\leq t_0\leq T}\sqrt{\int_Q |u(x,t_0)|^2dx}$$

on $C(Q\times[0,T],\mathbb{R}^n)$ where $T>0$. Whenever we want to emphasize the norm we use notations like $C_{L_2}(Q,\mathbb{R}^n)$, $C_s(Q,\mathbb{R}^n)$ or $C_{sL_2}(Q\times[0,T],\mathbb{R}^n)$.

Associate to any grid function $f_N : G_N\to\mathbb{Q}$ the continuous extension $\tilde{f}_N : Q\to\mathbb{R}$ of f obtained by the piecewise-linear interpolation on each coordinate. Such interpolations known also as *multilinear interpolations* are the simplest class of splines see e.g. [Sz59, So74, ZKM80, Ba86]). Note that the restriction of $\tilde{f}_N$ to any grid cell is a polynomial of degree m, see an example in Subsection 3.3. The extensions $\tilde{f}_N$ induce a countable dense set in $C(Q,\mathbb{R}^n)$ (or $C(Q\times[0,T],\mathbb{R}^n)$) with any of the three norms.

Let again $(\mathbb{B},\beta)$ be a strongly constructive real closed ordered subfield of $\mathbb{R}_c$. Define $\tilde{\beta},\tilde{\varkappa}:\mathbb{N}\to C(Q,\mathbb{R}^n)$ by $\tilde{\beta}\langle N,l\rangle = \widetilde{\beta^p_N(l)}$ where p is the number of grid points in G_N and β^p_N is the numbering of grid functions $f : G_N\to\mathbb{B}^n$ ($\tilde{\varkappa}$ is defined similarly). Define also $\tilde{\mu},\tilde{\nu}:\mathbb{N}\to C(Q\times[0,T],\mathbb{R}^n)$ in the same way, starting from the numberings μ,ν above and the natural numbering of all the grids G^τ_N with rational positive τ. The next fact follows from Lemmas 3.13, 3.14 and the well-known estimates $||\tilde{f}||\leq||f||$ where $||\cdot||$ is any of the three norms (see e.g. [Ba86, p. 187–189], [Sz59, p. 335]).

Lemma 3.15.
(1) *For any $n\geq 1$ and $d\in\{d_s,d_{L_2}\}$, $(C(Q,\mathbb{R}^n),d,\tilde{\varkappa})$ and $(C(Q,\mathbb{R}^n),d,\tilde{\beta})$ are equivalent computable metric spaces.*
(2) *In the notation before the formulation of lemma, $(C(Q\times[0,T],\mathbb{R}^n),d_{sL_2},\tilde{\mu})$ and $(C(Q\times[0,T],\mathbb{R}^n),d_{sL_2},\tilde{\nu})$ are equivalent computable metric spaces.*

Let again G be the grid in Q with step $h=\frac{1}{2^N}$ on each coordinate. From well-known facts of Computable Analysis [We00] it follows that the restriction $\varphi\mapsto\varphi|_G$ is a computable operator from $C_s(Q,\mathbb{R}^n)$ to $(\mathbb{R}^n)^G$. From well-known properties of the multilinear interpolations (see e.g. [God71, ZKM80]) it follows that $f\mapsto\tilde{f}$ is a computable operator from $((\mathbb{R}^n)^G)_s$ to $C_{L_2}(Q,\mathbb{R}^n)$ (see also the estimate (6.4) below).

Along with the mentioned norms, we sometimes use their A-modifications, for a given matrix A. In particular, the A-modification of the L_2-norms is defined by

$$||\varphi||_{A,L_2} = \sqrt{\int_Q\langle A\varphi,\varphi\rangle dx}$$

while the A-modification of the sL_2-norms is defined by

$$||u||_{A,sL_2} = \sup_{0\leq t_0\leq T}\sqrt{\int_Q\langle Au(x,t_0),u(x,t_0)\rangle dx},$$

and in a similar way for the grid norms.

4. Finite-dimensional approximations

In this section we describe the construction of difference operators approximating the differential problem considered in this paper and establish their basic properties. Subsection 4.1 recalls some relevant notions and general facts about difference schemes. In Subsection 4.2 we describe the difference scheme [God76] for the symmetric hyperbolic systems under consideration. In Subsection 4.3 we establish some basic properties of the corresponding difference operators.

4.1. Basic facts about difference schemes.

Here we briefly recall some relevant notions and facts about difference schemes (for more details see any book on the subject, e.g. [GR62, Jo66, St04, Tre96]).

Let us consider the boundary-value problem (1.1) for a (system of) PDEs. Difference approximations to (1.1) are written in the form

$$L_h\mathbf{u}^{(h)} = \mathbf{f}^{(h)},\ \mathcal{L}_h\mathbf{u}^{(h)} = \varphi^{(h)} \tag{4.1}$$

where $L_h, \mathcal{L}_h$ are difference operators (which are in our case linear), and all functions are defined on some grids in Ω or $\Gamma \subseteq \partial\Omega$ (the grids are not always uniform, as in our simplest case). For simplicity we will use the restriction notation $\mathbf{g}|_{G_k}$ to denote the restriction of $\mathbf{g} : \Omega \to \mathbb{R}^n$ to the grid G_k in Ω though in general the restriction operator may be more complicated. Both sides of (4.1) depend on the grid step h.

Note that in this paper we consider a little more complicated grids than the uniform grids discussed above, namely grids with the integer time steps $l\tau$, $l \geq 0$, (for some $\tau > 0$) and half-integer steps $x_{i+\frac{1}{2}} = (i + \frac{1}{2})h$ for the space variables. The theory for such slightly modified grids remains the same.

Let the space of grid functions defined on the same grid as $\mathbf{f}^{(h)}$ (resp. as $\mathbf{u}^{(h)}$, $\varphi^{(h)}$) carry some norm $||\cdot||_{F_h}$ (resp. some norms $||\cdot||_{U_h}$, $||\cdot||_{\Phi_h}$). Note that in our case these will be L_2 and sL_2-norms defined in Section 3.

Definition 4.1. Difference equations (4.1), also called difference schemes, *approximate the differential equation* (1.1) *with order of accuracy l (where l is a positive integer) on a solution* $\mathbf{u}(\mathbf{x}, t)$ *of* (1.1) if

$$||(L\mathbf{u})|_{G_k} - L_h\mathbf{u}^{(h)}||_{F_h} \leq M_1h^l,\ ||f|_{G_k} - f^{(h)}||_{F_h} \leq M_2h^l,$$
$$||(\mathcal{L}\mathbf{u})|_{G_k} - \mathcal{L}_h\mathbf{u}^{(h)}||_{\Phi_h} \leq M_3h^l \text{ and } ||\varphi|_{G_k} - \varphi^{(h)}||_{\Phi_h} \leq M_4h^l$$

for some constants M_1, M_2, M_3 and M_4 not depending on h and τ.

The definition is usually checked by working with the Taylor series for the corresponding functions, thus the constants M_i depend on the derivatives of the functions u and f. As a result, the degrees of smoothness of the functions become essential when one is interested in the order of accuracy of a difference scheme. Note that the definition assumes the existence of a solution of (1.1). For the problems (1.2) and (2.1) it is well-known (see e.g. [Fr54, God71, Mi73, Ev98]) that there is a unique solution.

The following notion identifies a property of difference schemes which is crucial for computing "good" approximations to the solutions of (1.1).

Definition 4.2. Difference scheme (4.1) is called *stable* if its solution $\mathbf{u}^{(h)}$ satisfies

$$||\mathbf{u}^{(h)}||_{U_h} \leq N_1||f^{(h)}||_{F_h} + N_2||\varphi^{(h)}||_{\Phi_h}$$

for some constants N_1 and N_2 not depending on h, τ, $f^{(h)}$ and $\varphi^{(h)}$.

For non-stationary processes (depending explicitly on the time variable t, as (1.2), (2.1)), the difference equation (4.1) may be rewritten in the equivalent recurrent form $\mathbf{u}^{[l+1]} = R_h\mathbf{u}^{[l]} + \tau\rho^{[l]}$ where $\mathbf{u}^{[0]}$ is known, $\mathbf{u}^{[l]}$ is the restriction of the solution to the time level $t = l\tau$, $l \geq 0$, $\rho^{[l]}$ depends only on f and φ, R_h is the difference operator obtained from L_h in a natural way. It is known (see e.g. [GR62]) that the stability of (4.1) on the interval $0 < t < T$ is equivalent to the uniform boundedness of the operators R_h and their powers: $||R_h^m|| < K$, $m = 1, 2, \ldots, \frac{T}{\tau}$, for some constant K not depending on h and τ. In general, the investigation of the stability of difference schemes is a hard task. The most popular tool is the so called Fourier method [GR62, God76, Tre96]; for problems (2), (3) and for the scheme from the next subsection this is done by using the discrete energy integral technique in [God76, p. 79]. We will briefly describe this idea below.

Our main results on the computability of solution operators for (1.2) and (2.1) make an essential use of the following basic fact from the theory of difference schemes (see e.g. [GR62, p. 172],):

Theorem 4.3. *Let the difference scheme* (4.1) *be stable and approximate* (1.1) *on the solution* $\mathbf{u}$ *with order* l. *Then the solution of* (4.1) *uniformly converges to the solution* $\mathbf{u}$ *in the sense that* $||\mathbf{u}|_{G_k^\tau} - \mathbf{u}^{(h)}||_{U_h} \leq Nh^l$ *for some constant* N *not depending on* h *and* τ.

4.2. **Constructing a difference scheme for symmetric hyperbolic systems.** The difference scheme for the boundary-value problem (2.1) and the Cauchy problem (1.2) may be chosen in various ways. The scheme we use is taken from [God76]. It can be applied to a broader class of systems, including some systems of nonlinear equations. We describe it in few stages, letting for simplicity the righthand part to be zero: $f = 0$.

1. First we describe some discretization details. To simplify notation, we stick to the 2-dimensional case $x_1 = x$, $x_2 = y$, $B_1 = B$, $B_2 = C$, i.e., $m = 2$. For $m \geq 3$ the difference scheme is obtained in the same way as for $m = 2$ but the step from $m = 1$ to $m = 2$ is nontrivial.

   Consider the uniform rectangular grid G on $Q = [0, 1]^2$ defined by the family of lines $\{x = x_i\}, \{y = y_j\}$ where $0 \leq i, j \leq 2^N$ for some natural number N. Let $h = x_i - x_{i-1} = y_j - y_{j-1} = 1/2^N$ be the step of the grid. Associate to any function $g \in \{u_1, \ldots, u_n\}$ and any fixed time point $t = l\tau$, $l \in \mathbb{N}$, the vector of dimension 2^{2N} with the components
$$g_{i-\frac{1}{2},j-\frac{1}{2}} = g\left(\frac{i-\frac{1}{2}}{2^N}, \frac{j-\frac{1}{2}}{2^N}, t\right) \tag{4.2}$$
   equal to the values of g in the centers of grid cells, and denote, as in the previous subsection,
$$\mathbf{u}^{(h)} = \{\mathbf{u}_{i-\frac{1}{2},j-\frac{1}{2}}\}_{1\leq i,j\leq 2^N, t=l\tau, l=1,\ldots,M}.$$
   The initial grid function, from which the iteration process starts, will be denoted as $\varphi^{(h)}$, which equals to $\mathbf{u}^{(h)}$ restricted to the time level $t = 0$.

   Note that, strictly speaking, we work with modifications of the grids G_k in Subsection 3.3 when the centers of grid cells are taken as nodes of the modified grids.

2. Consider the following two auxiliary one-dimensional systems with parameters obtained by fixing any of the variables x, y:
$$A\frac{\partial \mathbf{u}}{\partial t} + B\frac{\partial \mathbf{u}}{\partial x} = 0,\ A\frac{\partial \mathbf{u}}{\partial t} + C\frac{\partial \mathbf{u}}{\partial y} = 0, \tag{4.3}$$
where $\mathbf{u} = (u_1, u_2, \dots, u_n)^T$. Transform the systems into their canonical forms
$$\frac{\partial \mathbf{v}_x}{\partial t} + M_x\frac{\partial \mathbf{v}_x}{\partial x} = 0, \quad \frac{\partial \mathbf{v}_y}{\partial t} + M_y\frac{\partial \mathbf{v}_y}{\partial y} = 0 \tag{4.4}$$
via the linear transformations $\mathbf{v}_x = T_x^{-1}\mathbf{u}$ and $\mathbf{v}_y = T_y^{-1}\mathbf{u}$ defined as follows (see [God76, SS09] for additional details):
$$T_x = LDK_x,\ \mathbf{u} = T_x\mathbf{v}_x, \tag{4.5}$$
and in a similar way on the y-coordinate. Here the orthogonal matrix L transforms the matrix A to its canonical form $L^*AL = \Lambda = \mathrm{diag}\{\lambda_1, \lambda_2, \dots, \lambda_n\}$, $D = \Lambda^{-\frac{1}{2}}$. The orthogonal matrix K_x transforms the matrix D^*L^*BLD to its diagonal form M_x. And similarly for the second auxiliary system in (4.3).

Note that the matrices L, K_x, K_y consisting of eigenvectors of the corresponding symmetric matrices are not uniquely defined. We choose some orthonormal eigenvectors and keep them fixed for all iteration steps. The components of the vectors $\mathbf{v}_x$, $\mathbf{v}_y$ in (4.4) are called *Riemannian invariants*; they are invariant along the characteristics of the corresponding one-dimensional systems, see e.g. [God71, God76, Ev98, Mi73]. The existence of these invariants is a corollary of the hyperbolicity property.
3. Any of the systems (4.4) in the canonical form consists of n independent equations of the form
$$\frac{\partial w}{\partial t} + \mu\frac{\partial w}{\partial x} = 0, \tag{4.6}$$
where $w = w(x,t)$ is a scalar function and $\mu \in \mathbb{R}$. Consider for the equation (4.6) the following difference scheme. The function $w(t_0, x)$, already computed at time level $t = t_0$ (initially $t = 0$; the values on this level are taken from the initial conditions), is substituted by the piecewise-constant function with the values $w_{i-\frac{1}{2}}$ within the corresponding grid cell $x_{i-1} < x \le x_i$. Define for each $1 \le i \le 2^N - 1$ auxiliary "interior" values (called "large values" in [God76]) as follows:
$$\mathcal{W}_i = \begin{cases} w_{i-\frac{1}{2}}, & \text{if } \mu \ge 0, \\ w_{i+\frac{1}{2}}, & \text{if } \mu < 0. \end{cases} \tag{4.7}$$

In the case of the Cauchy problem (1.2), for the auxiliary "boundary" values $\mathcal{W}_0$ and $\mathcal{W}_{2^N}$ we use the same formula (4.7) where $w_{-\frac{1}{2}} = w_{2^N-\frac{1}{2}}$ and $w_{2^N+\frac{1}{2}} = w_{\frac{1}{2}}$. The case of the boundary-value problem is described in the step 4 below.

The values $\{w^{i-\frac{1}{2}}\}$ on the next time level $t = t_0 + \tau$ (τ is a time step depending on h as specified in the next subsection) are then computed as
$$w^{i-\frac{1}{2}} = w_{i-\frac{1}{2}} - \mu\frac{\tau}{h}(\mathcal{W}_i - \mathcal{W}_{i-1}). \tag{4.8}$$

Taking the scheme (4.8) for each equation of the systems (4.4), we obtain for them schemes of the following vector form:

$$\frac{\mathbf{v_x}^{i-\frac{1}{2}} - \mathbf{v_x}_{i-\frac{1}{2}}}{\tau} + M_x \frac{(\mathcal{V}_x)_i - (\mathcal{V}_x)_{i-1}}{h} = 0.$$

4. For the boundary-value problem (2.1), we compute the vector boundary values $(\mathcal{V}_x)_0, (\mathcal{V}_x)_{2^N}$ with the help of the boundary conditions. On the left boundary $x = 0$ we calculate m_+ components of $(\mathcal{V}_x)_0$, corresponding to the positive eigenvalues of the matrix $A^{-1}B$, from the system of linear equations $\Phi_1^{(1)}(T_x\mathcal{V}_x)_0 = 0$; for m_- components of $(\mathcal{V}_x)_0$, corresponding to the negative eigenvalues, we let $(\mathcal{V}_x)_0 := (\mathbf{v_x})_{\frac{1}{2}}$. The components corresponding to the zero eigenvalues of $A^{-1}B$ can be chosen arbitrary since they are multiplied by zero in the scheme. The values on the right boundary and on both boundaries by the y-coordinate are calculated in a similar way.
5. Finally, for finding the values $\{\mathbf{u}^{i-\frac{1}{2},j-\frac{1}{2}}\}$ on the next time step, we use the system of linear equations

$$A\frac{\mathbf{u}^{i-\frac{1}{2},j-\frac{1}{2}} - \mathbf{u}_{i-\frac{1}{2},j-\frac{1}{2}}}{\tau} + B\frac{\mathcal{U}_{i,j-\frac{1}{2}} - \mathcal{U}_{i-1,j-\frac{1}{2}}}{h} + C\frac{\mathcal{U}_{i-\frac{1}{2},j} - \mathcal{U}_{i-\frac{1}{2},j-1}}{h} = 0 \qquad (4.9)$$

where $\mathcal{U}_{i,j-\frac{1}{2}} = T_x(\mathcal{V}_x)_i$ and $\mathcal{U}_{i-\frac{1}{2},j} = T_y(\mathcal{V}_y)_j$ are obtained by applying the transformations inverse to (4.5).

The scheme (4.9) was invented by S.K. Godunov; it is described and analysed in all details in [God76], see also e.g. [KPS01, GV96]. It approximates the system (1.2) or (2.1) with the first order of accuracy (the proof of the approximation property is done by means of the Taylor decomposition).

4.3. **Properties of the difference operators.** Here we establish some properties of the difference operators from the previous subsection.

Lemma 4.4.
(1) *The difference operators* $\left\{\mathbf{u}_{i-\frac{1}{2},j-\frac{1}{2}}\right\} \mapsto \left\{\mathbf{u}^{i-\frac{1}{2},j-\frac{1}{2}}\right\}$ *and* $\varphi^{(h)} \mapsto u^{(h)}$ *are linear.*
(2) *Let* $(\mathbb{B}, \beta)$ *be a strongly constructive real closed ordered subfield of* $\mathbb{R}_c$. *Given symmetric matrices* $A, B_1, \ldots, B_m$ *with coefficients in* $\mathbb{B}$, *an initial grid function* $\varphi^{(h)}$ *on* G_k *with values in* $\mathbb{B}$, *and a number* $\tau \in \mathbb{B}$, *one can compute (w.r.t.* β*) the grid function* $u^{(h)}$ *on* G_k^τ *(which again has its values in* $\mathbb{B}$*).*
(3) *For any positive* $\tau \leq \left(\frac{1}{\tau_x} + \frac{1}{\tau_y}\right)^{-1}$, *where*

$$\tau_x = \max_i\{|\mu_i| : \mathit{det}(\mu_i A - B) = 0\} \cdot h \quad \mathit{and} \quad \tau_y = \max_i\{|\mu_i| : \mathit{det}(\mu_i A - C) = 0\} \cdot h \quad (4.10)$$

$(i = 1, 2, \ldots, n)$, the difference scheme (4.9) is stable in the sense of Definition 4.2.

Proof.
(1) The assertion follows from the fact that (4.9) is a linear system of equations and the operator computing the "large" values $\mathcal{U}_i$ is linear provided that the eigenvectors found at step 2 are fixed.

(2) The operator $(A, B_1, \dots, B_m, \varphi^{(h)}, \tau) \mapsto u^{(h)}$ involves only algebraic (iterative) computations, including the solution of linear systems of equations with (previously computed) coefficients in $\mathbb{B}$, finding of eigenvalues and eigenvectors of symmetric matrices with (previously computed) coefficients in $\mathbb{B}$, and comparing (previously computed) numbers in $\mathbb{B}$, in particular in the branching operators (4.7), (4.8). Therefore, the assertion follows from Lemma 3.7 and other remarks in Section 3.1.

(3) This assertion is a standard fact, we give some details for the reader not working with the difference schemes. Recall from Section 4.1 that a difference scheme is *stable* if the corresponding difference operators R_h (that send the grid function $[u^l]$ to the grid function $[u^{l+1}]$) are bounded uniformly on h, together with their powers. The investigation of stability of the difference scheme from the previous subsection can be done as follows (for more details see e.g. [God76, p. 78]).

First consider the one-dimensional scheme (4.8) and the case $\mu \geq 0$ (in case $\mu < 0$ the argument is similar). Denote by $\nu = |\mu|\dfrac{\tau}{h}$ the Courant number and check the scheme stability by the Fourier method. Substituting in (4.8) the values (where $i^2 = -1$)

$$w_{j-\frac{1}{2}} = w^* e^{ij\phi}, \quad w^{j-\frac{1}{2}} = \lambda w_{j-\frac{1}{2}},$$

we obtain the characteristic equation

$$\lambda(\phi) = 1 - \nu(1 - e^{-i\phi}).$$

The necessary and sufficient condition for the one-dimensional difference operator to be uniformly bounded, together with its powers, is the condition $|\lambda(\phi)| \leq 1$ for all $\phi \in [0, 2\pi)$, that is equivalent to the condition $\nu \leq 1$ for the Courant number (it follows from a rather technical, but simple argument).

For the n-dimensional scheme (4.9) approximating the boundary-value problem (2.1) (when $m = 2$), the stability condition is

$$\tau \left(\frac{1}{\tau_x} + \frac{1}{\tau_y} \right) \leq 1, \tag{4.11}$$

where τ_x, τ_y as in (4.10) are the maximal time steps guaranteeing the stability of the corresponding one-dimensional schemes.

The proof of this stability condition is based on the case of a one-dimensional scheme for one equation (described above), on the theory of matrix pencils $\lambda A - B_i$ [Ga67, HJ83] and on a difference energy integral inequality: under the restriction (4.11) one has

$$\begin{aligned} &\sum_{i,j=1}^{2^N} (A\mathbf{u}^{i-\frac{1}{2},j-\frac{1}{2}}, \mathbf{u}^{i-\frac{1}{2},j-\frac{1}{2}}) - \sum_{i,j=1}^{2^N} (A\mathbf{u}_{i-\frac{1}{2},j-\frac{1}{2}}, \mathbf{u}_{i-\frac{1}{2},j-\frac{1}{2}}) \leq \\ &\frac{\tau}{h} \left(\sum_{j=1}^{2^N} \left[(B\mathcal{U}_{0,j-\frac{1}{2}}, \mathcal{U}_{0,j-\frac{1}{2}}) - (B\mathcal{U}_{2^N,j-\frac{1}{2}}, \mathcal{U}_{2^N,j-\frac{1}{2}}) \right] \right) + \\ &+\frac{\tau}{h} \left(\sum_{i=1}^{2^N} \left[(C\mathcal{U}_{i-\frac{1}{2},0}, \mathcal{U}_{i-\frac{1}{2},0}) - (C\mathcal{U}_{i-\frac{1}{2},2^N}, \mathcal{U}_{i-\frac{1}{2},2^N}) \right] \right). \end{aligned} \tag{4.12}$$

Due to the dissipativity of the boundary conditions, the right-hand part of the inequality (4.12) is below zero, thus

$$\sum_{i,j=1}^{2^N}(A\mathbf{u}^{i-\frac{1}{2},j-\frac{1}{2}},\mathbf{u}^{i-\frac{1}{2},j-\frac{1}{2}}) \leq \sum_{i,j=1}^{2^N}(A\mathbf{u}_{i-\frac{1}{2},j-\frac{1}{2}},\mathbf{u}_{i-\frac{1}{2},j-\frac{1}{2}}) \tag{4.13}$$

which is equivalent to the stability condition.

We omit the proof of the energy inequality (4.12) since it is standard (see e.g. [God76, Fr54, GR62, GV96]) and rather technical. □

5. Computability of the solution operators

In this section we give precise formulations and proof schemes of our main results. The precise formulations are given in Subsection 5.1. In Subsections 5.2 and 5.3 we describe the proof schemes of the main results omitting technical details of the relevant estimates. The technical details are presented in the next section.

5.1. **Formulations of main results.** Let us formulate the main results of this paper. The first main result concerns computability of the boundary-value problem (2.1) posed in Section 2.

Theorem 5.1. *Let $Q=[0,1]^m$; $T>0$ be a computable real and $M_\varphi>0$, $p\geq 2$ be integers. Let $A,B_1,\dots,B_m$ be fixed computable symmetric matrices, such that $A=A^*>0$, $B_i=B_i^*$. Let $\Phi_i^{(1)}$, $\Phi_i^{(2)}$ $(i=1,2,\dots,m)$ be fixed computable rectangular real non-degenerate matrices, with their numbers of rows equal to the number of positive and negative eigenvalues of $A^{-1}B_i$, respectively, and such that inequalities* (2.2) *hold.*

If $\varphi\in C^{p+1}(Q)$ satisfies

$$||\frac{\partial\varphi}{\partial x_i}||_s\leq M_\varphi,\ ||\frac{\partial^2\varphi}{\partial x_i\partial x_j}||_s\leq M_\varphi,\ i,j=1,2,\dots,m, \tag{5.1}$$

and meets the boundary conditions, then the operator $R:\varphi\mapsto\mathbf{u}$ mapping the initial function to the unique solution $\mathbf{u}\in C^p(Q\times[0,T],\mathbb{R}^n)$ of the boundary-value problem (2.1) *is a computable partial function from $C_s(Q,\mathbb{R}^n)$ to $C_{sL_2}(Q\times[0,T],\mathbb{R}^n)$.*

A natural question is whether the computability of solution operator R is uniform on the matrices A,B_i. Currently we do not know the answer for the arbitrary real matrices (cf. Remark 5.5.2 below) but the uniformity holds when the coefficients of A,B_i range through an arbitrary strongly constructive real closed ordered subfield $(\mathbb{B},\beta)$ of $\mathbb{R}_c$. The next result extends the previous theorem because, by Lemma 3.5, for any computable real matrices $A,B_i,\Phi_i^{(1)}$, $\Phi_i^{(2)}$ there is a strongly constructive real closed ordered subfield $(\mathbb{B},\beta)$ of $\mathbb{R}_c$ that contains all coefficients of the matrices $A,B_1,\dots,B_m,\Phi_i^{(1)}$, $\Phi_i^{(2)}$.

Theorem 5.2. *Let $(\mathbb{B},\beta)$ be a strongly constructive real closed ordered subfield of $\mathbb{R}_c$. Then the operator R from the previous theorem is uniformly computable (w.r.t the numbering β) on the matrices $A,B_1,\dots,B_m,\Phi_i^{(1)}$, $\Phi_i^{(2)}$ with coefficients in $\mathbb{B}$.*

The second main result concerns the initial value problem (1.2). It improves the main result of [SS09] and is uniform on the arbitrary matrices A,B_i.

Theorem 5.3. *Let $M_\varphi > 0, M_A > 0, p \geq 2$ be integers, let $i = 1, \dots, m$, and let $n_A, n_1, \dots, n_m$ be cardinalities of spectra of A and of the matrix pencils $\lambda A - B_1, \dots, \lambda A - B_m$, respectively (i.e., n_i is the number of distinct roots of the characteristic polynomial $\det(\lambda A - B_i)$). Then the operator*

$$(A, B_1, \dots, B_m, n_A, n_1, \dots, n_m, \varphi) \mapsto \mathbf{u}$$

sending any sequence $A, B_1, \dots, B_m$ of symmetric real matrices with $A > 0$ such that the matrix pencils $\lambda A - B_i$ have no zero eigenvalues,

$$||A||_2, ||A^{-1}||_2, ||B_i||_2 \leq M_A, \quad \lambda^{(i)}_{\min} < 0 < \lambda^{(i)}_{\max}, \quad i = 1, 2, \dots, m, \tag{5.2}$$

the sequence $n_A, n_1, \dots, n_m$ of the corresponding cardinalities, and any $\varphi \in C^{p+1}(Q, \mathbb{R}^n)$ satisfying the conditions (5.1)*, to the unique solution $\mathbf{u} \in C^p(H, \mathbb{R}^n)$ of* (1.2) *is a computable partial function from the space $S_+ \times S^m \times \mathbb{N}^{m+1} \times C_s(Q, \mathbb{R}^n)$ to $C_{sL_2}(H, \mathbb{R}^n)$.*

In Theorem 5.3, $\lambda^{(i)}_{\min}, \lambda^{(i)}_{\max}$ are respectively the minimal and maximal eigenvalues of the matrix pencil $\lambda A - B_i$, $S \subseteq \mathbb{R}^{n \times n}$ is the space of symmetric $n \times n$ matrices equipped with the Euclidean norm, S_+ is the space of symmetric positively definite matrices with the Euclidean norm, and $H \subseteq \mathbb{R}^{m+1}$ is the domain of correctness of (1.2), i.e., the maximal set where, for any $p \geq 2$ and $\varphi \in C^{p+1}(Q, \mathbb{R}^n)$, there exists a unique solution $\mathbf{u} \in C^p(H, \mathbb{R}^n)$ of the initial value problem (1.2).

The set H is known to be (see e.g. [God71]) a nonempty intersection of the semi-spaces

$$t \geq 0,\ x_i - \lambda^{(i)}_{\max} t \geq 0,\ x_i - 1 - \lambda^{(i)}_{\min} t \leq 0,\ (i = 1, \dots, m)$$

of $\mathbb{R}^{m+1}$. We are especially interested in the case when H is a compact subset of $Q \times [0, +\infty)$ (obviously, a sufficient condition for this to be true is $\lambda^{(i)}_{\min} < 0 < \lambda^{(i)}_{\max}$ for all $i = 1, \dots, m$; this is often the case for natural physical systems. In [SS09] we observed that the domain H for the problem (1.2) is computable from $A, B_1, \dots, B_m$ (more exactly, the vector $(\lambda^{(1)}_{\max}, \dots, \lambda^{(m)}_{\max}, \lambda^{(1)}_{\min}, \dots, \lambda^{(m)}_{\min})$ is computable from $A, B_1, \dots, B_m$; this implies computability of H in the sense of computable analysis [We00]).

Since, for each $i = 1, \dots, m$, $\lambda^{(i)}_{\max}$ is the maximal and $\lambda^{(i)}_{\min}$ is the minimal eigenvalue of the matrix pencil $\lambda A - B_i$, and maximum and minimum of a vector of reals are computable [We00], it suffices to show that a vector $(\lambda_1, \dots, \lambda_n)$ consisting of all eigenvalues of a matrix pencil $\lambda A - B$ is computable from A, B. But $(\lambda_1, \dots, \lambda_n)$ is a vector of all roots of the characteristic polynomial of $\lambda A - B$, hence it is computable [We00, BHW03]. We immediately obtain

Lemma 5.4. *A rational number τ meeting* (4.11) *is computable from symmetric real matrices $A, B_1, \dots, B_m$ such that $A > 0$.*

Remarks 5.5.

1. Besides the condition $\lambda^{(i)}_{\min} < 0 < \lambda^{(i)}_{\max}$ in Theorem 5.3, some alternative natural conditions may be assumed. E.g., for one equation $\frac{\partial u}{\partial t} - \frac{\partial u}{\partial x} = 0$ the domain of correctness may be the intersection of the semi-planes $\{t \geq 0\}$, $\{x \leq t\}$, $\{x \leq 1 + t\}$, and we search the solution in the intersection of the semi-planes $\{t \geq 0\}$, $\{x \leq t\}$, $\{x \leq 1\}$. Our proof is adjusted to similar modifications in a straightforward way.
2. Note that Theorem 5.3 states computability on arbitrary matrices A, B_i while Theorem 5.1 (and Theorem 5.2) does not. The reason is that our proof of Theorem 5.3 (where we take rational fast Cauchy approximations to A, B_i) can not be straightforwardly adjusted to

that of Theorem 5.1 because the dissipativity conditions in the last theorem might hold for the given matrix but not hold for the approximate matrices. Currently we do not know whether Theorem 5.2 may be strengthened to include computability on the real coefficients A, B_i. If we strengthen the dissipativity conditions to strict inequalities then the solution operator in Theorem 5.1 will be computable on $A, B_i, \Phi_i^{(1)}$, $\Phi_i^{(2)}$ $(i = 1, 2, \dots, m)$, similarly to Theorem 5.3.

3. In [SS09], we established a weaker result that the solution is computable provided that $A, B_1, \dots, B_m$ are fixed computable matrices (in this case one can of course omit the conditions on spectra of A and of the matrix pencils). This weaker result is proved just in the same way as Theorem 5.1 below. In the stronger formulation above, the proof requires additional considerations described below.

5.2. **Scheme of proof of Theorem 5.1.** Here we provide outline of the proof of Theorem 5.1 (which also applies to Theorem 5.2) omitting the proofs of some technical estimates. The estimates are proved in the next section.

1. Consider a grid $G = G_k$ of step $h = 2^{-k}$ on Q, as described in Subsection 3.3, and choose a computable sequence $\{T_k\}$ of rational numbers that fast converges to T. Take a sequence $\{\varphi_k\}$ of grid functions $\varphi_k : G_{i_k} \to \mathbb{Q}^n$ such that their multilinear interpolations $\{\tilde{\varphi}_k\}$ form a fast Cauchy sequence converging in $C_s(Q, \mathbb{R}^n)$ to φ, so
$$||\tilde{\varphi}_k - \varphi||_s \leq \frac{1}{2^k}. \tag{5.3}$$
2. Choose $(\mathbb{B}, \beta)$ as in Remark 5.2. We compute a sequence $\{\upsilon_k\}$ of grid functions $\upsilon_k : G^{\tau}_{i_k} \to \mathbb{B}^n$ (for some sequence $\{i_k\}$ of natural numbers). Without loss of generality we may assume that the sequence $\{i_k\}$ is increasing (otherwise, choose a suitable subsequence of $\{\varphi_k\}$). Let the grid function υ_k be constructed from φ_k, A, $B_1, \dots, B_m$ and $\Phi_i^{(1)}$, $\Phi_i^{(2)}$ $(i = 1, 2, \dots, m)$, by the algorithm of the difference equation in Subsection 4.2. According to Lemma 4.4, the operation $\varphi_k \mapsto \upsilon_k$ is computable (w.r.t. β) and linear on φ_k.

   By Lemma 3.15, it suffices to show that for some constant c (depending only on $A, B_1, \cdots, B_m$, $\Phi_i^{(1)}$, $\Phi_i^{(2)}$ and M_φ) we have
$$||\tilde{\upsilon}_k - \mathbf{u}||_{sL_2} \leq c \cdot \frac{1}{2^k}. \tag{5.4}$$
   for all k, i.e. $\{\tilde{\upsilon}_k\}$ fast converges in $C_{sL_2}(H, \mathbb{R}^n)$ to $\mathbf{u}$.
3. We divide the proof of (5.4) into several parts. For any k, let $\tilde{u}_k$ be the interpolation of the grid function computed by the algorithm in Subsection 4.3 from the exact initial values $\varphi|_{G_k}$. By $\widetilde{u|_{G_k^{\tau_k}}}$ we denote the interpolation of the G_k-discretization of the solution $\mathbf{u}$ of the differential problem (2.1). We will estimate independently the following three summands:
$$||\tilde{\upsilon}_k - \mathbf{u}||_{sL_2} \leq ||\tilde{\upsilon}_k - \tilde{u}_k||_{sL_2} + ||\tilde{u}_k - \widetilde{u|_{G_k^{\tau_k}}}||_{sL_2} + ||\widetilde{u|_{G_k^{\tau_k}}} - \mathbf{u}||_{sL_2}. \tag{5.5}$$
4. The third summand is estimated with the help of the properties of interpolations:
$$||\widetilde{u|_{G_k^{\tau_k}}} - \mathbf{u}||_{sL_2} \leq c_{int} \cdot 2^{-k}. \tag{5.6}$$

   The constant c_{int} depends on the norms of derivatives of $\mathbf{u}$, which can be estimated by φ and its derivatives (hence by some expression involving M_φ and the norms of the coefficient matrices, following the lines of the proof of the uniqueness theorem for (2.1) in [God76, p. 194], see also [Ev98]).

5. The second summand is estimated with the help of Theorem 4.3 on convergence of the difference scheme in grid norms:
$$||\tilde{u}_k - \widetilde{u|_{G_k^{\tau_k}}}||_{sL_2} \leq c_{diff} \cdot 2^{-k}. \tag{5.7}$$
The constant c_{diff} also depends only on the derivatives of φ and on the coefficients of (2.1).
6. The first summand is estimated by means of stability of the difference scheme, with taking into account linearity of the difference and interpolation operators. The corresponding constant depends on the coefficients of the differential equations:
$$||\tilde{v}_k - \tilde{u}_k||_{sL_2} \leq c_A \cdot 2^{-k} \tag{5.8}$$

5.3. **Scheme of proof of Theorem 5.3.** Here we provide outline of the proof of Theorem 5.3 omitting the proofs of some technical estimates. The estimates are proved in the next section.

1. Consider again the grid $G = G_k$ of step $h = 2^{-k}$ on Q .Take a sequence $\{\varphi_k\}$ of grid functions $\varphi_k : G_{i_k} \to \mathbb{Q}^n$ such that their multilinear interpolations $\{\tilde{\varphi}_k\}$ form a fast Cauchy sequence converging in $C_s(Q, \mathbb{R}^n)$ to φ.
2. Let $A^{(k)}$ and $B_i^{(k)}$ be sequences of symmetric matrices, such that $A^{(k)} > 0$, with rational coefficients fast converging to A and B_i respectively, $i = 1, 2, \dots, m$, in the standard Euclidean norm
$$||A - A^{(k)}||_2 \leq 2^{-k},\ ||B_i - B_i^{(k)}||_2 \leq 2^{-k}. \tag{5.9}$$
From [ZB01, Theorem 2] (stating that, given a complex normal matrix and the cardinality of its spectrum one can compute the sequence of all its eigenvalues counted with their multiplicities, as well as an orthonormal basis of eigenvectors), we can without loss of generality assume that, for all k and $i = 0, \dots, m$, the matrix pencil $\lambda A^{(k)} - B_i^{(k)}$ has no zero eigenvalues. By item (2) of Lemma 4.4, the function $(A, B_1, \dots, B_m, k) \mapsto \tau$ is computable, so there is a computable sequence $\{\tau_k\}$ of positive rationals that fast converges to τ and, for any k, τ_k satisfies the stability condition (4.11) for matrices $A, B_1, \dots, B_m$, and also for matrices $A^{(k)}, B_1^{(k)}, \dots, B_m^{(k)}$.
Let $\{v_k\}$ be an α-computable sequence obtained as in the proof of Theorem 5.1, only with the strongly constructive ordered field $(\mathbb{A}, \alpha)$ of algebraic reals in place of $(\mathbb{B}, \beta)$ and with $A^{(k)}$, $B_i^{(k)}$ in place of A, B_i. It suffices to show that for some constant c (depending only on M_A and M_φ) we have (5.4) for all k, i.e. $\{\tilde{v}_k\}$ fast converges in $C_{sL_2}(H, \mathbb{R}^n)$ to $\mathbf{u}$.
3. We divide the proof of (5.4) again into several parts.
Let $\widehat{v}_k$ be the grid function obtained by the difference scheme of Subsection 4.2 from φ_k and A, B_i (the "exact" coefficient matrices); let $\widetilde{\widehat{v}_k}$ be its interpolation.
Let $\tilde{u}_k$ be the sequence of interpolations of the grid functions obtained by the difference scheme of Subsection 4.2 from the exact initial values $\varphi|_{G_k}$ and from the exact matrices A, B_i. By $\widetilde{u}|_{G_k^{\tau_k}}$ we denote the interpolation of the G_k-discretization of the solution $\mathbf{u}$ of the differential problem (1.2). Now (5.4) is naturally splitted to four summands:
$$||\tilde{v}_k - \mathbf{u}||_{sL_2} \leq ||\tilde{v}_k - \tilde{\widehat{v}}_k||_{sL_2} + ||\tilde{\widehat{v}}_k - \widetilde{u}_k||_{sL_2} + ||\tilde{u}_k - \widetilde{u}|_{G_k^{\tau_k}}||_{sL_2} + ||\widetilde{u}|_{G_k^{\tau_k}} - \mathbf{u}||_{sL_2}. \tag{5.10}$$

4. The last three summands are estimated in the same way as (5.6)–(5.8) in the previous subsection. The only difference is that all estimates are obtained in the domain H of uniqueness of the Cauchy problem rather than in the set $Q \times [0,T]$ in the proof of Theorem 5.1. The procedure of identifying the grid cells, which are in H, was described in [SS09].
5. The key technical tool for estimating the first summand
$$||\tilde{v}_k - \tilde{\tilde{v}}_k||_{sL_2} \leq c_{rat} \cdot 2^{-k} \tag{5.11}$$
is formal differentiation of the difference scheme. For doing this correctly, the assumption of Theorem 5.3 that the eigenvalues are non-zero and the cardinalities of spectra are known in advance, are needed. Note that the constant c_{rat} depends on the eigenvalues of the matrices A, B_i.

6. Proofs of the estimates

In this section we prove the technical estimates from the previous section.

6.1. **Interpolation and proof of the estimates** (5.6)**,** (5.8)**.** Recall the construction of the multilinear interpolations.

In the one-dimensional case, the interpolating function $\tilde{\mathbf{u}}$ is defined inside the grid rectangles
$$\left(i - \frac{1}{2}\right) h \leq x \leq \left(i + \frac{1}{2}\right) h;\ l\tau \leq t \leq (l+1)\tau$$
in the standard way as follows
$$\begin{aligned}\tilde{\mathbf{u}}(x,t) &= \mathbf{u}_{i-\frac{1}{2}} \cdot \left(l+1-\frac{t}{\tau}\right) \cdot \left(i+\frac{1}{2}-\frac{x}{h}\right) + \mathbf{u}_{i+\frac{1}{2}} \cdot \left(l+1-\frac{t}{\tau}\right) \cdot \left(\frac{x}{h} - \left(i-\frac{1}{2}\right)\right) \\ &+ \mathbf{u}^{i-\frac{1}{2}} \cdot \left(\frac{t}{\tau} - l\right) \cdot \left(i+\frac{1}{2}-\frac{x}{h}\right) + \mathbf{u}^{i+\frac{1}{2}} \cdot \left(\frac{t}{\tau} - l\right) \cdot \left(\frac{x}{h} - \left(i-\frac{1}{2}\right)\right)\end{aligned}$$
where $\mathbf{u}_{i\pm\frac{1}{2}}$ and $\mathbf{u}^{i\pm\frac{1}{2}}$ are the grid functions on time levels $t = l\tau$ and $t = (l+1)\tau$, respectively.

In the two-dimensional case (and for higher dimensions m) the interpolating function is defined in a similar way. Since the full expression is rather long we write down only two (of eight) summands, others are constructed in an obvious way (see [God71, p. 212]):
$$\begin{aligned}\tilde{\mathbf{u}}(x,y,t) &= \mathbf{u}_{i-\frac{1}{2},j-\frac{1}{2}} \cdot \left(l+1-\frac{t}{\tau}\right) \cdot \left(i+\frac{1}{2}-\frac{x}{h}\right) \cdot \left(j+\frac{1}{2}-\frac{y}{h}\right) \\ &+ \mathbf{u}_{i+\frac{1}{2},j-\frac{1}{2}} \cdot \left(l+1-\frac{t}{\tau}\right) \cdot \left(\frac{x}{h} - \left(i-\frac{1}{2}\right)\right) \cdot \left(j+\frac{1}{2}-\frac{y}{h}\right) \\ &+ \cdots\end{aligned}$$
where $\left(j - \frac{1}{2}\right) h \leq y \leq \left(j + \frac{1}{2}\right) h$.

From these formulas for multilinear interpolation, linearity of the interpolation operators $\mathbf{u} \mapsto \tilde{\mathbf{u}}$ and $\mathbf{u}|_{G_k} \mapsto \widetilde{\mathbf{u}|_{G_k}}$ is obvious.

Proposition 6.1. $||\widetilde{u|_{G_k}} - \mathbf{u}||_{sL_2} \leq c_{int} \cdot \frac{1}{2^k}$ *for some constant* c_{int} *depending only on* M_A, M_φ.

Proof. By a well-known estimate for the multilinear interpolations [Sz59, So74, ZKM80, Ba86],

$$||\widetilde{u|_{G_k}} - \mathbf{u}||_s \leq c_{int} \cdot \frac{1}{2^k} \tag{6.1}$$

for some constant c_{int} depending only on the s-norms of second derivatives of $\mathbf{u}$. Since the s-norm is stronger than the sL_2-norm, $||\tilde{u}_k - \mathbf{u}||_{sL_2} \leq c_{int} \cdot \frac{1}{2^k}$. It suffices to show that c_{int} depends in fact only on the second derivatives of φ and the norms of the matrices A, A^{-1}, B_i.

Considering the Cauchy problem, due to the smoothness assumptions, we can construct auxiliary Cauchy problems for partial derivatives of $\mathbf{u}$ (we write down a couple of them, as examples):

$$\begin{cases} A(\mathbf{u}_x)_t + B(\mathbf{u}_x)_x + C(\mathbf{u}_x)_y = 0, \\ \mathbf{u}_x|_{t=0} = \varphi_x, \end{cases}$$

$$\begin{cases} A(\mathbf{u}_t)_t + B(\mathbf{u}_t)_x + C(\mathbf{u}_t)_y = 0, \\ \mathbf{u}_t|_{t=0} = -A^{-1}(B\varphi_x + C\varphi_y), \end{cases}$$

$$\begin{cases} A(\mathbf{u}_{tt})_t + B(\mathbf{u}_{tt})_x + C(\mathbf{u}_{tt})_y = 0, \\ \mathbf{u}_{tt}|_{t=0} = -A^{-1}(B(\mathbf{u}_t|_{t=0})_x + C(\mathbf{u}_t|_{t=0})_y) = \psi(x,y). \end{cases} \tag{6.2}$$

As it is known, we have

$$||\mathbf{u}||_{A,sL_2} \leq ||\varphi||_{A,L_2} \tag{6.3}$$

The proof of this estimate is rather long and technical; it is presented in detail for the considered systems of PDEs (even with a nonzero righthand part f; in this case the estimate contains also the norm of f) in [God71] (the estimates are stated on p. 155 for the Cauchy problem and on p. 194 for the boundary-value problem, respectively), and also can be found in [Ev98, subsection 7.3]. Applying an analog of (6.3) to the systems for the second derivatives of $\mathbf{u}$ and using the equivalence of norms in $\mathbb{R}^n$, we obtain

$$||\frac{\partial^2 \mathbf{u}}{\partial x_i \partial x_j}||_{sL_2} \leq \sqrt{\frac{\lambda_{max}(A)}{\lambda_{min}(A)}} \cdot c(||A||_2, ||A^{-1}||_2, ||B_1||_2, \dots, ||B_1||_2,$$
$$\max_{i,j=1,\dots,m} ||\frac{\partial^2 \varphi}{\partial x_i \partial x_j}||_{L_2}) \leq c(M_A, M_\varphi). \qquad \square$$

Thus the estimate (5.6) needed in the proofs of Theorems 2 and 3 is established. Further we will also need the following

Lemma 6.2. *Let $u^{(h)}$ be calculated from $\varphi^{(h)}$ by means of the difference scheme in Subsection 4.2. Then $||\widetilde{u^{(h)}}||_{sL_2} \leq \sqrt{\frac{\lambda_{max}(A)}{\lambda_{min}(A)}} ||\widetilde{\varphi^{(h)}}||_s$.*

Proof. Let us first show that

$$\max_{0 \leq l\tau \leq T} \int\limits_{H \cap \{t = l\tau\}} |\tilde{u}(x,y,t)|^2 dxdy \leq \max_{0 \leq l\tau \leq T} \left(h^2 \sum_{i,j} u^2_{i-\frac{1}{2}, j-\frac{1}{2}} \right) = ||u^{(h)}||^2_{sL_2}. \tag{6.4}$$

For simplicity of notation consider the one-dimensional case (adding an additional variable is straightforward). In the i-th grid cell for a fixed $t = l\tau$ we have

$$\tilde{u}(x,t) = u_{il} \cdot (i + 1 - \frac{x}{h}) + u_{il+1} \cdot (\frac{x}{h} - i)$$

and

$$\int\limits_{ih}^{(i+1)h} \tilde{u}^2(x,t)dx = \int\limits_{ih}^{(i+1)h} [u_{j,l}(i+1-\frac{x}{h}) + u_{i,l+1}(\frac{x}{h}-i)]^2 dx =$$

$$h\int\limits_0^1 [u_{i,l}(1-\xi)+u_{i,l+1}\xi]^2 d\xi = \frac{h}{3}(u_{i,l}^2 + u_{i,l}u_{j,l+1} + u_{i,l+1}^2) \le h\frac{u_{i,l}^2 + u_{i,l+1}^2}{2}.$$

The summation by i yields $\int\limits_0^1 \tilde{u}^2(x,l\tau)dx \le h\sum\limits_i u_{i,l}^2(x,l\tau)$. Taking maximum over all l concludes the proof of (6.4).

It is well-known [Sz59, So74], that for the linear interpolations $||\widetilde{u^{(h)}}||_{sL_2} \le ||u^{(h)}||_{sL_2}$ where the right-hand part refers to the grid norm. Obviously,

$$||\varphi^{(h)}||_{L_2}^2 = h^2\sum_{i,j}\varphi^2_{i-\frac{1}{2},j-\frac{1}{2}} \le h^2\frac{1}{h^2}\max_{i,j}\varphi^2_{i-\frac{1}{2},j-\frac{1}{2}} \le \sup_{(x,y)\in Q}\widetilde{\varphi|_G}^2(x,y) = ||\varphi^{(h)}||_s^2. \quad (6.5)$$

The estimate (4.13) implies $||u^{(h)}||_{A,sL_2} \le ||\varphi^{(h)}||_{A,L_2}$. Taking into account the equivalence of the Euclidean norms $\lambda_{min}(A)\langle\mathbf{u},\mathbf{u}\rangle \le \langle A\mathbf{u},\mathbf{u}\rangle \le \lambda_{max}(A)\langle\mathbf{u},\mathbf{u}\rangle$ we obtain the desired estimate. □

Arguments similar to those in the proof of Lemma 6.2 can be found in [God71]. We have recalled them for the convenience of the reader.

Proposition 6.3. *The estimate* (5.8) *holds.*

Proof. Using linearity of the interpolation and difference operators and Lemma 6.2 we obtain

$$||\tilde{\upsilon}_k - \tilde{u}_k||_{sL_2} = ||\widetilde{\upsilon_k - u_k}||_{sL_2} \le \sqrt{\frac{\lambda_{max}(A)}{\lambda_{min}(A)}}||\widetilde{\varphi_k} - \widetilde{\varphi|_{G_k}}||_s$$

$$\le \sqrt{\frac{\lambda_{max}(A)}{\lambda_{min}(A)}}(||\widetilde{\varphi_k}-\varphi||_s + ||\widetilde{\varphi|_{G_k}} - \varphi||_s) \le c(M_A, M_\varphi)2^{-k}.$$

Here, $||\widetilde{\varphi|_{G_k}} - \varphi||_s$ is again estimated by (6.1). □

6.2. **Convergence of the difference scheme and proof of the estimate** (5.7)**.**

Lemma 6.4. *There is a constant* c_{diff} *depending only on* M_A, M_φ *such that for all* $k \ge 0$ *we have* $||u_k - u|_{G_k^\tau}||_{sL_2} \le c_{diff}\cdot\frac{1}{2^k}$ *where* u *is the solution of* (1.2) *or* (2.1).

Proof. The estimate follows from Theorem 4.3. The fact that c_{diff} depends only on M_A and M_φ follows from the proof of this theorem in [GR62, Chapter 5] according to which we can take $c_{\text{diff}} = c_1\cdot c_2$ where c_2 comes from the stability condition and c_1 is from the approximation $||L_h u_h - (Lu)|_{G_k^\tau}||_{sL_2} \le c_1 h$. Since we consider a first-order difference scheme, it follows from the Taylor decomposition of $\mathbf{u}$ that c_1 depends only on M_A and

$$||\frac{\partial\mathbf{u}}{\partial x_i}||_{sL_2},\ ||\frac{\partial\mathbf{u}}{\partial t}||_{sL_2},\ ||\frac{\partial^2\mathbf{u}}{\partial x_i\partial x_j}||_{sL_2},\ ||\frac{\partial^2\mathbf{u}}{\partial x_i\partial t}||_{sL_2}.$$

By the proof of the uniqueness theorem for (1.2) (see the proof of Proposition 6.1), the norms of the derivatives above are bounded by a constant depending only on M_A, M_φ. □

Proposition 6.5. *There is a constant c depending only on M_A, M_φ such that for all $k \geq 0$ we have $||\tilde{u}_k - \widetilde{u|_{G_k}}||_{sL_2} \leq c \cdot \frac{1}{2^k}$ where u is the solution of* (1.2) *or* (2.1)*.*

Proof. Since the operator of multilinear interpolation is linear, from (6.4) and the estimate of the previous lemma we obtain

$$||\tilde{u}_k - \widetilde{u|_{G_k^\tau}}||_{sL_2} \leq ||u_k - u|_{G_k^\tau}||_{sL_2} \leq c_{diff} 2^{-k}.$$

This implies the desired estimate, which is exactly the estimate (5.7) needed in the proofs of Theorems 5.1, 5.3. □

6.3. **Proof of the estimate** (5.11)**.** Finally, we prove the estimate (5.11) which is needed only for the Cauchy problem. Recall that υ_k and $\widehat{\upsilon}_k$ satisfy respectively the following difference schemes (see (4.9)) in which the index k in τ_k, υ_k, $\widehat{\upsilon}_k$ and w_k is omitted for simplicity (note that τ_k is chosen in such a way that both difference schemes below are stable)

$$A\frac{\widehat{\upsilon}^{i-\frac{1}{2},j-\frac{1}{2}} - \widehat{\upsilon}_{i-\frac{1}{2},j-\frac{1}{2}}}{\tau} + B\frac{\widehat{\Upsilon}_{i,j-\frac{1}{2}} - \widehat{\Upsilon}_{i-1,j-\frac{1}{2}}}{h} + C\frac{\widehat{\Upsilon}_{i-\frac{1}{2},j} - \widehat{\Upsilon}_{i-\frac{1}{2},j-1}}{h} = 0, \tag{6.6}$$

$$A^{(k)}\frac{\upsilon^{i-\frac{1}{2},j-\frac{1}{2}} - \upsilon_{i-\frac{1}{2},j-\frac{1}{2}}}{\tau} + B^{(k)}\frac{\Upsilon_{i,j-\frac{1}{2}} - \Upsilon_{i-1,j-\frac{1}{2}}}{h} + C^{(k)}\frac{\Upsilon_{i-\frac{1}{2},j} - \Upsilon_{i-\frac{1}{2},j-1}}{h} = 0 \tag{6.7}$$

with the initial conditions $\upsilon|_{t=0} = \widehat{\upsilon}|_{t=0} = \varphi_k$. Deducting the second system of equations from the first one we obtain the difference equations

$$A\frac{\mathbf{w}^{i-\frac{1}{2},j-\frac{1}{2}} - \mathbf{w}_{i-\frac{1}{2},j-\frac{1}{2}}}{\tau} + B\frac{\widetilde{\mathcal{W}}_{i,j-\frac{1}{2}} - \widetilde{\mathcal{W}}_{i-1,j-\frac{1}{2}}}{h} + C\frac{\widetilde{\mathcal{W}}_{i-\frac{1}{2},j} - \widetilde{\mathcal{W}}_{i-\frac{1}{2},j-1}}{h} = \widetilde{f}, \tag{6.8}$$

with the initial condition $\mathbf{w}|_{t=0} = 0$ where $\mathbf{w} = \widehat{\upsilon} - \upsilon$, $\widetilde{\mathcal{W}} = \widehat{\Upsilon} - \Upsilon$ and

$$\widetilde{f} = (A^{(k)} - A)\frac{\upsilon^{i-\frac{1}{2},j-\frac{1}{2}} - \upsilon_{i-\frac{1}{2},j-\frac{1}{2}}}{\tau} + (B^{(k)} - B)\frac{\Upsilon_{i,j-\frac{1}{2}} - \Upsilon_{i-1,j-\frac{1}{2}}}{h} + (C^{(k)} - C)\frac{\Upsilon_{i-\frac{1}{2},j} - \Upsilon_{i-\frac{1}{2},j-1}}{h}.$$

By definition of $\widehat{\Upsilon}$ and Υ, they are constructed by means of Riemannian invariants defined by the "exact" matrices A, B, C and by their rational approximations $A^{(k)}, B^{(k)}, C^{(k)}$, respectively. To work with the system (6.8) as with a difference scheme approximating the Cauchy problem (1.2), we would like to have in the left-hand part of (6.8) the "large values" $\mathcal{W}$ defined by means of Riemannian invariants for A, B, C and $\widehat{\upsilon} - \upsilon$, rather then $\widetilde{\mathcal{W}}$. Recall from Subsection 4.2 (stage 2) that the Riemannian invariants are constructed by means of eigenvectors of the corresponding matrix pencils. Using the fact that the cardinalities of their spectra and the spectrum of A are known as inputs, by [ZB01, Theorem 2] we have that the eigenvectors are computable, i.e. they can be chosen in such a way that, for an absolute constant c,

$$||T_{(A,B)} - T_{(A^{(k)},B^{(k)})}||_2 \leq \frac{c}{2^k}, \ ||T_{(A,C)} - T_{(A^{(k)},C^{(k)})}||_2 \leq \frac{c}{2^k}, \tag{6.9}$$

where by $T_{(A,B)}$ we denote the matrix T_x defined in (4.5), and, in a similar way for other pairs of matrices (cf. [SS09, Theorem 5]). The results of [ZB01] can be applied to this case since the procedure of finding $T_{(A,B)}$ consists of two spectral decompositions for symmetric matrices. Moreover, the fact that the matrix pencils $\lambda A - B$ and $\lambda A - C$ have no zero eigenvalues guarantees that the numbers of positive and negative eigenvalues in (13) are the same for $\lambda A - B$ and $\lambda A^{(k)} - B^{(k)}$, as well as for $\lambda A - C$ and $\lambda A^{(k)} - C^{(k)}$. Thus we can rewrite (6.8) as

$$A\frac{\mathbf{w}^{i-\frac{1}{2},j-\frac{1}{2}} - \mathbf{w}_{i-\frac{1}{2},j-\frac{1}{2}}}{\tau} + B\frac{\mathcal{W}_{i,j-\frac{1}{2}} - \mathcal{W}_{i-1,j-\frac{1}{2}}}{h} + C\frac{\mathcal{W}_{i-\frac{1}{2},j} - \mathcal{W}_{i-\frac{1}{2},j-1}}{h} = f,$$

where $\mathcal{W}$ are the desired "large values" for w while f differs from $\tilde{f}$ on a value involving the norms from (6.9) multiplied by difference derivatives analogous to the ones listed below in (6.10).

By the stability condition, $||\widehat{\upsilon} - \upsilon||_{sL_2} \leq c||f||_{L_2}$ for some constant c depending only on M_A, M_φ [God71, God76, Ev98]. By formal differentiation of the scheme (cf. [God71], similar arguments in the proof of the existence theorem for one-dimensional symmetric hyperbolic systems in the canonical form) it is possible to check that any of the norms

$$||\frac{\upsilon^{i-\frac{1}{2},j-\frac{1}{2}} - \upsilon_{i-\frac{1}{2},j-\frac{1}{2}}}{\tau}||_{sL_2},\ ||\frac{\Upsilon_{i,j-\frac{1}{2}} - \Upsilon_{i-1,j-\frac{1}{2}}}{h}||_{sL_2},\ ||\frac{\Upsilon_{i-\frac{1}{2},j} - \Upsilon_{i-\frac{1}{2},j-1}}{h}||_{sL_2} \tag{6.10}$$

is below the difference derivatives of φ_k, which are below a constant depending only on M_A, M_φ. We provide necessary arguments for the first norm, the arguments for the others being similar.

Direct computations (writing the system of difference equations (6.7) for two neighbour time levels, subtracting them and dividing by τ) show that the grid function

$$\frac{\Delta\upsilon^{(h)}}{\Delta t} = \left\{\frac{\upsilon^{l+1}_{i-\frac{1}{2},j-\frac{1}{2}} - \upsilon^{l}_{i-\frac{1}{2},j-\frac{1}{2}}}{\tau}\right\},$$

consisting of the "difference derivatives" of $\upsilon^{(h)}$, meets the system of difference equations (6.11) below.

For $t = l\tau$ we obtain:

$$\begin{cases} A^{(k)}\frac{\upsilon^{l+1}_{i-\frac{1}{2},j-\frac{1}{2}} - \upsilon^{l}_{i-\frac{1}{2},j-\frac{1}{2}}}{\tau} + B^{(k)}\frac{\Upsilon^{l}_{i,j-\frac{1}{2}} - \Upsilon^{l}_{i-1,j-\frac{1}{2}}}{h} + C^{(k)}\frac{\Upsilon^{l}_{i,j-\frac{1}{2}} - \Upsilon^{l}_{i-1,j-\frac{1}{2}}}{h} = 0, \\ \upsilon^{0}_{i-\frac{1}{2},j} = (\varphi_k)_{i-\frac{1}{2},j-1}. \end{cases}$$

For $t = (l+1)\tau$ we obtain

$$\begin{cases} A^{(k)}\frac{\upsilon^{l+2}_{i-\frac{1}{2},j-\frac{1}{2}} - \upsilon^{l+1}_{i-\frac{1}{2},j-\frac{1}{2}}}{\tau} + B^{(k)}\frac{\Upsilon^{l+1}_{i,j-\frac{1}{2}} - \Upsilon^{l+1}_{i-1,j-\frac{1}{2}}}{h} + C^{(k)}\frac{\Upsilon^{l+1}_{i-\frac{1}{2},j} - \Upsilon^{l+1}_{i-\frac{1}{2},j-1}}{h} = 0, \\ \upsilon^{1}_{i-\frac{1}{2},j-\frac{1}{2}} = (\varphi_k)_{i-\frac{1}{2},j-\frac{1}{2}} - (A^{(k)})^{-1}\tau\left(B^{(k)}\frac{\Upsilon^{0}_{i,j-\frac{1}{2}} - \Upsilon^{0}_{i-1,j-\frac{1}{2}}}{h} + C^{(k)}\frac{\Upsilon^{0}_{i-\frac{1}{2},j} - \Upsilon^{0}_{i-\frac{1}{2},j-1}}{h}\right). \end{cases}$$

Subtracting the first system from the second one and dividing by τ we obtain:

$$\begin{cases} A^{(k)}\frac{(\frac{\Delta\upsilon}{\Delta t})^{l+1} - (\frac{\Delta\upsilon}{\Delta t})^{l}}{\tau} + B^{(k)}\frac{(\frac{\Delta\Upsilon_x}{\Delta t})_i - (\frac{\Delta\Upsilon_x}{\Delta t})_{i-1}}{h} + C^{(k)}\frac{(\frac{\Delta\Upsilon_y}{\Delta t})_j - (\frac{\Delta\Upsilon_y}{\Delta t})_{j-1}}{h} = 0, \\ (\frac{\Delta\upsilon}{\Delta t})^0 = -(A^{(k)})^{-1}\left(B^{(k)}\frac{\Upsilon^{0}_{i,j-\frac{1}{2}} - \Upsilon^{0}_{i-1,j-\frac{1}{2}}}{h} + C^{(k)}\frac{\Upsilon^{0}_{i-\frac{1}{2},j} - \Upsilon^{0}_{i-\frac{1}{2},j-1}}{h}\right) =: \psi_{i-\frac{1}{2},j-\frac{1}{2}} \end{cases} \tag{6.11}$$

where

$$\left(\frac{\Delta\Upsilon_x}{\Delta t}\right)_i = \frac{\Upsilon^{l+1}_{i,j-\frac{1}{2}} - \Upsilon^l_{i,j-\frac{1}{2}}}{\tau}, \quad \left(\frac{\Delta\Upsilon_y}{\Delta t}\right)_j = (\frac{\Upsilon^{l+1}_{i-\frac{1}{2},j} - \Upsilon^l_{i-\frac{1}{2},j-1}}{\tau}),$$

$$\left(\frac{\Delta\upsilon}{\Delta t}\right)^l = \frac{\upsilon^{l+1}_{i-\frac{1}{2},j-\frac{1}{2}} - \upsilon^l_{i-\frac{1}{2},j-\frac{1}{2}}}{\tau}.$$

Note that, since the operator of difference differentiation is linear, $\frac{\Delta\Upsilon_x}{\Delta t}$ and $\frac{\Delta\Upsilon_y}{\Delta t}$ are the "large values" for $\frac{\Delta\upsilon}{\Delta t}$. The stability condition for this scheme looks as follows:

$$||\left\{\frac{\Delta\upsilon}{\Delta t}\right\}_{i-\frac{1}{2},j-\frac{1}{2}}||_{A^{(k)},sL_2} \le ||\{\psi_{i-\frac{1}{2},j-\frac{1}{2}}\}||_{A^{(k)},L_2}.$$

Since $A^{(k)}, B^{(k)}, C^{(k)}$ fast converge to A, B, C respectively, it remains to estimate

$$||\frac{\Delta U_x}{\Delta x}|| := ||\left\{\frac{\Upsilon^0_{i,j-\frac{1}{2}} - \Upsilon^0_{i-1,j-\frac{1}{2}}}{h}\right\}||_{L_2}, \quad ||\frac{\Delta U_y}{\Delta y}|| := ||\left\{\frac{\Upsilon^0{}_{i-\frac{1}{2},j} - \Upsilon^0{}_{i-\frac{1}{2},j-1}}{h}\right\}||_{L_2}. \tag{6.12}$$

Recall that the "large values" are calculated from the Riemannian invariants of auxiliary one-dimensional systems, see Subsection 4.2: $\Upsilon^0_{i,j-\frac{1}{2}} = T_x V^0_i$, where $V^0_i = v_{i\pm\frac{1}{2}}$, $V^0_{i-1} = v_{i-\frac{1}{2}}$ or $v_{i-\frac{3}{2}}$, depending on the eigenvalues of the corresponding matrix pencils (and in a similar way with $\Upsilon^0_{i-\frac{1}{2},j}$). Note also that the eigenvectors were chosen to be orthonormal, hence $||T_x||_2 = ||T^*_x||_2 = 1$. Therefore,

$$||\frac{\Delta U_x}{\Delta x}|| = ||T^*_x \frac{\Delta V}{\Delta x}|| \le 2||\frac{\Delta v}{\Delta x}|| \le c(M_A)||\frac{\Delta\varphi_k}{\Delta x}||.$$

The last estimate can be derived as an energy integral inequality for the auxiliary one-dimensional scheme like in (4.13) (see also [God76, p. 78], [Fr54, GR62, GV96]). We have

$$||\frac{\Delta\varphi_k}{\Delta x}||^2 = \sum_{i,j} h^2 \left(\frac{(\varphi_k)_{i+\frac{1}{2},j-\frac{1}{2}} - (\varphi_k)_{i-\frac{1}{2},j-\frac{1}{2}}}{h}\right)^2 =$$

$$\frac{1}{h^2}\sum_{i,j} h^2 \langle (\varphi_k)_{i+\frac{1}{2},j-\frac{1}{2}} - (\varphi_k)_{i-\frac{1}{2},j-\frac{1}{2}}, (\varphi_k)_{i+\frac{1}{2},j-\frac{1}{2}} - (\varphi_k)_{i-\frac{1}{2},j-\frac{1}{2}}\rangle.$$

Adding and subtracting the expression $\varphi_{i+\frac{1}{2},j-\frac{1}{2}} - \varphi_{i-\frac{1}{2},j-\frac{1}{2}}$ (where φ is the "exact" initial function) within the scalar product, we obtain

$$||\frac{\Delta U_x}{\Delta x}||^2 \le \frac{2||\varphi_k - \varphi||^2}{h^2} + \sum_{i,j} h^2 \langle \frac{\varphi_{i+\frac{1}{2},j-\frac{1}{2}} - \varphi_{i-\frac{1}{2},j-\frac{1}{2}}}{h}, \frac{\varphi_{i+\frac{1}{2},j-\frac{1}{2}} - \varphi_{i-\frac{1}{2},j-\frac{1}{2}}}{h}\rangle.$$

Since $\{\varphi_k\}$ fast converges to φ, the first summand is below 2. Passing to the limit in the above inequality when h tends to 0 and taking into account that the integration operator is computable we see that the second summand (which tends to $\int_Q \langle \frac{\partial\varphi}{\partial x}, \frac{\partial\varphi}{\partial x}\rangle dxdy$) is uniformly (on h) bounded by a universal constant depending only on M_φ. Therefore, $||\frac{\Delta U_x}{\Delta x}||$ is uniformly (on h) bounded by a universal constant depending only on M_A and M_φ. The expression $||\frac{\Delta U_y}{\Delta y}||$ is estimated in a similar way.

Since $||A^{(k)} - A||_2$, $||B^{(k)} - B||_2$, and $||C^{(k)} - C||_2$ are below $\frac{1}{2^k}$, the desired estimate for $||f||$ follows. This completes the proof of the estimate (5.11).

7. Conclusion

From the proof of Theorems 5.1, 5.3 we see that the same idea may be applied to a broader class of systems for which there is a stable difference scheme and which satisfy the existence and uniqueness theorem. In particular, most probably, analogs of our results hold for systems (1.2), (2.1) with the variable coefficients A, B_i depending on t, x_j. However, precise specification of the whole class of such systems and of corresponding difference schemes remains an open question.

As noted in Section 2, the wave equation (2.5) can be reduced (not in a unique way) to a symmetric hyperbolic system, thus its solution operator is computable which gives an affirmative answer to the question of paper [WZ02], for the case of dissipative boundary conditions and an initial function with uniformly bounded derivatives. Note, however, that we consider not Sobolev H^s spaces of generalized functions, but C^k spaces of continuously differentiable functions; the smoothness k, which is to be assumed, depends on the particular problem under consideration. To prove computability of the generalized solutions it would be probably suitable to use finite element methods.

The restriction on the initial function seems to be rather strong from Computable Analysis viewpoint, but it is very natural for Numerical Analysis (though we have never seen it explicitly formulated in Numerical Analysis theorems). Indeed, it is well known, that any initial (and right-hand part) function can be represented in the form of a Fourier series (consider for simplicity the one-dimensional case) $\varphi(x) = \sum_n a_n e^{inx}$, the differentiation of which gives $\sum_n n a_n e^{inx}$, i.e. "fast oscillating" functions lead to large derivatives and hence large convergence constants which can make the scheme not convergent on a real computer.

A similar situation is with the additional assumptions of Theorem 5.3 about the apriori knowledge of the spectra of A and of matrix pencils and the absence of zero eigenvalues: the assumptions correspond well to the experience of numerical analysts. Namely, violation of these assumptions may lead to computational instabilities. We currently do not know whether our results hold without these assumptions. Note, however, that the cardinalities are known (for physical reasons) for some important systems invariant under rotations [GM98].

Finally, we would like to point out that it would be interesting to study the computational complexity of the considered problems, in the spirit of [Ko91]. The algorithms suggested in our paper are very time- and space-consuming (in fact, our proofs provide no explicit complexity bounds at all, establishing only computability). Finding feasible algorithms to compute the solution operators seems to be a challenge.

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