ISSN 1995-0802, Lobachevskii Journal of Mathematics, 2021, Vol. 42, No. 1, pp. 30–43. © Pleiades Publishing, Ltd., 2021.

On the Existence of Solutions of Nonlinear Boundary Value Problems for Inhomogeneous Isotropic Shallow Shells of the Timoshenko Type with Free Edges

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(Submitted by A. M. Elizarov)

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Received February 25, 2020; revised June 8, 2020; accepted July 8, 2020

Abstract—The paper deals with the study of solvability to geometrically nonlinear boundary value problem for elastic inhomogeneous isotropic shallow shells with free edges within S. P. Timoshenko shear model. The problem is reduced to one nonlinear equation relative to deflection of shell in Sobolev space. Solvability of equation is proved with the use of contracting mappings principle.

DOI: 10.1134/S1995080221010054

Keywords and phrases: *nonlinear boundary value problem, inhomogeneous isotropic shell of Timoshenko type, equilibrium equations, generalized solution of a boundary value problem, holomorphic functions, integral of Cauchy type, integral equations, operator, existence theorem.*

INTRODUCTION

The solvability of nonlinear problems of equilibrium for thin elastic shells is currently sufficiently fully studied in the framework of the Kirchhoff–Love model (see [1–5] and by the quoted literature). The questions of existence of solutions of nonlinear problems of equilibrium within the more general models of the theory of the shells, not based on hypotheses of Kirchhoff–Love were included into the known list of unresolved problems of the mathematical theory of shells [1] and until recently they remained open. Today there is a number of works [6–15] in which the solvability of nonlinear problems are studied within the shear model of S. P. Timoshenko. The studies in [6–15] are based on integral representations for generalized displacements. The integral representations contain the arbitrary holomorphic functions. The holomorphic functions are defined so that the generalized displacements satisfy the given boundary conditions. For their construction two approaches are used. The first approach is based on application explicit representations of solutions of a problem of Riemann–Hilbert for the holomorphic functions in a unit disk. Therefore the flat domain which is homeomorphic to middle surface of shell, or is supposed by a unit disk [6, 7, 9, 10, 13], or conformally mapped onto a unit disk [8]. The second approach for determine the holomorphic functions uses the theory of integrals of Cauchy type with the real density. These densities are defined as the solutions of system of one-dimensional singular integral equations [11, 12, 14, 15]. In the present work, the second approach is used. Work is a direct development of the works [11, 12, 14] to the case of shallow shells with variable principal curvatures, which significantly complicates a obtaining of necessary and sufficient conditions for the solvability of the problem.

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1. STATEMENT OF THE PROBLEM

In a simply connected bounded flat domain Ω we consider a system of nonlinear differential equations

$$T^{i\lambda}_{\alpha^\lambda} + R^i = 0, \quad i = 1,2,$$

$$T^{\lambda 3}_{\alpha^\lambda} + k_\lambda T^{\lambda\lambda} + (T^{\lambda\mu} w_{3\alpha^\mu})_{\alpha^\lambda} + R^3 = 0,$$

$$M^{i\lambda}_{\alpha^\lambda} - T^{i3} + L^i = 0, \quad i = 1,2, \tag{1}$$

under the conditions

$$T^{j1}\frac{d\alpha^2}{ds} - T^{j2}\frac{d\alpha^1}{ds} = P^j(s), \quad j = 1,2,$$

$$T^{13}\frac{d\alpha^2}{ds} - T^{23}\frac{d\alpha^1}{ds} + T^{11}w_{3\alpha^1}\frac{d\alpha^2}{ds} - T^{22}w_{3\alpha^2}\frac{d\alpha^1}{ds} + T^{12}\left(w_{3\alpha^2}\frac{d\alpha^2}{ds} - w_{3\alpha^1}\frac{d\alpha^1}{ds}\right) = P^3(s),$$

$$M^{j1}\frac{d\alpha^2}{ds} - M^{j2}\frac{d\alpha^1}{ds} = N^j(s), \quad j = 1,2, \tag{2}$$

on its boundary Γ. In (1) and (2) the following notations is accepted:

$$T^{ij} \equiv T^{ij}_\gamma(a) = D^{ijkn}_0 \gamma^0_{kn}, \quad M^{ij} \equiv M^{ij}_\gamma(a) = D^{ijkn}_2 \gamma^1_{kn}, \quad a = (w_1, w_2, w_3, \psi_1, \psi_2),$$

$$D^{ijkn}_m = \int\limits_{-h_0/2}^{h_0/2} B^{ijkn}(\alpha^1,\alpha^2,\alpha^3)(\alpha^3)^m d\alpha^3, \quad B^{1111} = B^{2222} = \frac{E}{1-\nu^2}, \quad B^{1122} = \frac{\nu E}{1-\nu^2},$$

$$B^{1212} = \frac{E}{2(1+\nu)}, \quad B^{1313} = B^{2323} = \frac{Ek^2}{2(1+\nu)}, \quad \gamma^0_{jj} = w_{j\alpha^j} - k_j w_3 + \frac{w^2_{3\alpha^j}}{2} \quad (j = 1,2),$$

$$\gamma^0_{12} = w_{1\alpha^2} + w_{2\alpha^1} + w_{3\alpha^1}w_{3\alpha^2}, \quad \gamma^1_{jj} = \psi_{j\alpha^j} \quad (j=1,2), \quad \gamma^1_{12} = \psi_{1\alpha^2} + \psi_{2\alpha^1},$$

$$\gamma^0_{j3} = w_{3\alpha^j} + \psi_j \quad (j = 1,2), \quad \gamma^0_{33} = \gamma^1_{k3} \equiv 0, k = \overline{1,3}; \tag{3}$$

the remaining B^{ijkn} are zero; $\alpha^j = \alpha^j(s)$ $(j = 1,2)$ are equations of the curve Γ, variable s is the length of the arc of the curve Γ, subscript α^λ in (1)–(3) and further means differentiation with respect to α^λ, $\lambda = 1,2$.

System (1), together with boundary conditions (2), describes the equilibrium state of an elastic shallow isotropic inhomogeneous shell with free edges within Timoshenko shear model [16, pp. 168–170, 269]. Herewith T^{ij} are the stresses, M^{ij} are the moments; γ^k_{ij} $(i,j = \overline{1,3}$, $k = 0,1)$ are the strain components of middle surface S_0 of the shell, where S_0 is homeomorphic to Ω; w_j $(j = 1,2)$ are the tangential displacements of points of S_0, w_3 is the normal displacement of points of S_0; ψ_i $(i = 1,2)$ are the rotation angles of the normal cross-sections of surface S_0; a is the generalized displacement vector; R^j, P^j $(j = \overline{1,3})$, L^k, N^k $(k = 1,2)$ are the components of external forces acting on the shell; ν is the Poisson ratio; E is the Young modulus; $k_j = k_j(\alpha^j)$ $(j = 1,2)$ are the principal curvatures; k^2 is the shear coefficient; $h_0 = \text{const}$ is the shell thickness; α^1,α^2 are Cartesian coordinates of points of the domain Ω.

In (1)–(3) and further, on repeated Latin indices summation is carried out from 1 to 3, on repeated Greek indices summation is carried out from 1 to 2.

System (1) in the generalized displacements takes the form

$$(D^{1111}_0 w_{1\alpha^1} + D^{1122}_0 w_{2\alpha^2})_{\alpha^1} + [D^{1212}_0(w_{1\alpha^2} + w_{2\alpha^1})]_{\alpha^2} - (D^{11\mu\mu}_0 k_\mu w_3)_{\alpha^1} = f_1,$$

$$[D^{1212}_0(w_{1\alpha^2} + w_{2\alpha^1})]_{\alpha^1} + (D^{1122}_0 w_{1\alpha^1} + D^{2222}_0 w_{2\alpha^2})_{\alpha^2} - (D^{22\mu\mu}_0 k_\mu w_3)_{\alpha^2} = f_2,$$

$$D_0^{1313}w_{3\alpha^1\alpha^1}+D_0^{2323}w_{3\alpha^2\alpha^2}+D_{0\alpha^1}^{1313}w_{3\alpha^1}+D_{0\alpha^2}^{2323}w_{3\alpha^2}$$

$$+(D_0^{1313}\psi_1)_{\alpha^1}+(D_0^{2323}\psi_2)_{\alpha^2}+k_\lambda T_e^{\lambda\lambda}(a)=f_3,$$

$$(D_2^{1111}\psi_{1\alpha^1}+D_2^{1122}\psi_{2\alpha^2})_{\alpha^1}+[D_2^{1212}(\psi_{1\alpha^2}+\psi_{2\alpha^1})]_{\alpha^2}-D_0^{1313}(w_{3\alpha^1}+\psi_1)=f_4,$$

$$[D_2^{1212}(\psi_{1\alpha^2}+\psi_{2\alpha^1})]_{\alpha^1}+(D_2^{1122}\psi_{1\alpha^1}+D_2^{2222}\psi_{2\alpha^2})_{\alpha^2}-D_0^{2323}(w_{3\alpha^2}+\psi_2)=f_5, \tag{4}$$

where

$$f_j\equiv f_j(w_3)=-(D_0^{jj\mu\mu}\chi^0_{\mu\mu})_{\alpha^j}-(D_0^{1212}\chi^0_{12})_{\alpha^{3-j}}-R^j,\quad j=1,2,$$

$$f_3\equiv f_3(a)=-k_\lambda T_\chi^{\lambda\lambda}-(T^{\lambda\mu}w_{3\alpha^\lambda})_{\alpha^\mu}-R^3,\quad f_4=-L^1,\quad f_5=-L^2;$$

$$T_e^{ij}=D_0^{ij\delta\beta}e^0_{\delta\beta},\quad T_\chi^{ij}=D_0^{ij\delta\beta}\chi^0_{\delta\beta},\quad T^{ij}=T_e^{ij}+T_\chi^{ij}; \tag{5}$$

$e^0_{\delta\beta}$ and $\chi^0_{\delta\beta}$ denote linears and nonlinear summands of the strains components $\gamma^0_{\delta\beta}$: $\gamma^0_{\delta\beta}=e^0_{\delta\beta}+\chi^0_{\delta\beta}$, $\delta,\beta=1,2$.

Problem (4), (2). *Find a solution of system (4) satisfying the boundary conditions (2).*

We study the boundary value Problem (4), (2) in the generalized setting. Let the following conditions be satisfied:

(a) the elastic characteristics $B^{ijkn}(\alpha^1,\alpha^2,\alpha^3)$ are the even functions of a variable $\alpha^3\in[-h_0/2, h_0/2]$ and $B^{ijkn}\in W_p^{(1)}(\Omega)\times L_1[-h_0/2,h_0/2]$;

(b) $k_j\in W_p^{(1)}(\Omega)$, $j=1,2$;

(c) R^j $(j=\overline{1,3})$, L^k $(k=1,2)\in L_p(\Omega)$; P^j $(j=\overline{1,3})$, N^k $(k=1,2)\in C_\beta(\Gamma)$;

(d) Ω is an arbitrary simply connected domain with boundary $\Gamma\in C^1_\beta$;

(e) the external load is self-balanced.

Here and further: $2<p<4/(2-\beta)$, $0<\beta<1$.

Definition 1. *A vector $a=(w_1,w_2,w_3,\psi_1,\psi_2)$ is called a generalized solution of Problem (4),(2) if $a\in W_p^{(2)}(\Omega)$, satisfies the system (4) almost everywhere and satisfies the boundary conditions (2) pointwise.*

Here $W_p^{(i)}(\Omega)$ $(i=1,2)$ are the Sobolev spaces. By the embedding theorems for the Sobolev spaces $W_p^{(2)}(\Omega)$ with $p>2$, a generalized solution $a\in C^1_\alpha(\overline{\Omega})$. Here and further $\alpha=(p-2)/p$. Let us notice what if the condition $2<p<4/(2-\beta)$ is satisfied, then inequality $\alpha<\beta/2$ is holds.

2. CONSTRUCTION OF INTEGRAL REPRESENTATIONS FOR GENERALIZED DISPLACEMENTS

Let us introduce the two complex-valued functions $\omega_j=\omega_j(z)=D_{2(j-1)}^{1111}(f_{j1\alpha^1}+f_{j2\alpha^2})+iD_{2(j-1)}^{1212}(f_{j2\alpha^1}-f_{j1\alpha^2})$ $(j=1,2)$, $z=\alpha^1+i\alpha^2$, $f_{1j}=w_j$, $f_{2j}=\psi_j$, $j=1,2$. Relatively of functions $\omega_j(z)$ $(j=1,2)$ and normal displacement $w_3(z)$ we consider the equations

$$\omega_{j\overline{z}}=\rho^j,\quad j=1,2,\quad D_0^{1313}w_{3\alpha^1\alpha^1}+D_0^{2323}w_{3\alpha^2\alpha^2}=\rho^3, \tag{6}$$

where $\rho^1=\rho_1+i\rho_2$, $\rho^2=\rho_4+i\rho_5$, $\rho^3\equiv\rho_3$ are arbitrary fixed functions belonging to space $L_p(\Omega)$; $\omega_{j\overline{z}}=(\omega_{j\alpha^1}+i\omega_{j\alpha^2})/2$.

The first two equations in (6) are inhomogeneous Cauchy–Riemann equations. Therefore, their general solutions are given by formulas [17, p. 29]

$$\omega_j(z)=\Phi_j(z)+T\rho^j(z),\quad T\rho^j=-\frac{1}{\pi}\iint\limits_{\Omega}\frac{\rho^j(\zeta)}{\zeta-z}d\xi d\eta,\quad j=1,2,\quad \zeta=\xi+i\eta, \tag{7}$$

where $\Phi_j(z)$ are arbitrary holomorphic functions belonging to the space $C_\alpha(\overline{\Omega})$.

It is known [17, p. 29] that T is a completely continuous operator in spaces $L_p(\Omega)$ and $C_\alpha^k(\overline{\Omega})$ and he carry out mapping these spaces in $C_\alpha(\overline{\Omega})$ and $C_\alpha^{k+1}(\overline{\Omega})$, respectively. In addition, there exist generalized derivatives

$$\frac{\partial Tf}{\partial \bar{z}} = f, \quad \frac{\partial Tf}{\partial z} \equiv Sf = -\frac{1}{\pi}\iint\limits_{\Omega} \frac{f(\zeta)}{(\zeta - z)^2} d\xi d\eta, \tag{8}$$

where S is a linear bounded operator in $L_p(\Omega)$, $p > 1$ and in $C_\alpha^k(\overline{\Omega})$.

In turn, using the functions $\omega_1^0 = w_2 + iw_1$, $\omega_2^0 = \psi_2 + i\psi_1$, the relations (7) can be written in the form of the inhomogeneous Cauchy–Riemann equations

$$\omega_{j\bar{z}}^0 = i(d_j^1 \omega_j + d_j^2 \overline{\omega}_j) \equiv i d_j[\omega_j], \quad d_j^k = \frac{1}{4}\left(\frac{1}{D_{2(j-1)}^{1111}} + \frac{(-1)^k}{D_{2(j-1)}^{1212}}\right), \quad j, k = 1, 2.$$

The general solution of Cauchy–Riemann equations has the form

$$\omega_j^0(z) = \Psi_j(z) + iTd_j[\omega_j](z), \quad j = 1, 2, \tag{9}$$

where $\Psi_j(z)$ are arbitrary holomorphic functions of the space $C_\alpha^1(\overline{\Omega})$.

The third equation in (6), taking into account $D_0^{1313} = D_0^{2323}$, we presented in the form

$$w_{3z\bar{z}} = \widetilde{\rho}_3/2, \quad \widetilde{\rho}_3 = \rho_3/(2D_0^{1313}), \quad w_{3z} = (w_{3\alpha^1} - iw_{3\alpha^2})/2.$$

Wherefrom we obtain the representation $w_{3z} = \Phi_0(z) + T\widetilde{\rho}_3(z)/2$, where $\Phi_0(z)$ is an arbitrary holomorphic function of the space $C_\alpha(\overline{\Omega})$.

By integrating the last relation with respect to z, we will have

$$w_3(z) = \operatorname{Re}\Phi_3(z) - \widetilde{T}\widetilde{\rho}_3, \quad \widetilde{T}\widetilde{\rho}_3 = -\frac{1}{\pi}\iint\limits_{\Omega} \widetilde{\rho}_3(\zeta) \ln\left|1 - \frac{z}{\zeta}\right| d\xi d\eta, \tag{10}$$

where $\Phi_3(z) \in C_\alpha^1(\overline{\Omega})$ is an arbitrary holomorphic function.

Relations (9) and (10) are the desired integral representations for generalized displacements. For their first-order partial derivatives, from (9) and (10), taking into account (6), (7) and (8), we will obtain

$$f_{kj\alpha^j} = \operatorname{Re}\omega_k/\left(2D_{2(k-1)}^{1111}\right) - (-1)^j \operatorname{Im}\left\{\Psi_k'(z) + iSd_k[\omega_k](z)\right\}, \quad \omega_k = \Phi_k(z) + T\rho^k(z),$$

$$f_{kj\alpha^n} = \operatorname{Re}\left\{\Psi_k'(z) + iSd_k[\omega_k](z)\right\} + (-1)^j \operatorname{Im}\omega_k/\left(2D_{2(k-1)}^{1212}\right), \quad k, j, n = 1, 2, \quad j \neq n,$$

$$f_{1j} = w_j, \quad f_{2j} = \psi_j; \quad w_{3\alpha^j} = \operatorname{Re}\left\{i^{j-1}[\Phi_3'(z) + T\widetilde{\rho}_3(z)]\right\}, \quad j = 1, 2. \tag{11}$$

The generalized displacements w_j $(j = \overline{1,3}), \psi_1, \psi_2$ and their derivatives defined by formulas (9), (10) and (11) we will represent in form convenient for further studies:

$$w_j = w_j^1 + w_j^2, \quad \psi_k = \psi_k^1 + \psi_k^2, \quad w_{j\alpha^n} = w_{j\alpha^n}^1 + w_{j\alpha^n}^2, \quad \psi_{k\alpha^n} = \psi_{k\alpha^n}^1 + \psi_{k\alpha^n}^2,$$
$$j = \overline{1,3}, \quad k, n = 1, 2,$$

where the summands with a superscript "1" contain only the functions $\rho = (\rho^1, \rho^2, \rho^3)$, and the summands with a superscript "2" contain only holomorphic functions $\Phi = (\Phi_1, \Phi_2, \Phi_3, \Psi_1, \Psi_2)$.

3. SOLUTION OF PROBLEM (4), (2)

Integral representations (9) and (10) for generalized displacements $a=(w_1,w_2,w_3,\psi_1,\psi_2)$ contain arbitrary holomorphic functions $\Phi_j(z)$ $(j=\overline{1,3})$, $\Psi_k(z)$ $(k=1,2)$ and arbitrary functions $\rho^j(z)$ $(j=\overline{1,3})$. We find them so that the generalized displacements satisfy the system (4) of equilibrium equations and the boundary conditions (2). For this purpose, we substitute relations (9), (10), and (11) into (4) and (2). As a result, the system of equations (4) takes the form

$$\rho^j(z)+h_{j1}(\rho)(z)+h_{j2}(\Phi)(z)=f^j(z),\quad j=\overline{1,3}, \tag{12}$$

where

$$h_{1j}=D^{1212}_{0\alpha^2}w^j_{2\alpha^1}-D^{1212}_{0\alpha^1}w^j_{2\alpha^2}+i\left(D^{1212}_{0\alpha^1}w^j_{1\alpha^2}-D^{1212}_{0\alpha^2}w^j_{1\alpha^1}\right)-\frac{(D^{11\mu\mu}_0k_\mu w^j_3)_{\alpha^1}+i(D^{22\mu\mu}_0k_\mu w^j_3)_{\alpha^2}}{2},$$

$$h_{2j}=D^{1212}_{2\alpha^1}\psi^j_{2\alpha^1}-D^{1212}_{2\alpha^1}\psi^j_{2\alpha^2}+i\left(D^{1212}_{2\alpha^1}\psi^j_{1\alpha^2}-D^{1212}_{2\alpha^2}\psi^j_{1\alpha^1}\right)-\frac{D^{1313}_0\left[\psi^j_1+w^j_{3\alpha^1}+i(\psi^j_2+w^j_{3\alpha^2}\right]}{2},$$

$$h_{3j}=D^{1313}_{0\alpha^1}w^j_{3\alpha^1}+D^{2323}_{0\alpha^2}w^j_{3\alpha^2}+\left(D^{\mu3\mu3}_0\psi^j_\mu\right)_{\alpha^\mu}+k_\lambda T^{\lambda\lambda}_e(a^j),\quad a^j=(w^j_1,w^j_2,w^j_3,\psi^j_1,\psi^j_2),\quad j=1,2;$$

$$f^1=(f_1+if_2)/2,\quad f^2=(f_4+if_5)/2,\quad f^3\equiv f_3;\quad h_{j1}\equiv h_{j1}(\rho),\quad h_{j2}\equiv h_{j2}(\Phi),\quad j=\overline{1,3}.$$

The boundary conditions (2) are transformed to the form

$$\operatorname{Re}\left\{i^j[\overline{t}'+(-1)^jt']a_{2k-1}(t)\omega_k(t)-i^{j-1}a_{2k}(t)t'[\Psi'_k(t)+iSd_k[\omega_k]^+(t)]\right\}$$
$$+(2-k)l_j(w_3)(t)=\varphi_{j+3(k-1)}(t),\quad j,k=1,2,$$

$$D^{1313}_0\operatorname{Im}\left\{t'[\Phi'_3(t)+T\widetilde{\rho}_3(t)]\right\}+l_3(\psi)(t)=\varphi_3(a)(t), \tag{13}$$

where

$$l_j(w_3)(t)=(-1)^jD^{jj\mu\mu}_0k_\mu w_3\frac{d\alpha^{3-j}}{ds},\quad j=1,2,$$

$$l_3(\psi)(t)=D^{1313}_0\psi_1\frac{d\alpha^2}{ds}-D^{2323}_0\psi_2\frac{d\alpha^1}{ds},\quad \psi=(\psi_1,\psi_2);$$

$$\varphi_j(t)=P^j(s)+(-1)^j\left[D^{jj\mu\mu}_0\chi^0_{\mu\mu}\frac{d\alpha^{3-j}}{ds}-D^{1212}_0\chi^0_{12}\frac{d\alpha^j}{ds}\right],\quad j=1,2$$

$$\varphi_3(a)(t)=P^3(s)-\left\{T^{11}w_{3\alpha^1}\frac{d\alpha^2}{ds}-T^{22}w_{3\alpha^2}\frac{d\alpha^1}{ds}+T^{12}\left(w_{3\alpha^2}\frac{d\alpha^2}{ds}-w_{3\alpha^1}\frac{d\alpha^1}{ds}\right)\right\},$$

$$\varphi_4(t)=N^1(s),\quad \varphi_5(t)=N^2(s);\quad a_{2j-1}=\frac{1}{4}\left(1+\frac{D^{1122}_{2(j-1)}}{D^{1111}_{2(j-1)}}\right),\quad a_{2j}=2D^{1212}_{2(j-1)},$$

$$\omega_j(t)=\Phi_j(t)+T\rho^j(t),\quad j=1,2;\quad \Phi_k(t)\equiv\Phi^+_k(t),\quad \Psi'_k(t)\equiv\Psi'^+_k(t),\quad k=1,2,\quad \Phi'_3(t)\equiv\Phi'^+_3(t).$$

Here and below, the symbol $\Psi^+(t)$ denotes the limit of a function $\Psi(z)$ as $z\to t\in\Gamma$ from the interior of the domain Ω.

Thus, to determine the functions $\rho^j\in L_p(\Omega)$, $j=\overline{1,3}$, $\Phi_k(z)\in C_\alpha(\overline{\Omega})$, $k=1,2$, $\Phi_3(z)$, $\Psi_k(z)\in C^1_\alpha(\overline{\Omega})$, $k=1,2$, we have the system of equations (12) and (13). We will find the holomorphic functions in the form of Cauchy type integrals with real densities:

$$\Phi_j(z)=\theta(\mu_{2j-1})(z),\quad \Psi'_j(z)=\theta(\mu_{2j})(z),\quad j=1,2,$$

$$\Phi_3'(z) = i\theta(\mu_5)(z), \quad \theta(f)(z) = \frac{1}{2\pi i}\int\limits_\Gamma \frac{f(\tau)d\tau}{\tau'(\tau - z)}, \tag{14}$$

where $\mu_j(t) \in C_\alpha(\Gamma)$ $(j = \overline{1,5})$ are arbitrary real functions, $\tau' = d\tau/d\sigma$, $d\sigma$ is an element of the arc length of the curve Γ.

For functions $(\Psi_j(z)$ $(j = 1, 2)$, $\Phi_3(z))$, we have representations:

$$\Psi_j(z) = -\frac{1}{2\pi i}\int\limits_\Gamma \frac{\mu_{2j}(\tau)}{\tau'}\ln\left(1 - \frac{z}{\tau}\right)d\tau + c_{2j-1} + ic_{2j} \equiv \Psi_j(\mu_{2j})(z) + c_{2j-1} + ic_{2j}, \quad j = 1, 2,$$

$$\Phi_3(z) = -\frac{1}{2\pi}\int\limits_\Gamma \frac{\mu_5(\tau)}{\tau'}\ln\left(1 - \frac{z}{\tau}\right)d\tau + c_5 + ic_6 \equiv \Phi_3(\mu_5)(z) + c_5 + ic_6, \tag{15}$$

where c_j $(j = \overline{1,6})$ are arbitrary real constants, under $\ln(1 - z/\tau)$ we mean a one-valued branch that vanishes when condition $z = 0$.

Using formulas of Sokhotsky [18, p. 66], we find $\Phi_j(t)$, $\Psi_j'(t)$, $j = 1, 2$, $\Phi_3'(t)$, $t \in \Gamma$. We substitute their expressions, as well as (15) in (12) and (13), taking into account representation

$$Sd_j[\omega_j]^+(t) = -\frac{1}{2}(\bar{t}')^2 d_j[\Phi_j(t)] - \frac{d_j^1(t)}{2\pi i}\int\limits_\Gamma \frac{\overline{\tau} - \overline{t}}{(\tau - t)^2}\Phi_j(\tau)d\tau - \frac{d_j^2(t)}{2\pi i}\int\limits_\Gamma \frac{\overline{\Phi_j(\tau)}}{\tau - t}d\overline{\tau}$$

$$-\frac{1}{\pi}\iint\limits_\Omega \frac{d_j^1(\zeta) - d_j^1(t)}{(\zeta - t)^2}\Phi_j(\zeta)d\xi d\eta - \iint\limits_\Omega \frac{d_j^2(\zeta) - d_j^2(t)}{(\zeta - t)^2}\overline{\Phi_j(\zeta)}d\xi d\eta + Sd_j[T\rho^j]^+(t), \quad j = 1, 2.$$

Representation for $Sd_j[\omega_j]^+(t)$ is obtained using relations (7) and (8), formulas (4.7) and (4.9) from [17, p. 28] and formulas of Sokhotsky. As a result, after simple transformations, we arrive at a system of equations with respect to functions $\rho = (\rho^1, \rho^2, \rho^3) \in L_p(\Omega)$ and $\mu_0 = (\mu_1, \mu_2, \mu_3, \mu_4, \mu_5) \in C_\alpha(\Gamma)$:

$$\rho^j(z) + h_{j1}(\rho)(z) + h_{j2}(\mu_0)(z) = f^j(z) - h_{j3}(c)(z), \quad z \in \Omega, \quad j = \overline{1,3},$$

$$\sum_{k=1}^{5}\left\{a_{jk}(t)\mu_k(t) + b_{jk}(t)\int\limits_\Gamma \frac{\mu_k(\tau)}{\tau - t}d\tau\right\} + K_j\mu_0(t) + H_j\rho(t) = g_j(a)(t), \quad t \in \Gamma, \quad j = \overline{1,5}. \tag{16}$$

In system (16) accepted of notations:

$$h_{13}(c)(z) = -c_5\left[(D_0^{11\mu\mu}k_\mu)_{\alpha^1} + i(D_0^{22\mu\mu}k_\mu)_{\alpha^2}\right]/2, \quad h_{23}(c)(z) = -(c_4 + ic_3)D_0^{1313}/2,$$

$$h_{33}(c)(z) = c_4 D_{0\alpha^1}^{1313} + c_3 D_{0\alpha^2}^{2323} + k_\mu T_e^{\mu\mu}(c), \quad h_{j2}(\mu_0)(z) \equiv h_{j2}(\Phi(\mu_0))(z), \quad j = 1, 2;$$

$$K_j\mu_0(t) = -\frac{a_j(t)}{2\pi}\operatorname{Re}\left(t'\int\limits_\Gamma \frac{\mu_j(\tau)}{\tau - t}d\sigma\right) + \frac{a_j(t)}{2\pi}\int\limits_\Gamma \frac{\mu_j(\tau)}{\tau - t}d\tau - \widetilde{a}_j(t)\operatorname{Re}\left(\frac{t'}{2\pi i}\int\limits_\Gamma \frac{\widetilde{\mu}_j(\tau)}{\tau - t}d\sigma\right)$$

$$-\operatorname{Re}\left\{i^{n_j}a_{2m_j}(t)t'K_{0m_j}\mu_{2m_j-1}(t)\right\} + \widetilde{l}_{j2}(\mu_5)(t), \quad j = \overline{1,4},$$

$$K_{0j}\mu_{2j-1}(t) = \frac{d_j^2(t)}{2\pi i}\int\limits_\Gamma \frac{\psi_0(\tau, t) - \psi_0(\tau, \tau)}{\tau - t}\mu_{2j-1}(\tau)\overline{\tau}'d\tau - \frac{d_j^1(t)}{2\pi i}\int\limits_\Gamma \frac{\psi(\tau, t) - \psi(t, t)}{\tau - t}\Phi_j(\mu_{2j-1})(\tau)d\tau$$

$$-\frac{1}{\pi}\iint\limits_\Omega \frac{d_j^1(\zeta) - d_j^1(t)}{(\zeta - t)^2}\Phi_j(\mu_{2j-1})(\zeta)d\xi d\eta - \frac{1}{\pi}\iint\limits_\Omega \frac{d_j^2(\zeta) - d_j^2(t)}{(\zeta - t)^2}\overline{\Phi_j(\mu_{2j-1})(\zeta)}d\xi d\eta, \quad j = 1, 2,$$

$$\psi_0(\tau,t)=\frac{1}{2\pi i}\int\limits_{\Gamma}\frac{\overline{\psi(\tau,\tau_1)}}{\tau_1-t}d\overline{\tau}_1,\quad \psi(\tau,t)=\frac{\overline{\tau}-\overline{t}}{\tau-t},\quad \psi(t,t)=\frac{d\overline{t}}{dt}=(\overline{t}')^2;$$

$$\Phi_j(\mu_{2j-1})(z)\equiv\theta(\mu_{2j-1})(z);$$

$$K_5\mu_0(t)=\frac{D_0^{1313}}{2\pi}\operatorname{Im}\left(t'\int\limits_{\Gamma}\frac{\mu_5(\tau)}{\tau-t}d\sigma\right)+l_3\left(\psi^2(\Phi(\mu_0))\right)(t),$$

$$H_j\rho(t)=\operatorname{Re}\left\{i^{n_j}[\overline{t}'+(-1)^j t']a_{2m_j-1}(t)T\rho^j(t)-i^{n_j}a_{2m_j}(t)t'Sd_j[T\rho^j]^+(t)\right\}+\widetilde{l}_{j1}(\rho_3)(t),\quad j=\overline{1,4},$$

$$H_5\rho(t)=D_0^{1313}\operatorname{Im}(t'T\widetilde{\rho}_3)+l_3(\psi^1(\rho))(t);\quad g_j(a)(t)=\varphi_j(w_3)(t)-l_{j3}(c)(t),\quad j=1,2,$$

$$g_j(a)(t)=\varphi_{1+j}(t),\quad j=3,4,\quad g_5(a)(t)=\varphi_3(a)(t)-l_3(c)(t);\tag{17}$$

where

$$l_{j3}(c)(t)=(-1)^j c_5 k_\lambda D_0^{jj\lambda\lambda}\frac{d\alpha^{3-j}}{ds},\quad j=1,2,\quad l_3(c)(t)=c_4D_0^{1313}\frac{d\alpha^2}{ds}-c_3D_0^{2323}\frac{d\alpha^1}{ds};$$

$$\widetilde{l}_{j2}(\mu_5)(t)\equiv l_j\left(w_3^2(\Phi_3(\mu_5))\right)(t)\quad\text{at}\quad j=1,2\quad\text{and}\quad \widetilde{l}_{j2}(\mu_5)(t)\equiv 0\quad\text{at}\quad j=3,4;$$

$$\widetilde{l}_{j1}(\rho_3)(t)\equiv l_j\left(w_3^1(\rho_3)\right)(t)\quad\text{at}\quad j=1,2\quad\text{and}\quad \widetilde{l}_{j1}(\rho_3)(t)\equiv 0\quad\text{at}\quad j=3,4;$$

$$\psi^k=(\psi_1^k,\psi_2^k),\quad k=1,2;$$

$$\widetilde{a}_{2j-1}=a_{2j},\widetilde{a}_{2j}=1,\widetilde{\mu}_{2j-1}=\mu_{2j},\widetilde{\mu}_{2j}=\mu_{2j-1},n_{2j-1}=1,n_{2j}=0,j=1,2;m_1=m_2=1,m_3=2,$$

$$m_4=2,\quad a_{12}=-\frac{a_2}{2},\quad a_{21}=-\frac{a_1}{2},\quad a_{34}=-\frac{a_4}{2},\quad a_{43}=-\frac{a_3}{2},\quad a_{55}=\frac{D_0^{1313}}{2},$$

$$b_{jj}=-\frac{a_j}{2\pi},\quad j=\overline{1,4};$$

the remaining $a_{ij}(t)$, $b_{ij}(t)$ are equal to zero; $a_j(t)$ $(j=\overline{1,4})$ and $\varphi_j(t)$ $(j=\overline{1,5})$ are defined in (13).

Lemma 1. *Let conditions (a), (b), (c), (d) be satisfied. Then:*

1) $h_{j1}(\rho)$ $(j=\overline{1,3})$ are linear completely continuous operators in $L_p(\Omega)$;

2) $h_{j2}(\mu_0)(t)$ $(j=\overline{1,3})$ are linear completely continuous operators from $C_\nu(\Gamma)$ in $L_p(\Omega)$, where ν is any number from the interval (0,1);

3) $K_j\mu_0(t)$ $(j=\overline{1,5})$ are linear completely continuous operators from $C_\nu(\Gamma)$ in $C_{\alpha'}(\Gamma)$ $\forall\alpha'<\alpha$ and are bounded operators from $C_\nu(\Gamma)$ in $C_\alpha(\Gamma)$;

4) $H_j\rho(t)$ $(j=\overline{1,5})$ are linear completely continuous operators from $L_p(\Omega)$ in $C_{\alpha'}(\Gamma)$ $\forall\alpha'<\alpha$ and are bounded operators from $L_p(\Omega)$ in $C_\alpha(\Gamma)$;

5) the have place inclusions $f^j(z),h_{j3}(c)(z)\in L_p(\Omega),j=\overline{1,3}$; $g_j(a)(t)\in C_\alpha(\Gamma)$, $j=\overline{1,5}$; $a_{jk},b_{jk}\in C_\alpha(\Gamma)$.

Proof. It is known [17, pp. 26–27] that the Cauchy type integral $\theta(f)(z)$ in (14) is a bounded operator from $C_\alpha(\Gamma)$ in $C_\alpha(\overline{\Omega})$ and its derivative $\theta'(f)(z)$ is a bounded operator from $C_\alpha(\Gamma)$ in $L_q(\Omega)$, $1<q<2/(1-\alpha)$. In addition, can easily be shown that $\theta(f)$ is a completely continuous operator from $C_\alpha(\Gamma)$ in $L_p(\Omega)$ $\forall p>1$ and in $C_{\alpha'}(\overline{\Omega})$ $\forall\alpha'<\alpha$. Since $\psi(\tau,t)\in C_\beta(\Gamma)\times C_\beta(\Gamma)$ [18, pp. 28–32] then takes place $\psi_0(\tau,t)\in C_{\beta-\varepsilon}(\Gamma)\times C_\beta(\Gamma)$ [18, pp. 61–62] ($\varepsilon>0$ is arbitrarily small number). It follows from condition (a) that $d_j^k(t)\in C_\alpha(\Gamma)$ holds. Then it is easily shown that $K_{0j}(\mu_{2j-1})(t)$ $(j=1,2)$ are the linear completely continuous operators from $C_\nu(\Gamma)$ in $C_{\alpha'}(\Gamma)$ $\forall\alpha'<\alpha$ and bounded operators from $C_\nu(\Gamma)$ in $C_\alpha(\Gamma)$ (ν is any number from the interval (0,1)). Now we will obtain the validity of

Lemma 1 immediately from (17), if we also take into account the estimate [18, pp. 31–32, 55–56] $|\mathrm{Im}(t'/(\tau - t))| \leq c|\tau - t|^{\alpha-1}$, the properties of the operators T, S in (7), (8) and the representation

$$Sd_j[T\rho^j]^+(t) = T\left(\frac{\partial}{\partial\zeta}d_j[T\rho^j]\right)(t) - \frac{1}{2}(\bar{t}')^2 d_j[T\rho^j](t) - \frac{1}{2\pi i}\int\limits_\Gamma \frac{d_j[T\rho^j](\tau)}{\tau - t}d\overline{\tau}.$$

Representation for $Sd_j[T\rho^j]^+(t)$ is obtained using formulas (8.20) from [17, p. 58] and formulas of Sokhotsky.

We investigate the solvability the system of equations (16) in the space $L_p(\Omega) \times C_{\alpha'}(\Gamma)$, $\alpha' < \alpha$. Note that, by virtue of Lemma 1, the solution $(\rho, \mu_0) \in L_p(\Omega) \times C_{\alpha'}(\Gamma)$ of system (16) belongs to the space $L_p(\Omega) \times C_\alpha(\Gamma)$. Using the expressions for $a_{jk}(t)$, $b_{jk}(t)$ in (17), we calculate $det(a(t) + \pi i b(t)) = D_0^{1313}a_1a_2a_3a_4/8 \neq 0$, $t \in \Gamma$, where $a = (a_{jk})_{5\times5}$, $b = (b_{jk})_{5\times5}$ are fifth-order matrices. Then for the index of system (16) we have

$$æ = \frac{1}{2\pi}\left[arg\frac{det(a - \pi i b)}{det(a + \pi i b)}\right]_\Gamma = 0$$

(here, the symbol $[arg\varphi]_\Gamma$ means the increment of function argument φ when traversing the curve Γ once in the positive direction). Consequently, the Fredholm alternative is applicable to system (16). Let $(\rho, \mu_0) \in L_p(\Omega) \times C_{\alpha'}(\Gamma)$ is a solution of system (16) with $f^j - h_{j3}(c) \equiv 0$, $j = \overline{1,3}$, $g_j(a) \equiv 0$, $j = \overline{1,5}$. To solution (ρ, μ_0) by formulas (14) and (15), where the constants c_j $(j = \overline{1,6})$ are zero, correspond to the holomorphic functions $\Phi_j(z)$ $(j = \overline{1,3})$, $\Psi_k(z)$ $(k = 1,2)$. Functions $\Phi_j(z)$ $(j = \overline{1,3})$, $\Psi_k(z)$ $(k = 1,2)$ together with $\rho(z)$, in turn, the determine generalized displacements w_j $(j = \overline{1,3})$, ψ_k $(k = 1,2)$ using formulas (9) and (10). It is easy to see that these displacements satisfy the homogeneous system of linear equations ($f_j \equiv 0$, $j = \overline{1,5}$) in (4) and the homogeneous linear boundary conditions

$$T_e^{j1}\frac{d\alpha^2}{ds} - T_e^{j2}\frac{d\alpha^1}{ds} = 0, \quad j = 1,2, \quad T^{13}\frac{d\alpha^2}{ds} - T^{23}\frac{d\alpha^1}{ds} = 0, \quad M_e^{j1}\frac{d\alpha^2}{ds} - M_e^{j2}\frac{d\alpha^1}{ds}, \quad j = 1,2.$$

We multiply these equations in the system (4) onto $w_1, w_2, w_3, \psi_1, \psi_2$, respectively, integrate the resulting relations over the domain Ω and add up the results of integrations. Taking into account homogeneous boundary conditions, we obtain system $e^0_{\lambda\mu} = 0$, $\gamma^1_{\lambda\mu} = 0$, $\gamma^0_{\lambda3} = 0$, $\lambda, \mu = 1,2$, the solution of which, taking into account $\Psi_2(0) = 0$, $w_3(0) = 0$, we obtain in the form $w_1 = -c_0\alpha^2 + c_1$, $w_2 = c_0\alpha^1 + c_2$, $w_3 = \psi_k = 0$, $k = 1,2$, where c_0, c_1, c_2 are arbitrary real constants. Then $\omega_1(z) = 2ic_0D_0^{1212}$, $\omega_2(z) \equiv 0$ and from (6) and (10) it follows that

$$\rho^1(z) = 2ic_0D_{0\bar{z}}^{1212}, \quad \rho^2(z) \equiv 0, \quad \rho^3(z) \equiv 0. \tag{18}$$

Using formulas (7), (10) and relations $\omega^0_{jz} = \Psi'_j(z) + iSd_j[\omega_j](z)$, $j = 1,2$, obtained from (9) by differentiation with respect to z, we find

$$\Phi_1(z) = c_0\alpha_0(z), \quad \Psi'_1(z) = c_0\beta'_0(z), \quad \Phi_2(z) = \Psi'_2(z) = \Phi'_3(z) \equiv 0,$$

$$\alpha_0(z) = \frac{1}{\pi}\int\limits_\Gamma \frac{D_0^{1212}(t)}{t - z}dt, \quad \beta'_0(z) = \frac{1}{2\pi i}\int\limits_\Gamma \frac{d\bar{t}}{t - z}.$$

Substituting the found expressions in (14), we obtain

$$\mu_1(t)/t' - 2ic_0D_0^{1212}(t) = F_1^-(t), \quad \mu_2(t)/t' - c_0(\bar{t}')^2 = F_2^-(t), \quad \mu_j(t)/t' = F_j^-(t), \quad j = \overline{3,5},$$

where $F_j^-(t)$ are the boundary values of the function $F_j^-(z)$, which is holomorphic function in the exterior Ω and disappears at infinity. Therefore, for the function $F_j^-(z)$ in the exterior Ω, we arrive at the Riemann–Hilbert problem with the boundary condition $\mathrm{Re}\left[it'F_j^-(t)\right] = f_j^-(t)$, $j = \overline{1,5}$, where $f_1^-(t) = 2c_0D_0^{1212}(t)\,\mathrm{Re}(t')$, $f_2^-(t) = c_0\,\mathrm{Re}(it')$, $f_j^-(t) = 0$, $j = \overline{3,5}$. The solution to this problem has the form [19, p. 253] $F_j^-(z) = c_0 f_j^0(z) + \beta_{0j} f_j^1(t)$, $j = 1, 2$, $F_j^-(z) = \beta_{0j} f_j^1(t)$, $j = \overline{3,5}$; here $f_j^k(z)$ are the known holomorphic functions in the exterior Ω, c_0, β_{0j} are arbitrary real constants. Then for the functions $\mu_j(t)$, we obtain equalities

$$\mu_j(t) = c_0 \mu_j^0(t) + \beta_{0j} \mu_j^1(t), \quad j = 1, 2, \quad \mu_j(t) = \beta_{0j} \mu_j^1(t), \quad j = \overline{3,5}, \tag{19}$$

in which $\mu_j^k(t)$ are the known real functions.

Solutions (18) and (19) show that the homogeneous system of equations (16) has six linearly independent solutions. Then the system of equations adjoint with system (16) will also have six linearly independent solutions. To derive the adjoint system, we multiply the real and imaginary parts of the left parts of the equations in (12) onto the real functions $v_1, v_2, v_3, v_4, v_5 \in L_p(\Omega)$, respectively, and integrate over the domain Ω; we will multiply the left parts of the equations in (13) onto the real functions $\nu_1, \nu_2, \nu_3, \nu_4, \nu_5 \in C_\alpha(\Gamma)$ and integrate the result over curve Γ. After this, we sum them and we equate to zero. Replacing the holomorphic functions $\Phi_j(z)$ $(j = \overline{1,3})$, $\Psi_k(z)$, $\Psi_k'(z)$ $(k = 1, 2)$, $\Phi_3(z)$, $\Phi_3'(z)$ by their expressions from (14) and (15) with constants equal to zero, interchanging the order of integration in the obtained repeated integrals, after simple but cumbersome transformations, we arrive to the desired adjoint system of equations

$$\overline{v^j(z)} + (-1)^j T d_j[S_j v](z) + 2\theta(\tau' \overline{\nu^j})(z) = 0, \quad j = 1, 2,$$

$$2D_0^{1313} v_3(z) + \operatorname{Re}\left[T p_2(v)(z) + 2\theta(\tau' D_0^{1313} \nu_3)(z)\right] + \widetilde{T}^* p_1(v)(z) + \widetilde{T}_\Gamma p_3(\nu)(z) = 0, \quad z \in \Omega,$$

$$\operatorname{Re}\left\{iT d_j[S_j v](t) + 2(-1)^j i\theta^-(\tau' \overline{\nu^j})(t)\right\} = 0, \quad j = 1, 2,$$

$$\operatorname{Re}\left\{T\left[a_{2\overline{\zeta}} v^1 + 2(d_0^{11} - d_0^{22}) v_3\right](t) - 2\theta^-(\tau' a_2 \nu^1)(t)\right\} = 0,$$

$$\operatorname{Re}\left\{T^0\left(2D_{0\overline{\zeta}}^{1313} v_3 - \frac{1}{2} D_0^{1313} v^2\right)(t) + T(a_{4\overline{\zeta}} v^2)(t) + 2T_\Gamma^0(\tau' D_0^{1313} \nu_3)(t) - 2\theta^-(\tau' a_4 \nu^2)(t)\right\} = 0,$$

$$\operatorname{Re}\left\{T^0 p_1(v)(t) + T p_2(v)(t) + 2iT_\Gamma^0 p_3(\nu)(t) + 2\theta^-(\tau' D_0^{1313} \nu_3)(t)\right\} = 0, \quad t \in \Gamma, \tag{20}$$

where we use the notations

$$S_1 v(z) = S(a_{2\overline{\zeta}} v^1 + 2(d_0^{11} - d_0^{22}) v^3)(z) - 2\theta'(\tau' a_2 \nu^1)(z) - a_{2z} v^1(z) + 2(d_0^{11} + d_0^{22}) v_3(z),$$

$$S_2 v(z) = T\left(2D_{0\overline{\zeta}}^{1313} v_3 - \frac{1}{2} D_0^{1313} v^2\right)(z) - S(a_{4\overline{\zeta}} v^2)(z) - 2D_0^{1313} v_3(z) + a_{4z} v^2(z)$$

$$+ 2\theta'(\tau' a_4 \nu^2)(z) - 2\theta(D_0^{1313} \tau' \nu_3)(z),$$

$$T^0 f(z) = -\frac{1}{\pi} \iint_\Omega f(\zeta) \ln\left(1 - \frac{\zeta}{z}\right) d\xi d\eta, \quad \widetilde{T}^* f(z) = -\frac{1}{\pi} \iint_\Omega f(\zeta) \ln\left|1 - \frac{\zeta}{z}\right| d\xi d\eta,$$

$$T_\Gamma^0 f(z) = -\frac{1}{2\pi i} \int_\Gamma f(\tau) \ln\left(1 - \frac{\tau}{z}\right) d\sigma, \quad \widetilde{T}_\Gamma f(z) = -\frac{1}{\pi} \int_\Gamma f(\tau) \ln\left|1 - \frac{\tau}{z}\right| d\sigma,$$

$$\theta'(f)(z) = \frac{1}{2\pi i} \int_\Gamma \frac{f(\tau) d\tau}{\tau'(\tau - z)^2}; p_1(v)(z) = d_{0\alpha^1}^{11} v_1 + d_{0\alpha^2}^{22} v_2 + k_\lambda k_\mu D_0^{\lambda\lambda\mu\mu} v_3,$$

$$p_2(v)(z) = d_0^{11} v_1 + i d_0^{22} v_2 - 2D_{0\overline{z}}^{1313} v_3 + \frac{1}{2} D_0^{1313} v^2, \quad p_3(\nu)(\tau) = i^j d_0^{jj} (\overline{\tau}' + (-1)^j \tau') \nu_j(\tau),$$

$$d_0^{jj} = \frac{1}{2} D_0^{jj\mu\mu} k_\mu, \quad j = 1, 2; \quad v^1 = v_1 + iv_2, \quad v^2 = v_4 + iv_5, \quad \nu^1 = \nu_1 + i\nu_2, \quad \nu^2 = \nu_4 + i\nu_5;$$

$$v=(v^1,v^2,v_3),\quad \nu=(\nu^1,\nu^2,\nu_3); \tag{21}$$

$\theta^-(f)(t)$ are the boundary values of the function $\theta(f)(z)$ as $z\to t\in\Gamma$ from the exterior of the domain Ω; the operators $Tf, Sf, \theta(f)$ are defined in (7), (8) and (14), respectively.

As was already noted, the system of equations (20) has six linearly independent solutions. Let us obtain their explicit representations. Further we will treat $v=(v^1,v^2,v_3)\in L_p(\Omega)$, $\nu=(\nu^1,\nu^2,\nu_3)\in C_\alpha(\Gamma)$ as some solution to the system of equations (20). Note that the operators T, T^0, T^0_Γ in (21) determine the functions $Tf(z), T^0f(z), T^0_\Gamma f(z)$ that are holomorphic in the exterior of the domain Ω and vanish at infinity. The functions $\theta(f)(z)$ have the same properties. Therefore, the equalities in (20) are the boundary conditions of the Riemann–Hilbert problem with zero index for functions holomorphic outside Ω and vanishing at infinity. It is well known that such a problem has only the zero solution. Therefore, they are transformed to the form

$$Td_j[S_jv](z)+2(-1)^j\theta(\tau'\overline{\nu^j})(z)=0,\quad j=1,2,$$
$$T\left[a_{2\overline{\zeta}}v^1+2(d_0^{11}-d_0^{22})v_3\right](z)-2\theta(\tau' a_2\nu^1)(z)=0,$$
$$T^0\left(2D^{1313}_{0\overline{\zeta}}v_3-(D_0^{1313}v^2)/2\right)(z)+T(a_{4\overline{\zeta}}v^2)(z)+2T^0_\Gamma(\tau' D_0^{1313}\nu_3)(z)-2\theta(\tau' a_4\nu^2)(z)=0,$$
$$T^0p_1(v)(z)+Tp_2(v)(z)+2iT^0_\Gamma p_3(\nu)(z)+2\theta(\tau' D_0^{1313}\nu_3)(z)=0,\quad z\in\overline{\Omega}_1\equiv\mathbb{C}\backslash\Omega, \tag{22}$$

where $\mathbb{C}$ is the complex plane.

The inclusion v_j $(j=\overline{1,5})\in C_\alpha(\overline{\Omega})$ follows from (20). Passing in the first two equalities in (20) to the limit as $z\to t\in\Gamma$ from the interior of the domain Ω and in the first two equalities in (22) from the exterior of the domain Ω, taking into account the continuity of the functions $Td_j[S_jv](z)$ $(j=1,2)$ on $\mathbb{C}$, and using the Sokhotskii formulas, we obtain

$$v^j(t)=-2\nu^j(t),\quad t\in\Gamma,\quad j=1,2. \tag{23}$$

Similarly, passing in the third equality in (20) to the limit as $z\to t\in\Gamma$ from the interior of the domain Ω, and subtracting the last equality in (20) from the resulting equality, taking into account relations $\widetilde{T}^*p_1(t)=\operatorname{Re}T^0p_1(t)$, $\widetilde{T}_\Gamma p_3(t)=2\operatorname{Re}(iT^0_\Gamma p_3(t))$ and the formulas of Sokhotskii, we obtain

$$v_3(t)=-\nu_3(t),\quad t\in\Gamma. \tag{24}$$

Now we differentiate the first two equalities in (20) with respect to $\overline{z}$. Taking into account relations (8), we obtain

$$\overline{v^j_z(z)}+(-1)^jd_j[S_jv](z)=0,\quad j=1,2,$$

whence for $S_jv(z)$ we will have

$$S_jv(z)=(-1)^{j-1}\left[D^{1111}_{2(j-1)}(v_{3j-2,\alpha^1}+v_{3j-1,\alpha^2})+iD^{1212}_{2(j-1)}(v_{3j-1,\alpha^1}-v_{3j-2,\alpha^2})\right],\quad j=1,2. \tag{25}$$

The relations (25) we again differentiate with respect to $\overline{z}$, and the third equality in (20) with respect to z and $\overline{z}$. Then, after simple transformations, we come to the conclusion that functions v_1, v_2, $2v_3$, v_4, v_5 satisfies a homogeneous system of linear equilibrium equations (4) under conditions $f_j\equiv 0$, $j=\overline{1,5}$.

Further, in relations (25) we pass to the limit as $z\to t\in\Gamma$ from the interior of the domain Ω; the last three equalities in (22) first we differentiate with respect to z, then in them we pass to the limit as $z\to t\in\Gamma$ from the exterior of the domain Ω; the third equality in (20) we differentiate with respect to z, then we pass to the limit as $z\to t\in\Gamma$ from the interior the domain Ω. Then, we use the equalities on curve Γ obtained in this way, the relations (23), (24) and equality

$$(Sf)^+(t)-(Sf)^-(t)=-f(t)\cdot(\overline{t}')^2,\quad \theta'^+(\tau' f)(t)-\theta'^-(\tau' f)(t)=f_t+f_{\overline{t}}\cdot(\overline{t}')^2,\quad t\in\Gamma,$$

where Sf, $\theta'(f)$ are the operators, defined in (8), (21), $f_t=\partial f/\partial t$, $f_{\overline{t}}=\partial f/\partial\overline{t}$. As a result, after simple transformations, we obtain that the functions v_1, v_2, $2v_3$, v_4, v_5 also satisfy homogeneous linear boundary conditions.

Thus, the functions v_1, v_2, $2v_3$, v_4, v_5 are solutions homogeneous system of linear equilibrium equations satisfying homogeneous boundary conditions. Then, acting in the same way as in deriving relations (18) and (19), we obtain for v_j $(j=\overline{1,5})$ the expressions

$$v_1 = c_4[\widetilde{k}_2(\alpha^2) - k_1^1(\alpha^1)] - c_5\alpha^2 k_1^0(\alpha^1) + c_6 k_1^0(\alpha^1) - c_0\alpha^2 + c_1,$$

$$v_2 = -c_4\alpha^1 k_2^0(\alpha^2) + c_5[\widetilde{k}_1(\alpha^1) - k_2^1(\alpha^2)] + c_6 k_2^0(\alpha^2) + c_0\alpha^1 + c_2,$$

$$v_3 = -c_4\alpha^1/2 - c_5\alpha^2/2 + c_6/2, \quad v_4 = c_4, \quad v_5 = c_5,$$

where

$$k_j^m(\alpha^j) = \int\limits_0^{\alpha^j} x^m k_j(x)dx, \quad m=0,1, \quad \widetilde{k}_j(\alpha^j) = \int\limits_0^{\alpha^j} k_j^0(x)dx, \quad j=1,2;$$

c_j are arbitrary real constants.

Functions $\nu_j(t)$ $(j=\overline{1,5})$ are expressed through functions v_k $(k=\overline{1,5})$ according to formulas (23) and (24). Therefore, the solution (v,ν), $v=(v_1,v_2,v_3,v_4,v_5)$, $\nu=(\nu_1,\nu_2,\nu_3,\nu_4,\nu_5)$ of the adjoint system (20) can be represented in the form $(v,\nu)=c_0\gamma_1+c_1\gamma_2+c_2\gamma_3+c_4\gamma_4+c_5\gamma_5+c_6\gamma_6$, where $\gamma_k=(\gamma_{k1},\gamma_{k2},...,\gamma_{k10})$ $(k=\overline{1,6})$ are linearly independent solutions of the system (20). Then, for the solvability of system (16), it is necessary and sufficient that conditions

$$\iint\limits_\Omega \left\{\mathrm{Re}[(f^1-h_{13})(\gamma_{k1}-i\gamma_{k2})] + \mathrm{Re}[(f^2-h_{23})(\gamma_{k4}-i\gamma_{k5})] + (f^3-h_{33})\gamma_{k3}\right\} d\alpha^1 d\alpha^2$$

$$+\int\limits_\Gamma \{g_1\gamma_{k6}+g_2\gamma_{k7}+g_5\gamma_{k8}+g_3\gamma_{k9}+g_4\gamma_{k10}\}\,ds = 0, \quad k=\overline{1,6},$$

be satisfied. After simple transformations, these conditions take the form

$$\iint\limits_\Omega R^j d\alpha^1 d\alpha^2 + \int\limits_\Gamma P^j ds = 0, \quad j=1,2, \quad \iint\limits_\Omega (R^1\alpha^2 - R^2\alpha^1)d\alpha^1 d\alpha^2 + \int\limits_\Gamma (P^1\alpha^2 - P^2\alpha^1)ds = 0,$$

$$\iint\limits_\Omega [R^1 k_1^0(\alpha^1) + R^2 k_2^0(\alpha^2) + R^3]d\alpha^1 d\alpha^2 + \int\limits_\Gamma [P^1 k_1^0(\alpha^1) + P^2 k_2^0(\alpha^2) + P^3]ds = 0,$$

$$\iint\limits_\Omega [R^1(\widetilde{k}_2(\alpha^2) - k_1^1(\alpha^1)) - R^2\alpha^1 k_2^0(\alpha^2) - R^3\alpha^1 + L^1 + R^1 w_3]d\alpha^1 d\alpha^2$$

$$+\int\limits_\Gamma [P^1(\widetilde{k}_2(\alpha^2) - k_1^1(\alpha^1)) - P^2\alpha^1 k_2^0(\alpha^2) - P^3\alpha^1 + N^1 + P^1 w_3]ds = 0,$$

$$\iint\limits_\Omega [R^1\alpha^2 k_1^0(\alpha^1) - R^2(\widetilde{k}_1(\alpha^1) - k_2^1(\alpha^2)) + R^3\alpha^2 - L^2 - R^2 w_3]d\alpha^1 d\alpha^2$$

$$+\int\limits_\Gamma [P^1\alpha^2 k_1^0(\alpha^1) - P^2(\widetilde{k}_1(\alpha^1) - k_2^1(\alpha^2)) + P^3\alpha^2 - N^2 - P^2 w_3]ds = 0, \qquad (26)$$

where R^j, P^j $(j=\overline{1,3})$, L^k, N^k $(k=1,2)$ are the components of external forces acting on the shell.

If conditions (26) are satisfied, then the general solution of system (16) can be represented in the form

$$\rho^j(z) = \mathfrak{F}_j(f(w_3) - f_c)(z) + \widetilde{\rho}^j(z), \quad j=\overline{1,3},$$

$$\mu_k(t) = \mathfrak{F}_{k+3}(f(w_3) - f_c)(t) + \widetilde{\mu}_k(t), \quad k = \overline{1,5}, \tag{27}$$

where

$$\widetilde{\rho}^1(z) = 2ic_0 D_{0\bar{z}}^{1212}, \quad \widetilde{\rho}^2 = \widetilde{\rho}^3 \equiv 0, \quad \widetilde{\mu}_j(t) = c_0\mu_j^0(t) + \beta_{0j}\mu_j^1(t), \quad j = 1,2,$$
$$\widetilde{\mu}_j(t) = \beta_{0j}\mu_j^1(t), \quad j = \overline{3,5},$$

$$f(w_3) = (f^1, f^2, f^3, \varphi_1, \varphi_2, \varphi_3, \varphi_4, \varphi_5), \quad f_c = (h_{13}(c), h_{23}(c), h_{33}(c), l_{13}(c), l_{23}(c), l_3(c), 0, 0);$$

$\mathfrak{F}_j$ $(j = \overline{1,3})$ and $\mathfrak{F}_k$ $(k = \overline{4,8})$ are linear bounded operators from $L_p(\Omega) \times C_\alpha(\Gamma)$ in $L_p(\Omega)$ and in $C_\alpha(\Gamma)$, respectively; c_0, β_{0j} are arbitrary real constants; functions $\mu_j^k(t)$ are defined in (19); f^j, φ_k, $h_{j3}(c)$, $l_{j3}(c)$, $l_3(c)$ are defined in (12), (13), and (17).

If we substitute the representations for $\mu_k(t)$ from (27) in (14) and (15), then we obtain

$$\Phi_j(z) = \Phi_j(w_3)(z) + \Phi_{jc}(z), \quad j = \overline{1,3}, \quad \Psi_k(z) = \Psi_k(w_3)(z) + \Psi_{kc}(z),$$
$$\Psi_k'(z) = \Psi_k'(w_3)(z) + \Psi_{kc}'(z), \quad k = 1,2, \quad \Phi_3'(z) = \Phi_3'(w_3)(z) + \Phi_{3c}'(z), \tag{28}$$

where the notations

$$\Phi_j(w_3)(z) = \theta(\mathfrak{F}_{2j+2}(f(w_3)))(z), \quad \Phi_{jc}(z) = -\theta(\mathfrak{F}_{2j+2}(f_c))(z) + c_0\widetilde{\alpha}_j(z), \quad j = 1,2;$$
$$\widetilde{\alpha}_1(z) = \alpha_0(z), \quad \widetilde{\alpha}_2(z) \equiv 0; \quad \Phi_3(w_3)(z) = \Phi_3(\mathfrak{F}_8(f(w_3)))(z),$$
$$\Phi_{3c}(z) = -\Phi_3(\mathfrak{F}_8(f_c))(z) + c_5 + ic_6;$$
$$\Psi_k(w_3)(z) = \Psi(\mathfrak{F}_{2k+3}(f(w_3)))(z), \quad \Psi_{kc}(z) = -\Psi_k(\mathfrak{F}_{2k+3}(f_c))(z) + c_{2k-1} + ic_{2k},$$
$$\Psi_k'(w_3)(z) = \theta(\mathfrak{F}_{2k+3}(f(w_3)))(z), \quad \Psi_{kc}'(z) = -\theta(\mathfrak{F}_{2k+3}(f_c))(z), \quad k = 1,2;$$
$$\Phi_3'(w_3)(z) = i\theta(\mathfrak{F}_8(f(w_3)))(z), \quad \Phi_{3c}'(z) = -i\theta(\mathfrak{F}_8(f_c))(z)$$

are accepted; here the operators $\theta(f)$, $\Psi_k(f)$, $\Phi_3(f)$ are defined in (14) and (15).

Now if we substitute the expressions for $\rho^j(z)$ from (27) and (28) in (7), (9), and (10), then Problem (4), (2) reduce to a system of equations for generalized displacements w_1, w_2, w_3, ψ_1, ψ_2:

$$\omega_j^0(z) = \omega_j^0(w_3)(z) + \omega_{jc}^0(z), \quad j = 1,2, \quad w_3 - Gw_3 = w_{3c}, \quad \omega_1^0 = w_2 + iw_1, \quad \omega_2^0 = \psi_2 + i\psi_1, \tag{29}$$

where

$$\omega_j^0(w_3)(z) = \Psi_j(w_3)(z) + iTd_j[\omega_j(w_3)](z), \quad \omega_{jc}^0(z) = \Psi_{jc}(z) + iTd_j[\omega_{jc}](z),$$
$$\omega_j(w_3) = \Phi_j(w_3)(z) + T\rho^j(w_3)(z), \quad \omega_{jc} = \Phi_{jc}(z) + T\rho_c^j(z), \quad j = 1,2;$$
$$\rho^k(w_3) = \mathfrak{F}_k(f(w_3))(z), \quad \rho_c^k(z) = -\mathfrak{F}_k(f_c)(z) + \widetilde{\rho}^k(z), \quad k = \overline{1,3}; \quad Gw_3 = \operatorname{Re}\Phi_3(w_3) - \widetilde{T}\widetilde{\rho}_3(w_3),$$
$$w_{3c} = \operatorname{Re}\Phi_{3c}(z) - \widetilde{T}\widetilde{\rho}_{3c}(z); \quad \widetilde{\rho}_3(w_3) = \rho^3(w_3)/(2D_0^{1313}), \quad \widetilde{\rho}_{3c} = \rho_c^3/(2D_0^{1313}).$$

We note that functions $\omega_{jc}^0(z)$, $j = 1,2$, w_{3c} are solutions of a homogeneous system linear of equilibrium equations and also satisfy homogeneous linear boundary conditions. Therefore, for them, as above, explicit expressions can be obtained. In particular, for function w_{3c} we have $w_{3c} = -c_4\alpha^1 - c_5\alpha^2 + c_6$, where c_4, c_5, c_6 are arbitrary real constants.

Let us study the solvability of the third equation in (29) in the space $W_p^{(2)}(\Omega)$, which we write in the form

$$\widetilde{w}_3 - \widetilde{G}\widetilde{w}_3 = 0, \tag{30}$$

where $\widetilde{G}\widetilde{w}_3 = G(\widetilde{w}_3 + w_{3c})$, $\widetilde{w}_3 = w_3 - w_{3c}$.

Using the relations in (5), (11), the representations for tangential displacements ω_1^0 and for rotation angles ω_2^0 and expression of operator G in (29), easily to show that $\widetilde{G}$ is a nonlinear bounded operator in $W_p^{(2)}(\Omega)$. Moreover, for any $\widetilde{w}_3^j \in W_p^{(2)}(\Omega)$, $(j=1,2)$ belonging to the ball $||\widetilde{w}_3||_{W_p^{(2)}(\Omega)} < r$, we have the estimate

$$\left|\left|\widetilde{G}\widetilde{w}_3^1 - \widetilde{G}\widetilde{w}_3^2\right|\right|_{W_p^{(2)}(\Omega)} \le q \left|\left|\widetilde{w}_3^1 - \widetilde{w}_3^2\right|\right|_{W_p^{(2)}(\Omega)},$$

where $q = c\left[||R^1||_{L_p(\Omega)} + ||R^2||_{L_p(\Omega)} + (r + |c_4| + |c_5|)(1+r) + c_4^2 + c_5^2\right]$; c is the known positive constant, depending on the physico-geometric characteristics of the shell; c_4, c_5 are the constants included in the expression of function w_{3c}.

Suppose that the radius r of the ball, the external forces $R^j (j=1,2)$, and the constants c_4, c_5 are such that inequalities

$$q < 1, \quad ||\widetilde{G}0||_{W_p^{(2)}(\Omega)} < (1-q)r \tag{31}$$

are satisfied. Then we can apply the contraction mappings principle [20, p. 146] to equation (30); as a result, in the ball $||\widetilde{w}_3||_{W_p^{(2)}(\Omega)} < r$, for fixed constants c_4, c_5 using conditions (31), equation (30) has a unique solution $\widetilde{w}_3 \in W_p^{(2)}(\Omega)$, which can be represented as $\widetilde{w}_3 = \mathfrak{R}\widetilde{G}0$, where $\mathfrak{R}$ is the resolvent of operator $\widetilde{G}\widetilde{w}_3 - \widetilde{G}0$. Therefore, the deflection w_3 has the form $w_3 = \widetilde{w}_3 + w_{3c}$. Knowing w_3, according to the first two formulas in (29) we find the tangential displacements ω_1^0 and the rotation angles ω_2^0, which, as can be easily verified, belong to the space $W_p^{(2)}(\Omega)$. As a result, we obtain the generalized solution $a = (w_1, w_2, w_3, \psi_1, \psi_2)$ of Problem (4), (2), which can be written as

$$a = a_0 + a_*, \tag{32}$$

where a_0 is the vector with components $\operatorname{Im}\omega_1^0(\widetilde{w}_3)$, $\operatorname{Re}\omega_1^0(\widetilde{w}_3)$, $\widetilde{w}_3, \operatorname{Im}\omega_2^0(\widetilde{w}_3)$, $\operatorname{Re}\omega_2^0(\widetilde{w}_3)$; $a_* = (w_{1*}, w_{2*}, w_{3*}, \psi_{1*}, \psi_{2*})$ is the vector with components determined by the formulas

$$\omega_{j*}^0 = \omega_j^0(w_{3c}) + \omega_{jc}^0, \quad j = 1,2, \quad w_{3*} = w_{3c},$$

where $\omega_{1*}^0 = w_{2*} + i w_{1*}$, $\omega_{2*}^0 = \psi_{2*} + i\psi_{1*}$.

Let us notice that a_* is the vector of rigid displacements of the shell as an absolutely rigid body, i.e. it sets to zero the components of strains γ_{ij}^k, $i, j = \overline{1,3}$, $k = 0, 1$. For the components of the vector a_*, one can obtain explicit expressions that have the form

$$w_{1*} = c_4[\widetilde{k}_2(\alpha^2) - k_1^1(\alpha^1)] - c_5\alpha^2 k_1^0(\alpha^1) + c_6 k_1^0(\alpha^1) - c_0\alpha^2 + c_1 - c_4^2\alpha^1/2 - c_4 c_5 \alpha^2,$$

$$w_{2*} = c_5[\widetilde{k}_1(\alpha^1) - k_2^1(\alpha^2)] - c_4\alpha^1 k_2^0(\alpha^2) + c_6 k_2^0(\alpha^2) + c_0\alpha^1 + c_2 - c_5^2\alpha^2/2,$$

$$w_{3*} = -c_4\alpha^1 - c_5\alpha^2 + c_6, \quad \psi_{1*} = c_4, \quad \psi_{2*} = c_5,$$

where c_j are arbitrary real constants.

Note that in the last two conditions in (26), w_3 means $w_3 = \widetilde{w}_3 + w_{3*}$. In the case of linear problems, these summands containing w_3 are absent.

It is easy to see that conditions (26) are not only sufficient, but also necessary conditions for the solvability of Problem (4), (2). Note that they mean that external load acting on the shell is self-balanced.

Thus, we have proved the theorem.

Theorem 1. *Assume that conditions (a), (b), (c), (d), (e) and inequalities (31) are satisfied.Then it is necessary and sufficient for the solvability of the Problem (4), (2) that conditions (26) be satisfied. If they are satisfied, then Problem (4), (2) has a generalized solution* $a = (w_1, w_2, w_3, \psi_1, \psi_2) \in W_p^{(2)}(\Omega)$, $2 < p < 4/(2-\beta)$, *of the form (32) up to rigid displacements* a_* *of the shell as an absolutely rigid body.*

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