The Marcinkiewicz exponents with applications

David B. Katz

Kazan Federal University

2016

Introduction 1

A great body of recent works is dealing with various characteristics of point sets of sophisticated structure: fractals, non-rectifiable curves and so on. The most known are Hausdorff and Minkowskii dimensions, and a number of new ones: approximation dimension, refined metric dimension and others.

In 2013 the author introduced a family of new metric characteristics for plane sets and called them Marcinkiewicz exponents for closed non-rectifiable Jordan curves. Now we introduce their weighted and local versions for any compact sets, but mainly we are interested in non-rectifiable curves and arcs on the complex plane. First of all, we study their properties and relations with known dimensions.

Introduction 2

Then we consider certain applications. We use these characteristics to solve the Riemann boundary value problems in domains with non-rectifiable boundaries. As known, there exists a lot of applications of that problems in mechanics, elasticity theory and others. We also study so called semi-continuous version of the Riemann boundary value problem. For both these problems we obtain new solvability conditions which are sharper than the known ones.

In particular, the improvement of known results is connected with the fact that the new characteristics allow more precise description of local properties of non-rectifiable curves, including a phenomenon of local asymmetry. Let us explain this phenomenon. A curve Γ on the complex plane divides small disk $B(t, r) = \{z : |z - t| < r\}$ with center at it's interior point t into left and right components B^+ and B^-. If Γ is smooth at point t, and S is mapping of symmetry relatively its tangent at point t, then the area of set $B^+ \triangle S(B^-)$ is $o(r^2)$ for $r \to 0$, i.e., it decreases faster than the area of $B(t, r)$. We understand this fact as local symmetry. Generally speaking, for a non-rectifiable curve we cannot find a symmetry axis with this property, i.e., non-rectifiable curves are locally asymmetric. In the present paper we introduce left and right (inner and outer) characteristics of plane curves, what allows us to improve solvability conditions for the locally asymmetric curves.

The Marcinkiewicz exponents 1

Let compact set E be a subset of a fixed open domain $Y \subset X$. We put

$$I_p(E,\mu) := \int\limits_{Y\setminus E} \frac{d\mu}{\operatorname{dist}^p(z,E)}.$$

Definition

The Marcinkiewicz exponent of set E with respect to measure μ is the least upper bound of set $\{p : I_p(E,\mu) < \infty\}$. We denote it $\mathfrak{m}(E,\mu)$.

Definition

The inner and outer Marcinkiewicz exponents of a closed plane curve Γ with respect to measure μ are $\mathfrak{m}^+(\Gamma,\mu) := \mathfrak{m}(\Gamma,\mu^+)$ and $\mathfrak{m}^-(\Gamma,\mu) := \mathfrak{m}(\Gamma,\mu^-)$, where μ^+ and μ^- are restrictions of measure μ on inner and outer domains of Γ.

Clearly, in this case $\mathfrak{m}(\Gamma, \mu)$ is the least of inner and outer exponents. We name all these values the Marcinkiewicz exponents in connection with the Marcinkiewicz's idea to characterize sets by means of certain integrals over their complements. We introduce also a local version of these values. We put

$$I_p(E, t, r, \mu) := \int\limits_{B(t,r)\setminus E} \frac{d\mu}{\mathrm{dist}^p(z, E)},$$

where $B(t, r)$ is ball of radius r with center $t \in E$.

Definition

The local Marcinkiewicz exponent of set E with respect to measure μ is the least upper bound of set $\{p : \lim\limits_{r\to 0} I_p(E, t, r, \mu) < \infty\}$. We denote it $\mathfrak{m}(E, t, \mu)$.

In what follows we consider mainly the case $d\mu = w(z)d\mathcal{L}$ where w is a non-negative function, $\mathcal{L}$ is Lebesgue measure on the plane, and write $\mathfrak{m}(E, w)$ and $\mathfrak{m}(E, t, w)$ instead of $\mathfrak{m}(E, \mu)$ and $\mathfrak{m}(E, t, \mu)$.
If Γ is a closed plane curve, then we consider also its inner and outer local Marcinkiewicz exponents $\mathfrak{m}^+(\Gamma, t, \mu) := \mathfrak{m}(\Gamma, t, \mu^+)$ and $\mathfrak{m}^-(\Gamma, t, \mu) := \mathfrak{m}(\Gamma, t, \mu^-)$. Clearly, the local properties of a curve cannot depend on its closeness. Hence, we can introduce analogs of the inner and outer exponents for open arcs, too.

If $\mu = \mathcal{L}$, then $\mathfrak{m}(E, \mu) \geq 2 - \overline{\mathrm{dm}}(E)$, where $\overline{\mathrm{dm}}$ is upper Minkowskii dimension; its definition is given below. If $E \subset \mathbb{C}$ is a continuum, then $\mathfrak{m}(E, \mu) \leq 1$. The Marcinkiewicz exponents of rectifiable curves on the complex plane are equal to 1.

Lemma

If the set E is compact then $\inf\{\mathfrak{m}(E; t; \mu) : t \in E\} = \mathfrak{m}(E; \mu)$. *For a closed curve Γ we have* $\inf\{\mathfrak{m}^+(\Gamma; t; \mu) : t \in \Gamma\} = \mathfrak{m}^+(\Gamma; \mu)$ *and* $\inf\{\mathfrak{m}^-(\Gamma; t; \mu) : t \in \Gamma\} = \mathfrak{m}^-(\Gamma; \mu)$.

Definition

Let Γ be a closed Jordan curve. We put

$$\mathfrak{m}^*(\Gamma; \mu) := \inf\{t \in \Gamma : \max\{\mathfrak{m}^+(\Gamma; t; \mu), \mathfrak{m}^-(\Gamma; t; \mu)\}\}.$$

The Riemann boundary value problem on a closed non-rectifiable curve

The Riemann boundary value problem is well known in complex analysis. Now we consider it on a closed Jordan curve.
Let a curve Γ divide the complex plane $\mathbb{C}$ into domains D^+ and $D^- \ni \infty$. We seek a holomorphic in $\overline{\mathbb{C}} \setminus \Gamma$ function $\Phi(z)$ satisfying equality

$$\Phi^+(t) = G(t)\Phi^-(t) + g(t), \quad t \in \Gamma. \tag{1}$$

The boundary values $\Phi^+(t)$ and $\Phi^-(t)$ of desired function are limits of $\Phi(z)$ for z tending to point $t \in \Gamma$ from domains D^+ and D^- correspondingly. The simplest case is so called jump problem

$$\Phi^+(t) - \Phi^-(t) = g(t), \quad t \in \Gamma. \tag{2}$$

The classical results here base on the properties of Cauchy integral

$$\Phi(z) = \frac{1}{2\pi i} \int\limits_{\Gamma} \frac{g(t)dt}{t - z}, \quad z \notin \Gamma,$$

over piecewise-smooth curve Γ. If g satisfy the Hölder condition

$$\sup\left\{ \frac{|g(t') - g(t'')|}{|t' - t''|^{\nu}} : t', t'' \in \Gamma, t' \neq t'' \right\} := h_{\nu}(g, \Gamma) < \infty$$

with exponent $\nu \in (0, 1]$, then the function $\Phi(z)$ is holomorphic in $\overline{\mathbb{C}} \setminus \Gamma$ and has boundary values $\Phi^{+}(t)$ and $\Phi^{-}(t)$ satisfying equality (2). Thus, the Cauchy integral with density $g \in H_{\nu}(\Gamma)$ is a solution of jump problem (2). Problem (1) is reducible to the jump problem. Hence, the whole theory of the Riemann boundary value problem on piecewise-smooth curves reduces to application of the cited above result on the boundary behavior of the Cauchy integral. But this integral is not defined for non-rectifiable curves.

The known result on solvability of the jump problem on on-rectifiable curve

In earlier 80-th B.A. Kats obtained the following result on solvability of the jump problem on non-rectifiable curves.

Theorem

If $g \in H_\nu(\Gamma)$, $\nu > \overline{\mathrm{dm}}(\Gamma)/2$, *then the jump problem (2) is solvable.*

As above, symbol $\overline{\mathrm{dm}}\,\Gamma$ stands for the upper Minkowskii dimension of the curve Γ, i.e.,

$$\overline{\mathrm{dm}}\,\Gamma = \overline{\lim}_{\varepsilon\to 0}\frac{\ln N(\Gamma;\varepsilon)}{-\ln\varepsilon},$$

where $N(\Gamma;\varepsilon)$ is the least number of disks of radius ε covering Γ. This characteristics is global, and does not take into account the asymmetry of Γ.

New result on solvability of the jump problem on non-rectifiable curve

Theorem

If $g \in H_\nu(\Gamma)$ and $\nu > (2-m)/2$, where $m = \mathfrak{m}^*(\Gamma, \mathcal{L})$, then the jump problem on a closed curve Γ has a solution Φ. This solution satisfies in domains D^+ and D^- the Hölder condition with any exponent lesser than $1 - 2(1-\nu)/m$.

The following result show that the last theorem sharpens the known one.

Theorem

There exist curves Γ such that $\mathfrak{m}^*(\Gamma; \mathcal{L}) > 2 - \overline{\mathrm{dm}}\,\Gamma$.

Uniqueness

We will consider now the uniqueness of obtained solution by means of the E.P. Dolzhenko theorem. Let $\mathfrak{dh}\,E$ stand for the Hausdorff dimension of compact set E, which is a subset of domain $G \subset \mathbb{C}$. The Dolzhenko theorem claims that if $\nu > \mathfrak{dh}\,E - 1$, then any holomorphic in $G \setminus E$ function $F \in H_\nu(\overline{G})$ is holomorphic in G; otherwise the Hölder space $H_\nu(\overline{G})$ contains a function which is holomorphic in $G \setminus E$, but not in G. By virtue of this theorem difference of two solutions of the problem (2) is a constant, if the both solutions satisfy Hölder condition with exponent $\nu > \mathfrak{dh}\,\Gamma - 1$ in D^+ and in D^-. In this connection we will study the Riemann boundary problem (1) in what follows under additional assumptions

$$\Phi|_{D^+} \in H_\lambda(D^+), \quad \Phi|_{D^-} \in H_\lambda(D^-), \quad \Phi(\infty) = 0. \tag{3}$$

Thus, solution of the jump problem (2) in class (3) is unique for $\lambda > \mathfrak{dh}\,\Gamma - 1$.

Theorem

Let $m^* := \mathfrak{m}^*(\Gamma; \mathcal{L})$, $\nu > (2 - m^*)/2$ *and* $1 - 2(1-\nu)/m^* > \lambda > \mathfrak{dh}\,\Gamma - 1$. *If* $G, g \in H_\nu(\Gamma)$ *and* $G(t)$ *does not vanish, then there are valid the following propositions on solvability of the Riemann boundary problem (1) in the class (3):*
— for $\varkappa = 0$ *the problem has a unique solution;*
— for $\varkappa > 0$ *the problem has a family of solutions depending on* $\varkappa$ *arbitrary complex constants;*
— for $\varkappa < 0$ *the problem has a unique solution if* $-\varkappa$ *solvability conditions are fulfilled.*

In other words, under assumptions of this theorem the problem has just the same solvability properties as in the classical case.

The semi-continuous version of Riemann boundary problem allows violation of equality (1) at a finite set of points of the curve Γ. We restrict ourselves to the jump problem, i.e., we seek holomorphic in $\overline{\mathbb{C}} \setminus \Gamma$ function $\Phi(z)$ such that it has limit values $\Phi^+(t)$ and $\Phi^-(t)$ from D^+ and D^- correspondingly at any point $t \in \Gamma' := \Gamma \setminus \tau$, $\tau := \{t_1, t_2, \dots, t_m\} \subset \Gamma$, and these values satisfy relation

$$\Phi^+(t) - \Phi^-(t) = g(t), \quad t \in \Gamma'. \tag{4}$$

For definiteness we have to assume the desired function having prescribed behavior at the points of set τ,

$$\Phi(z) = O(|z - t_j|^{-\gamma}), \quad \gamma = \gamma(\Phi) < 1, \quad j = 1, 2, \dots, m. \tag{5}$$

The set τ consists of discontinuity points of both Φ and g. If the curve is of piecewise-smooth, then all discontinuities of solution Φ are caused by singularities of jump g. But in the case under consideration the desired function can loss continuity at a point where the jump is continuous. The Marcinkiewicz exponents seem to be convenient for description of this phenomenon. Put

$$w(z) = \prod_{j=1}^{m} |z - t_j|^{-\gamma_j}, \quad 0 \leq \gamma_j < 1, \quad j = 1, 2, \ldots, m,$$

$g = wg_0$, $g_0 \in H_\nu(\Gamma)$.

As a result, we obtain

Theorem

Let $g = wg_0$, $g_0 \in H_\nu(\Gamma)$, $\nu > 1 - \mathfrak{m}^*(\Gamma; w)$, *and* $\nu > (2 - m')/2$ *where* $m' := \inf\{\max\{\mathfrak{m}^+(\Gamma; t; \mathcal{L}), \mathfrak{m}^-(\Gamma; t; \mathcal{L})\} : t \in \Gamma'\}$, *then the jump problem (4) has a solution satisfying (5).*

This result enables us to solve the Riemann boundary value problem in semi-continuous formulation, too.
The Marcienkewicz exponents are useful also for solving of the Riemann boundary value problem on non-closed arcs, on Riemann surfaces, certain Beltrami equations and so on.

Thank you for attention!