Applied Numerical Mathematics 93 (2015) 206–214

Contents lists available at ScienceDirect

Applied Numerical Mathematics

www.elsevier.com/locate/apnum

Finite element approximation with numerical integration for differential eigenvalue problems ☆

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ARTICLE INFO

Article history:
Available online 6 March 2014

Keywords:
Eigenvalue
Eigenfunction
Eigenvalue problem
Sturm–Liouville problem
Finite element method
Numerical integration

ABSTRACT

Error estimates of the finite element method with numerical integration for differential eigenvalue problems are presented. More specifically, refined results on the eigenvalue dependence for the eigenvalue and eigenfunction errors are proved. The theoretical results are illustrated by numerical experiments for a model problem.

1. Introduction

Finite element approximation of differential eigenvalue problems leads to the matrix eigenvalue problem $Ax = \lambda Bx$ with the matrices A and B whose elements involve integrals which are usually evaluated by numerical integration. The effect of numerical integration on the eigenvalue and eigenfunction errors has been investigated in the papers [18,13,12,14,22,38–40,7,6,25,28,30,32,33,35,36,34].

In the present paper, we establish refined results on the eigenvalue dependence for error in the finite element approximation of the following differential eigenvalue problem $-(pu')' + qu = \lambda ru$, $x \in (0,1)$, $u(0) = u(1) = 0$, with positive smooth coefficients p, q, r. The weak statement of the differential eigenvalue problem in the Sobolev space $V = \{v\colon\ v \in W_2^1(0,1),\, v(0) = v(1) = 0\}$ with norm $|\cdot|_1$ can be written in the form $a(u,v) = \lambda b(u,v)$ for any $v \in V$. This problem has a sequence of positive simple eigenvalues λ_k, $k = 1, 2, \ldots$, $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_k < \cdots$, $\lambda_k \to \infty$ as $k \to \infty$. To these eigenvalues there corresponds an orthonormal system of eigenfunctions u_k, $k = 1, 2, \ldots$, such that $a(u_i, u_j) = \lambda_i \delta_{ij}$, $b(u_i, u_j) = \delta_{ij}$, $i, j = 1, 2, \ldots$. The functions u_k, $k = 1, 2, \ldots$, form a complete system in the space V.

The original eigenvalue problem is approximated by the following scheme of the finite element method with numerical integration $a_h(u^h, v^h) = \lambda^h b_h(u^h, v^h)$ in the space V_h of finite elements of order n. For sufficiently small h, the finite dimensional eigenvalue problem has N_h eigenvalues λ_k^h, $k = 1, 2, \ldots, N_h$, $0 < \lambda_1^h \leqslant \lambda_2^h \leqslant \cdots \leqslant \lambda_{N_h}^h$. To these eigenvalues there corresponds an orthonormal system of eigenfunctions u_k^h, $k = 1, 2, \ldots, N_h$, $a_h(u_i^h, u_j^h) = \delta_{ij}\lambda_i^h$, $b_h(u_i^h, u_j^h) = \delta_{ij}$, $i, j = 1, 2, \ldots, N_h$. The functions u_k^h, $k = 1, 2, \ldots, N_h$, form a complete system in the space V_h.

☆ This work was supported by Alexander von Humboldt-Stiftung (Alexander von Humboldt Foundation) and Russian Foundation for Basic Research (Project nos. 11-01-00864, 12-01-97026, 13-01-00908, 14-01-00755).

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http://dx.doi.org/10.1016/j.apnum.2014.02.009
0168-9274/

In this paper, we prove the following results. If the algebraic precision of the quadrature rule is $2n-1$, then for sufficiently small h the error estimates

$$\left|\lambda_k^h-\lambda_k\right|\leqslant ch^{2n}\lambda_k^{n+1},\qquad \left|u_k^h-u_k\right|_1\leqslant ch^n\lambda_k^{(n+1)/2},$$

hold, where c is a positive constant independent of h and λ_k, the signs of eigenfunctions u_k^h and u_k are chosen as $b(u_k^h,u_k)>0$, $b_h(u_k^h,u_k^h)=1$, $b(u_k,u_k)=1$. Error estimates of the paper are identical with the error estimates for the finite element scheme with exact computation of integrals [37] and for finite element schemes with quadrature in the particular case when $n=1$ [7,8]. Theoretical results of the paper can be generalized for multidimensional case.

Results similar to the ones obtained in the paper hold for nonlinear eigenvalue problems [34,31,25,29,26,27,42,19–21,9] as well. In the present paper, we use the technique suggested in [32,33,35,36,34,31,25,28,30].

An efficient method for solving Sturm–Liouville problems has been studied in the papers [5,3,4]. Unfortunately, this method is limited by the particular case when we know the Green's function of the problem. However, the approach of the present paper is free from this limitation and can be applied in the multidimensional case. For further results on Sturm–Liouville problems, we refer the reader to [37,23,24,15,2].

In what follows, we use standard results on Sobolev spaces, weak statements for differential eigenvalue problems, the finite element method [1,37,17,23,41,16,10,11].

2. Weak formulation of the problem

Let $\Omega=(0,1)$, $\overline{\Omega}=[0,1]$, and let G be an interval of the real line $\mathbb{R}$. Denote by $L_2(G)$ the real Lebesgue space equipped with the respective norm

$$|u|_{0,G}=\left(\int\limits_G\left|u(x)\right|^2dx\right)^{1/2},$$

denote by $W_2^m(G)$ the real Sobolev space equipped with the respective norm

$$\|u\|_{m,G}=\left(\sum_{i=0}^m|u|_{i,G}^2\right)^{1/2},$$

where

$$|u|_{i,G}=\left|u^{(i)}\right|_{0,G},\quad i=0,1,\dots,m,$$

$u^{(i)}=d^iu(x)/dx^i$, $i=1,2,\dots,m$, $u^{(0)}=u$, m is a positive integer. Put $W_2^0(G)=L_2(G)$. We drop the subscript $G=\Omega$ for norms and seminorms.

We define sufficiently smooth functions $p(x)$, $q(x)$, $r(x)$, $x\in\overline{\Omega}$, for which there exist positive constants p_1, p_2, q_2, r_1, r_2 such that

$$p_1\leqslant p(x)\leqslant p_2,\qquad 0\leqslant q(x)\leqslant q_2,\qquad r_1\leqslant r(x)\leqslant r_2,$$

for $x\in\overline{\Omega}$.

Consider the differential eigenvalue problem: find numbers λ and nonzero functions $u(x), x\in\overline{\Omega}$ such that

$$-(pu')'+qu=\lambda ru,\quad x\in(0,1),$$
$$u(0)=u(1)=0.$$

Introduce the Hilbert space

$$V=\left\{v:\ v\in W_2^1(\Omega),\, v(0)=v(1)=0\right\}$$

with norm $|\cdot|_1$. For $u,v\in V$ we define the bilinear forms

$$a(u,v)=\int\limits_0^1\left(pu'v'+quv\right)dx,$$

$$b(u,v)=\int\limits_0^1 ruv\,dx.$$

Let us formulate a weak statement for the differential eigenvalue problem: find $\lambda \in \mathbb{R}$, $u \in V \setminus \{0\}$ such that

$$a(u, v) = \lambda b(u, v) \quad \forall v \in V. \tag{1}$$

Denote by c_0 the constant occurring in the Friedrichs inequality

$$|v|_0 \leqslant c_0 |v|_1 \quad \forall v \in V.$$

One can readily see that the bilinear form $a(.,.)$ is positive definite and bounded with constants $\alpha_1 = p_1$, $\alpha_2 = p_2 + c_0 q_2$, i.e.,

$$\alpha_1 |v|_1^2 \leqslant a(v, v) \leqslant \alpha_2 |v|_1^2 \quad \forall v \in V.$$

Similarly, we have

$$\beta_1 |v|_0^2 \leqslant b(v, v) \leqslant \beta_2 |v|_0^2 \quad \forall v \in L_2(\Omega),$$

where $\beta_1 = r_1$, $\beta_2 = r_2$.

According to [23], problem (1) has a sequence of positive simple eigenvalues λ_k, $k = 1, 2, \ldots$:

$$0 < \lambda_1 < \lambda_2 < \cdots < \lambda_k < \cdots, \qquad \lim_{k \to \infty} \lambda_k = \infty.$$

To these eigenvalues there corresponds an orthonormal system of eigenfunctions u_k, $k = 1, 2, \ldots$ such that

$$a(u_i, u_j) = \lambda_i \delta_{ij}, \qquad b(u_i, u_j) = \delta_{ij}, \quad i, j = 1, 2, \ldots$$

The functions u_k, $k = 1, 2, \ldots$ form a complete system in the space V. Moreover, the functions u_k, $k = 1, 2, \ldots$ are sufficiently smooth and for $m = 0, 1, \ldots$ the following estimate

$$\|u_k\|_m \leqslant c_1 \lambda_k^{m/2} |u_k|_0 \tag{2}$$

is valid, where c_1 is a positive constant independent of λ_k [37].

3. Approximation of the problem

An approximate scheme for problem (1) is specified by the definition of a finite dimensional subspace V_h and approximate bilinear forms a_h and b_h. We divide the interval $\overline{\Omega}$ by equidistant points $x_i = ih$, $i = 0, 1, \ldots, m$, into elements $e_i = [x_{i-1}, x_i]$, $i = 1, 2, \ldots, m$, $h = 1/m$. By V_h we denote the subspace of V that consists of functions v^h in the space $\mathcal{P}_n(e_i)$ of polynomials of degree less than or equal to n on each element $e_i = [x_{i-1}, x_i]$, $i = 1, 2, \ldots, m$, $v^h(0) = v^h(1) = 0$, $N_h = \dim V_h = mn - 1$. By using a quadrature formula, we define bilinear forms a_h and b_h.

To compute the integral

$$\hat{I}(\hat{\varphi}) = \int_{\hat{e}} \hat{\varphi}(\hat{x})\, d\hat{x},$$

where $\hat{\varphi}(\hat{x})$ is a continuous function over the reference element $\hat{e} = [0, 1]$, we introduce a quadrature formula with L nodes

$$\hat{S}(\hat{\varphi}) = \sum_{l=1}^{L} \hat{\alpha}_l \hat{\varphi}(\hat{\beta}_l).$$

Here $\hat{\alpha}_l$, $\hat{\beta}_l \in \hat{e}$, $l = 1, 2, \ldots, L$, are weights and nodes of the quadrature formula.

This quadrature formula induces a quadrature formula over the each element e_i, $i = 1, 2, \ldots, m$. Indeed, using the mapping $\hat{e}$ onto e_i defined by

$$x = (\hat{x} + i - 1)h, \quad \hat{x} \in \hat{e},$$

and setting

$$\hat{\varphi}(\hat{x}) = \varphi\big((\hat{x} + i - 1)h\big), \quad \hat{x} \in \hat{e},$$

we get

$$\int_{e_i} \varphi(x)\, dx = h \int_{\hat{e}} \hat{\varphi}(\hat{x})\, d\hat{x} \approx h \sum_{l=1}^{L} \hat{\alpha}_l \hat{\varphi}(\hat{\beta}_l) = \sum_{l=1}^{L} \alpha_l^{(i)} \varphi\big(\beta_l^{(i)}\big).$$

Consequently, for computing the integral

$$I_i(\varphi) = \int_{e_i} \varphi(x)\,dx,$$

we obtain the quadrature formula

$$S_i(\varphi) = \sum_{l=1}^{L} \alpha_l^{(i)} \varphi\big(\beta_l^{(i)}\big),$$

where

$$\alpha_l^{(i)} = h\hat{\alpha}_l, \qquad \beta_l^{(i)} = (\hat{\beta}_l + i - 1)h, \quad l = 1, 2, \ldots, L.$$

Now, for computing the integral

$$I(\varphi) = \sum_{i=1}^{m} I_i(\varphi),$$

we derive the composite quadrature formula

$$S(\varphi) = \sum_{i=1}^{m} S_i(\varphi).$$

Applying this quadrature formula we define the approximate bilinear forms

$$a_h\big(u^h, v^h\big) = \sum_{i=1}^{m} \sum_{l=1}^{L} \alpha_l^{(i)} \big(p\big(u^h\big)'\big(v^h\big)' + qu^h v^h\big)\big(\beta_l^{(i)}\big),$$
$$b_h\big(u^h, v^h\big) = \sum_{i=1}^{m} \sum_{l=1}^{L} \alpha_l^{(i)} \big(ru^h v^h\big)\big(\beta_l^{(i)}\big)$$

for any $u^h, v^h \in V_h$. Assume that $b_h(v^h, v^h) > 0$ for any $v^h \in V_h \setminus \{0\}$.

The variational problem (1) is approximated by the finite element method with numerical integration: find $\lambda^h \in \mathbb{R}$, $u^h \in V_h \setminus \{0\}$ such that

$$a_h\big(u^h, v^h\big) = \lambda^h b_h\big(u^h, v^h\big) \quad \forall v^h \in V_h. \tag{3}$$

Problem (3) has N_h positive eigenvalues λ_k^h, $k = 1, 2, \ldots, N_h$,

$$0 < \lambda_1^h \leqslant \lambda_2^h \leqslant \cdots \leqslant \lambda_{N_h}^h.$$

To these eigenvalues there corresponds an orthonormal system of eigenfunctions u_k^h, $k = 1, 2, \ldots, N_h$, such that

$$a_h\big(u_i^h, u_j^h\big) = \lambda_i^h \delta_{ij}, \qquad b_h\big(u_i^h, u_j^h\big) = \delta_{ij}, \quad i, j = 1, 2, \ldots, N_h.$$

The functions u_k^h, $k = 1, 2, \ldots, N_h$, form a complete system in the space V_h.

For $v \in W_2^n(e_k)$, $k = 1, 2, \ldots, n$, we set

$$\|v\|_{n,h} = \left(\sum_{k=1}^{m} \|v\|_{n,e_k}^2\right)^{1/2}.$$

If $v^h \in V_h$, $i = 1, 2, \ldots, n$, then the inverse inequality [17]

$$\big\|v^h\big\|_{i,h} \leqslant ch^{1-i}\big|v^h\big|_1 \tag{4}$$

holds. For $u \in W_2^{n+1}(\Omega)$, $i = 0, 1, \ldots, n$, the following estimate is valid [17]

$$\|u - P_h u\|_{i,h} \leqslant ch^{n+1-i}\|u\|_{n+1}, \tag{5}$$

where an operator $P_h : V \to V_h$ is defined by the rule $a(u - P_h u, v^h) = 0$ for any $v^h \in V_h$.

By c we denote various positive constants independent of h and λ_k. Suppose that the algebraic precision of the numerical quadrature $\hat{S}$ is $2n-1$, i.e., $\hat{I}(\hat{\varphi}) - \hat{S}(\hat{\varphi}) = 0$ for any $\hat{\varphi} \in \mathcal{P}_{2n-1}(\hat{e})$, where $\mathcal{P}_i(\hat{e})$ denotes the space of polynomials on $\hat{e}$ of degree less than or equal to i, $i \geqslant 1$. Then for any $v^h, w^h \in V_h$ we have the following estimates [36]

$$\left|a_h\left(v^h, w^h\right) - a\left(v^h, w^h\right)\right| \leqslant ch^{2n}\left\|v^h\right\|_{n,h}\left\|w^h\right\|_{n,h}, \tag{6}$$

$$\left|b_h\left(v^h, w^h\right) - b\left(v^h, w^h\right)\right| \leqslant ch^{2n}\left\|v^h\right\|_{n,h}\left\|w^h\right\|_{n,h}. \tag{7}$$

Theorem 1. *Let $0 < \lambda_1^h \leqslant \lambda_2^h \leqslant \cdots \leqslant \lambda_{N_h}^h$ be the eigenvalues of the approximate scheme (3), which correspond to eigenfunctions u_k^h, $k = 1, 2, \ldots, N_h$, $b_h(u_i^h, u_j^h) = \delta_{ij}$, $i, j = 1, 2, \ldots, N_h$. Then we have the convergence $\lambda_k^h \to \lambda_k$ as $h \to 0$. If the signs of eigenfunctions u_k^h are chosen so as to ensure that $b(u_k^h, u_k) > 0$, then $u_k^h \to u_k$ in V as $h \to 0$, $1 \leqslant k \leqslant N_h$. Here λ_k and u_k, $k = 1, 2, \ldots$, are eigenvalues and eigenfunctions of problem (1), satisfying the relations $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_k < \cdots$, $b(u_i, u_j) = \delta_{ij}$, $i, j = 1, 2, \ldots$.*

Proof. The desired results follow from [36]. □

Theorem 2. *Assume that, for fixed $k \geqslant 1$, the signs of eigenfunctions u_k^h and u_k are chosen as $b(u_k^h, u_k) > 0$, $b_h(u_k^h, u_k^h) = 1$, $b(u_k, u_k) = 1$. Then the estimate*

$$\left|u_k^h - u_k\right|_1 \leqslant ch^n \lambda_k^{(n+1)/2}$$

holds for sufficiently small h.

Proof. Suppose that u_i, $i = 1, 2, \ldots$, and u_i^h, $i = 1, 2, \ldots, N_h$, are eigenfunctions of problems (1) and (3) respectively, $b(u_k^h, u_k) > 0$, $b_h(u_k^h, u_k^h) = 1$, $b(u_k, u_k) = 1$. Set $\beta_i^h = b_h(P_h u_k, u_i^h)$, $i = 1, 2, \ldots, N_h$. Since the functions u_i^h, $i = 1, 2, \ldots, N_h$, form an orthonormal basis in the space V_h, it follows that the function $P_h u_k \in V_h$ can be represented in the form $P_h u_k = Q_k^h u_k + v_k^h + w_k^h$, where

$$Q_k^h u_k = \beta_k^h u_k^h, \qquad v_k^h = \sum_{i=1}^{k-1} \beta_i^h u_i^h, \qquad w_k^h = \sum_{i=k+1}^{N_h} \beta_i^h u_i^h.$$

Here we adopt the following convention: if $n < m$, then the sum $\sum_{i=m}^{n}$ is zero.

Set

$$\zeta_h(u_k) = \sup_{v^h \in V_h \setminus \{0\}} \frac{|a_h(P_h u_k, v^h) - \lambda_k b_h(P_h u_k, v^h)|}{|v^h|_1}.$$

Taking into account (1), (2), (4), (5), (6), (7), we obtain

$$\begin{aligned}
&\left|a_h\left(P_h u_k, v^h\right) - \lambda_k b_h\left(P_h u_k, v^h\right)\right| \\
&\quad \leqslant \left|a_h\left(P_h u_k, v^h\right) - a\left(P_h u_k, v^h\right)\right| + \lambda_k\left|b_h\left(P_h u_k, v^h\right) - b\left(P_h u_k, v^h\right)\right| + \lambda_k\left|b\left(u_k - P_h u_k, v^h\right)\right| \\
&\quad \leqslant c\lambda_k h^{2n}\|P_h u_k\|_{n,h}\left\|v^h\right\|_{n,h} + c\lambda_k|u_k - P_h u_k|_0\left|v^h\right|_0 \\
&\quad \leqslant c\lambda_k h^{n+1}\|u_k\|_{n+1}\left|v^h\right|_1 \leqslant ch^{n+1}\lambda_k\lambda_k^{(n+1)/2}\left|v^h\right|_1 \\
&\quad = ch^{n+1}\lambda_k^{(n+3)/2}\left|v^h\right|_1
\end{aligned}$$

for any $v^h \in V_h$. Here we use the relation

$$\begin{aligned}
\|P_h u_k\|_{n,h} &\leqslant \|u_k\|_n + \|u_k - P_h u_k\|_{n,h} \\
&\leqslant \|u_k\|_n + ch\|u_k\|_{n+1} \leqslant c\|u_k\|_n.
\end{aligned}$$

As a result, we derive the estimate $\zeta_h(u_k) \leqslant ch^{n+1}\lambda_k^{(n+3)/2}$.

For $k \geqslant 1$, $\lambda_0 = 0$, we denote

$$\rho_k = \frac{\lambda_{k-1}}{\lambda_k - \lambda_{k-1}} + \frac{\lambda_{k+1}}{\lambda_{k+1} - \lambda_k}.$$

According to Theorem 1, we have the convergence $\lambda_i^h \to \lambda_i$ as $h \to 0$, $i = 1, 2, \dots, k+1$. Hence it follows that the inequalities $\lambda_k - \lambda_{k-1}^h > 0$, $\lambda_{k+1}^h - \lambda_k > 0$, where $k \geqslant 1$, $\lambda_0^h = 0$, are satisfied for sufficiently small h and for some positive constant c the relations

$$\frac{\lambda_{k-1}^h}{\lambda_k - \lambda_{k-1}^h} \leqslant c\rho_k, \qquad \frac{\lambda_{k+1}^h}{\lambda_{k+1}^h - \lambda_k} \leqslant c\rho_k,$$

are valid.

For $k \geqslant 1$, let us prove the estimate $|v_k^h|_1 \leqslant c\rho_k \zeta_h(u_k)$. This estimate is obviously true for $k = 1$. Let $k \geqslant 2$. Then the following relations hold: $a_h(P_h u_k, v_k^h) = a_h(v_k^h, v_k^h)$, $b_h(P_h u_k, v_k^h) = b_h(v_k^h, v_k^h)$, $a_h(v_k^h, v_k^h) \leqslant \lambda_{k-1}^h b_h(v_k^h, v_k^h)$, and

$$\begin{aligned}|v_k^h|_1 \zeta_h(u_k) &\geqslant -a_h(P_h u_k, v_k^h) + \lambda_k b_h(P_h u_k, v_k^h)\\ &= -a_h(v_k^h, v_k^h) + \lambda_k b_h(v_k^h, v_k^h) \geqslant (\lambda_k - \lambda_{k-1}^h) b_h(v_k^h, v_k^h)\\ &\geqslant \frac{\lambda_k - \lambda_{k-1}^h}{\lambda_{k-1}^h} a_h(v_k^h, v_k^h) \geqslant (c\rho_k)^{-1} |v_k^h|_1^2, \quad k \geqslant 2.\end{aligned}$$

This implies the desired estimate.

For $k \geqslant 1$, let us prove the estimate $|w_k^h|_1 \leqslant c\rho_k \zeta_h(u_k)$. One can readily see that $a_h(P_h u_k, w_k^h) = a_h(w_k^h, w_k^h)$, $b_h(P_h u_k, w_k^h) = b_h(w_k^h, w_k^h)$, $a_h(w_k^h, w_k^h) \geqslant \lambda_{k+1}^h b_h(w_k^h, w_k^h)$. Then

$$\begin{aligned}|w_k^h|_1 \zeta_h(u_k) &\geqslant a_h(P_h u_k, w_k^h) - \lambda_k b_h(P_h u_k, w_k^h)\\ &= a_h(w_k^h, w_k^h) - \lambda_k b_h(w_k^h, w_k^h) \geqslant \frac{\lambda_{k+1}^h - \lambda_k}{\lambda_{k+1}^h} a_h(w_k^h, w_k^h)\\ &\geqslant (c\rho_k)^{-1} |w_k^h|_1^2, \quad k \geqslant 1,\end{aligned}$$

which yields the required estimates.

Now, using the estimates we have proved, we arrive at the inequalities

$$|P_h u_k - Q_k^h u_k|_1 \leqslant |v_k^h|_1 + |w_k^h|_1 \leqslant c\rho_k \zeta_h(u_k) \leqslant c\rho_k h^{n+1} \lambda_k^{(n+3)/2}$$

for sufficiently small h.

For sufficiently small h the following inequality is valid $|u_k^h|_1 \leqslant c\lambda_k^{1/2}$, since

$$\begin{aligned}\alpha_1 |u_k^h|_1^2 - \lambda_k^h &\leqslant a(u_k^h, u_k^h) - \lambda_k^h = a(u_k^h, u_k^h) - a_h(u_k^h, u_k^h)\\ &\leqslant |a_h(u_k^h, u_k^h) - a(u_k^h, u_k^h)| \leqslant ch^{2n} \|u_k^h\|_{n,h}^2 \leqslant ch^2 |u_k^h|_1^2,\\ (\alpha_1 - ch^2)|u_k^h|_1^2 &\leqslant \lambda_k^h \leqslant c\lambda_k.\end{aligned}$$

Let us prove that

$$|\beta_k^h u_k^h - u_k^h|_1 \leqslant c\rho_k h^{n+1} \lambda_k^{(n+4)/2}$$

for sufficiently small h. Denote $\|v\|_b^2 = b(v, v)$, $\|v^h\|_{b_h}^2 = b_h(v^h, v^h)$, for $v \in V$, $v^h \in V_h$. Then we get

$$\begin{aligned}\beta_k^h &= b_h(P_h u_k, u_k^h) = b(P_h u_k, u_k^h) + \big(b_h(P_h u_k, u_k^h) - b(P_h u_k, u_k^h)\big)\\ &= b(u_k, u_k^h) + b(P_h u_k - u_k, u_k^h) + \big(b_h(P_h u_k, u_k^h) - b(P_h u_k, u_k^h)\big)\\ &\geqslant b(u_k, u_k^h) - |b(P_h u_k - u_k, u_k^h)| - \big|\big(b_h(P_h u_k, u_k^h) - b(P_h u_k, u_k^h)\big)\big|\\ &\geqslant b(u_k, u_k^h) - ch^{n+1} \lambda_k^{(n+2)/2} > 0\end{aligned}$$

for sufficiently small h, where

$$\begin{aligned}|b(P_h u_k - u_k, u_k^h)| &\leqslant c|P_h u_k - u_k|_0 |u_k^h|_0\\ &\leqslant c|P_h u_k - u_k|_0 |u_k^h|_1 \leqslant ch^{n+1} \lambda_k^{(n+2)/2},\\ |b_h(P_h u_k, u_k^h) - b(P_h u_k, u_k^h)| &\leqslant ch^{2n} \|P_h u_k\|_{n,h} \|u_k^h\|_{n,h}\\ &\leqslant ch^{n+1} \|P_h u_k\|_{n,h} |u_k^h|_1 \leqslant ch^{n+1} \lambda_k^{(n+1)/2}.\end{aligned}$$

Therefore we have

$$\beta_k^h = \|\beta_k^h u_k^h\|_{b_h} \leqslant 1 + \|u_k - P_h u_k\|_b + \|P_h u_k - \beta_k^h u_k^h\|_{b_h} + \big|\|P_h u_k\|_{b_h} - \|P_h u_k\|_b\big|,$$
$$\beta_k^h = \|\beta_k^h u_k^h\|_{b_h} \geqslant 1 - \|u_k - P_h u_k\|_b - \|P_h u_k - \beta_k^h u_k^h\|_{b_h} - \big|\|P_h u_k\|_{b_h} - \|P_h u_k\|_b\big|.$$

Hence we obtain

$$\big|\beta_k^h - 1\big| \leqslant \|u_k - P_h u_k\|_b + \|P_h u_k - \beta_k^h u_k^h\|_{b_h} + \big|\|P_h u_k\|_{b_h} - \|P_h u_k\|_b\big|,$$

where

$$\begin{aligned}\big|\|P_h u_k\|_{b_h} - \|P_h u_k\|_b\big| &= \frac{|\|P_h u_k\|_{b_h}^2 - \|P_h u_k\|_b^2|}{|\|P_h u_k\|_{b_h} + \|P_h u_k\|_b|} \\ &\leqslant c\big|\|P_h u_k\|_{b_h}^2 - \|P_h u_k\|_b^2\big|\end{aligned}$$

for sufficiently small h. Now the desired estimate follows from the relations

$$\begin{aligned}\big|\beta_k^h u_k^h - u_k^h\big|_1 &= \big|\beta_k^h - 1\big|\big|u_k^h\big|_1 \leqslant c\lambda_k^{1/2}\big|\beta_k^h - 1\big| \\ &\leqslant c\lambda_k^{1/2}\big(|u_k - P_h u_k|_0 + \big|P_h u_k - Q_k^h u_k\big|_1 + \big|b_h(P_h u_k, P_h u_k) - b(P_h u_k, P_h u_k)\big|\big) \\ &\leqslant c\rho_k h^{n+1}\lambda_k^{(n+4)/2}.\end{aligned}$$

Thus, we conclude

$$\big|u_k - u_k^h\big|_1 \leqslant |u_k - P_h u_k|_1 + \big|P_h u_k - Q_k^h u_k\big|_1 + \big|\beta_k^h u_k^h - u_k^h\big|_1 \leqslant ch^n\lambda_k^{(n+1)/2}.$$

The proof of the theorem is complete. □

Theorem 3. *The estimate*

$$\big|\lambda_k^h - \lambda_k\big| \leqslant ch^{2n}\lambda_k^{n+1}$$

is satisfied for sufficiently small h.

Proof. Suppose that u_i, $i = 1, 2, \ldots,$ and u_i^h, $i = 1, 2, \ldots, N_h$, are eigenfunctions of problems (1) and (3) respectively, $b(u_k^h, u_k) > 0$, $b_h(u_k^h, u_k^h) = 1$, $b(u_k, u_k) = 1$. Then the following equality

$$\begin{aligned}\big(\lambda_k^h - \lambda_k\big)b_h\big(P_h u_k, u_k^h\big) &= a\big(P_h u_k - u_k, u_k^h - u_k\big) - \lambda_k b\big(P_h u_k - u_k, u_k^h - u_k\big) \\ &\quad + \big(a_h\big(P_h u_k, u_k^h\big) - a\big(P_h u_k, u_k^h\big)\big) - \lambda_k\big(b_h\big(P_h u_k, u_k^h\big) - b\big(P_h u_k, u_k^h\big)\big)\end{aligned}$$

is valid. Let us estimate the expressions on the right-hand side:

$$\begin{aligned}&\big|a\big(P_h u_k - u_k, u_k^h - u_k\big)\big| \leqslant c|P_h u_k - u_k|_1\big|u_k^h - u_k\big|_1, \\ &\big|b\big(P_h u_k - u_k, u_k^h - u_k\big)\big| \leqslant c|P_h u_k - u_k|_0\big|u_k^h - u_k\big|_0, \\ &\big|a_h\big(P_h u_k, u_k^h\big) - a\big(P_h u_k, u_k^h\big)\big| \leqslant ch^{2n}\|P_h u_k\|_{n,h}\big\|u_k^h\big\|_{n,h}, \\ &\big|b_h\big(P_h u_k, u_k^h\big) - b\big(P_h u_k, u_k^h\big)\big| \leqslant ch^{2n}\|P_h u_k\|_{n,h}\big\|u_k^h\big\|_{n,h}.\end{aligned}$$

By Theorem 2 and (2), (4), (5), (6), (7), we get

$$\begin{aligned}&|u_k - P_h u_k|_0 \leqslant ch^{n+1}\lambda_k^{(n+1)/2}, \qquad |u_k - P_h u_k|_1 \leqslant ch^n\lambda_k^{(n+1)/2}, \\ &\big|u_k^h - u_k\big|_0 \leqslant ch^n\lambda_k^{(n+1)/2}, \qquad \big|u_k^h - u_k\big|_1 \leqslant ch^n\lambda_k^{(n+1)/2}, \\ &\|P_h u_k\|_{n,h} \leqslant c\lambda_k^{n/2}, \qquad \big\|u_k^h\big\|_{n,h} \leqslant c\lambda_k^{n/2}.\end{aligned}$$

In addition, for sufficiently small h, we have

$$\begin{aligned}\big|b_h\big(P_h u_k, u_k^h\big)\big| &= \big|b_h\big(u_k^h, u_k^h\big) + b_h\big(P_h u_k - u_k^h, u_k^h\big)\big| \\ &= \big|1 + b_h\big(P_h u_k - u_k^h, u_k^h\big)\big| \geqslant 1 - c\big|P_h u_k - u_k^h\big|_1 \geqslant 1 - ch^n\lambda_k^{(n+1)/2}.\end{aligned}$$

Table 1
Eigenvalues.

k	λ_k
1	4.4411509
2	19.4103933
3	44.5374762
4	79.7633978
5	125.0723767

Hence there exists a positive constant c_2 such that

$$\left|b_h\left(P_h u_k, u_k^h\right)\right| \geqslant c_2$$

for sufficiently small h. As a result, we obtain the desired estimate

$$c_2\left|\lambda_k^h - \lambda_k\right| \leqslant \left|\left(\lambda_k^h - \lambda_k\right) b_h\left(P_h u_k, u_k^h\right)\right| \leqslant c h^{2n} \lambda_k^{n+1}.$$

The proof of the theorem is complete. □

4. Numerical experiments

Consider the differential eigenvalue problem (1) with coefficients

$$p(x) = \cos(x), \qquad q(x) = x e^{-x}, \qquad r(x) = 1 + \sin(\pi x), \quad x \in \overline{\Omega}.$$

We apply the scheme of the finite element method with quadratic elements and the Gauss quadrature formula with two nodes. This quadrature formula $\hat{S}$ over $\hat{e} = [0, 1]$ is defined by

$$\hat{\alpha}_1 = \frac{1}{2}, \qquad \hat{\alpha}_2 = \frac{1}{2}, \qquad \hat{\beta}_1 = \frac{1}{2}\left(1 - \frac{1}{\sqrt{3}}\right), \qquad \hat{\beta}_2 = \frac{1}{2}\left(1 + \frac{1}{\sqrt{3}}\right).$$

The algebraic precision of the Gauss quadrature formula $\hat{S}$ with two nodes is equal to 3. We have computed the value

$$c = \frac{|\lambda_k^h - \lambda_k|}{h^{2n} \lambda_k^{n+1}}$$

for $k = 1, 2, 3, 4, 5$, $n = 2$, $h = 1/m$, $m = 50, 100, 200$. Our computations show that $c \approx 0.01$. The experimental results are consistent with the theoretical results in Theorem 3. The exact eigenvalues λ_k, $k = 1, 2, 3, 4, 5$, of the differential eigenvalue problem are obtained as the limit values of the approximations λ_k^h, $k = 1, 2, 3, 4, 5$, and represented in Table 1. Note that the exact eigenvalues λ_k, $k = 1, 2, 3, 4, 5$, coincide with the approximate eigenvalues λ_k^h, $k = 1, 2, 3, 4, 5$, with accuracy shown in Table 1 for $n = 13$, $m = 2$, and for the Gauss quadrature formula with $L = 13$ nodes.

Acknowledgements

The author is grateful to Professor S.D. Algazin for discussing the results of the present paper and to an anonymous referee for very important suggestions to improve the first version of the paper.

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