Differential Equations, Vol. 36, No. 7, 2000, pp. 1109–1111. Translated from Differentsial'nye Uravneniya, Vol. 36, No. 7, 2000, pp. 998–999.

SHORT COMMUNICATIONS

The Fundamental Wave Problem for Cylindrical Dielectric Waveguides

E. M. Karchevskii
Kazan State University, Kazan, Russia
Received December 4, 1998

A nonlinear spectral problem for a system of singular integral equations was constructed in [1] for the numerical solution of the fundamental wave problem for cylindrical dielectric waveguides on the basis of the representation of unknown functions via single layer potentials. The present paper is a continuation of [1] and deals with the investigation of qualitative properties of the spectrum.

The problem of finding propagation constants for fundamental waves in cylindrical dielectric waveguides can be reduced (e.g., see [2]) to finding the values of a complex parameter β for which the system

$$\Delta u + \chi_j^2(\beta)u = 0, \qquad \Delta v + \chi_j^2(\beta)v = 0, \qquad M \in S_j, \qquad j = 1,2, \tag{1}$$

has nontrivial solutions satisfying the transmission conditions

$$\begin{gathered} u^+ - u^- = 0, \qquad v^+ - v^- = 0, \\ \chi_1^{-2}(\beta)\left(\beta\,\partial v/\partial\tau + \varepsilon_1\omega\,\partial u^-/\partial\nu\right) - \chi_2^{-2}(\beta)\left(\beta\,\partial v/\partial\tau + \varepsilon_2\omega\,\partial u^+/\partial\nu\right) = 0, \\ \chi_1^{-2}(\beta)\left(\beta\,\partial u/\partial\tau - \mu_0\omega\,\partial v^-/\partial\nu\right) - \chi_2^{-2}(\beta)\left(\beta\,\partial u/\partial\tau - \mu_0\omega\,\partial v^+/\partial\nu\right) = 0, \qquad M \in \Gamma, \end{gathered} \tag{2}$$

and the corresponding condition at infinity. Here S_1 is the domain bounded by the contour Γ, $S_2 = R^2 \backslash \bar{S}_1$, $\partial u/\partial\nu$ (respectively, $\partial u/\partial\tau$) is the normal (respectively, tangent) derivative on Γ, u^- (respectively, u^+) is the limit value of a function u from the interior (respectively, the exterior) of Γ, $\chi_j^2(\beta) = k_0^2 n_j^2 - \beta^2$, $k_0^2 = \omega^2\varepsilon_0\mu_0$, ε_0 is the dielectric constant, μ_0 is the magnetic constant, $\omega > 0$ is the frequency of electromagnetic oscillations, $n_1, n_2 > 0$ are the refraction coefficients of the waveguide and the ambient medium ($n_2 < n_1$), and $\varepsilon_j = \varepsilon_0 n_j^2$.

Following [3], we assume that the functions u and v satisfy the partial condition at infinity, i.e., can be represented in the form

$$u = \sum_{n=-\infty}^{\infty} \alpha_n H_n^{(1)}(\chi_2 r)\exp(in\varphi), \qquad v = \sum_{n=-\infty}^{\infty} \gamma_n H_n^{(1)}(\chi_2 r)\exp(in\varphi) \tag{3}$$

for sufficiently large r, where r and φ are the polar coordinates of the point M and $H_n^{(1)}$ is the first-kind Hankel function of order n.

We seek nontrivial solutions of problem (1)–(3) with a twice continuously differentiable contour Γ in the class of functions continuous and continuously differentiable in $\bar{S}_1$ and $\bar{S}_2$ and twice continuously differentiable in S_1 and S_2. Following [4, p. 228], we can readily show that the spectrum of problem (1)–(3) lies in the set Λ that is the intersection of the Riemann surfaces Λ_j of the functions $\ln\chi_j(\beta)$, $j = 1,2$. (The *spectrum* is the set of values of $\beta \in \Lambda$ for which problem (1)–(3) has nontrivial solutions.) By Λ_0 we denote the intersection of the principal (physical) sheets of the surfaces Λ_j; we also write

$$\begin{gathered} \Lambda_j^- = \{\beta \in \Lambda_0 : \ \operatorname{Im}\chi_j < 0\}, \qquad j = 1,2, \qquad \Lambda_2^+ = \{\beta \in \Lambda_0 : \ \operatorname{Im}\chi_2 > 0, \ \operatorname{Im}\beta \neq 0\}, \\ G = \{\beta \in \Lambda_0 : \operatorname{Im}\chi_2 > 0, \ \operatorname{Im}\beta = 0, \ k_0 n_2 < |\beta| < k_0 n_1\}. \end{gathered}$$

Theorem. *The spectrum of problem* (1)–(3) *consists only of isolated points. The spectral points of problem* (1)–(3) *on* Λ_0 *can lie only on* $G \cup \Lambda_2^- \cup \Lambda_2^+$.

Proof. The second assertion of the theorem follows from Theorem 45 in [4, p. 230]. Note that the values $\beta \in G \cup \Lambda_2^+$ correspond to surface waves (u and v exponentially decay as $r \to \infty$), and the values $\beta \in \Lambda_2^-$ correspond to leaking waves (u and v exponentially grow as $r \to \infty$).

To prove the first assertion of the theorem, we reduce problem (1)–(3) to the spectral problem for a Fredholm holomorphic operator function. Let the contour Γ be specified parametrically: $r = r(t)$, $t \in [0, 2\pi]$. Using the representations of u and v in the domains S_j by single layer potentials with densities $\varphi_j, \psi_j \in C^{0,\alpha}$, respectively ($C^{0,\alpha}$ is the space of Hölder continuous functions), we readily obtain the nonlinear spectral problem

$$\begin{aligned} &A(\beta)z \equiv (C(\beta) + R(\beta))z = 0, \\ &A(\beta): \ H \to H, \qquad H = C^{0,\alpha} \times C^{0,\alpha} \times C^{0,\alpha} \times C^{0,\alpha}, \qquad \beta \in \Lambda, \end{aligned} \tag{4}$$

where the operators R and C are given by the relations

$$\begin{aligned} R(\beta)z = \Big(& L^{-1}R_2^{(1)}(\beta)x_1 + L^{-1}\left(R_1^{(1)}(\beta) - R_2^{(1)}(\beta)\right)x_2, \\ & L^{-1}R_2^{(1)}(\beta)y_1 + L^{-1}\left(R_1^{(1)}(\beta) - R_2^{(1)}(\beta)\right)y_2, \\ & -\omega\varepsilon_2\chi_2^{-2}R_2^{(2)}(\beta)x_1 + \beta\chi_2^{-2}R_2^{(3)}(\beta)y_1 - \omega\left(\varepsilon_1\chi_1^{-2}R_1^{(2)}(\beta) + \varepsilon_2\chi_2^{-2}R_2^{(2)}(\beta)\right)x_2 \\ & + \beta\left(\chi_1^{-2}R_1^{(3)}(\beta) - \chi_2^{-2}R_2^{(3)}(\beta)\right)y_2, \\ & \beta\chi_2^{-2}R_2^{(3)}(\beta)x_1 + \omega\mu_0\chi_2^{-2}R_2^{(2)}(\beta)y_1 + \beta\left(\chi_1^{-2}R_1^{(3)}(\beta) - \chi_2^{-2}R_2^{(3)}(\beta)\right)x_2 \\ & + \omega\mu_0\left(\chi_1^{-2}R_1^{(2)}(\beta) + \chi_2^{-2}R_2^{(2)}(\beta)\right)y_2\Big), \\ C(\beta)z = \Big(& x_1, \ y_1, \ \omega\varepsilon_2\chi_2^{-2}x_1 + \beta\chi_2^{-2}Sy_1 - \omega\left(\varepsilon_1\chi_1^{-2} + \varepsilon_2\chi_2^{-2}\right)x_2 + \beta\left(\chi_1^{-2} - \chi_2^{-2}\right)Sy_2, \\ & \beta\chi_2^{-2}Sx_1 - \omega\mu_0\chi_2^{-2}y_1 + \beta\left(\chi_1^{-2} - \chi_2^{-2}\right)Sx_2 + \omega\mu_0\left(\chi_1^{-2} + \chi_2^{-2}\right)y_2\Big), \end{aligned}$$

$$z = (x_1, y_1, x_2, y_2), \qquad x_1(t) = (\varphi_1(M) - \varphi_2(M))\,|r'(t)|,$$

$$y_1(t) = (\psi_1(M) - \psi_2(M))\,|r'(t)|, \qquad x_2(t) = \varphi_1(M)\,|r'(t)|, \qquad y_2(t) = \psi_1(M)\,|r'(t)|,$$

$$Sx = \frac{1}{2\pi}\int_0^{2\pi} \cot\frac{t_0 - t}{2}x(t_0)\,dt_0 + \frac{i}{2\pi}\int_0^{2\pi} x(t_0)\,dt_0, \qquad Lx = -\frac{1}{2\pi}\int_0^{2\pi} \ln\left|\sin\frac{t - t_0}{2}\right| x(t_0)\,dt_0,$$

$$R_j^{(k)}(\beta)x = \frac{1}{2\pi}\int_0^{2\pi} h_j^{(k)}(\beta; t, t_0)\,x(t_0)\,dt_0, \qquad k = 1,2,3, \qquad j = 1,2,$$

$$\begin{aligned} h_j^{(1)}(\beta; t, t_0) &= 2\pi\Phi_j(\beta; M, M_0) + \ln|\sin((t - t_0)/2)|, \\ h_j^{(2)}(\beta; t, t_0) &= 4\pi\,|r'(t)|\,\partial\Phi_j(\beta; M, M_0)/\partial\nu_M, \\ h_j^{(3)}(\beta; t, t_0) &= 2\,|r'(t)|\,\partial h_j^{(1)}(\beta; t, t_0)/\partial\tau_M - i, \\ \Phi_j(\beta; M, M_0) &= (i/4)H_0^{(1)}(\chi_j\,|M - M_0|). \end{aligned}$$

The operator $L : C^{0,\alpha} \to C^{1,\alpha}$ (where $C^{1,\alpha}$ is the space of Hölder continuously differentiable functions) is continuously invertible [5, p. 10]. The operators $R_j^{(1)}(\beta) : C^{0,\alpha} \to C^{1,\alpha}$ and $R_j^{(k)}(\beta) : C^{0,\alpha} \to C^{0,\alpha}$, $k = 2,3$, $j = 1,2$, are compact for any $\beta \in \Lambda$; consequently, $R(\beta) : H \to H$ is a compact operator. Using the fact that $S : C^{0,\alpha} \to C^{0,\alpha}$ is a compact operator (e.g., see [6, p. 118]), we can readily show that the operator $C(\beta) : H \to H$ is continuously invertible for any $\beta \in \Lambda$. Therefore, $A(\beta)$ is a Fredholm operator function. The functions $h_j^{(k)}(\beta; t, t_0)$, $k = 1,2,3$, $j = 1,2$, are analytic in Λ for each point $(t, t_0) \in [0, 2\pi] \times [0, 2\pi]$. Consequently [7, p. 71], $A(\beta)$ is a holomorphic operator function in Λ.

Let us show that there exist $\beta \in \Lambda$ such that $A(\beta)$ is invertible. To this end, we study the relationship between problems (1)–(3) and (4). Let us introduce the following four problems: find the values of the parameter $\beta \in \Lambda$ such that there exist nontrivial solutions of the Helmholtz equation $\Delta u + \chi_j^2 u = 0$, $M \in S_i$, $i, j = 1, 2$, which are continuous in $\bar{S}_i$, twice continuously differentiable in S_i, satisfy the homogeneous Dirichlet boundary conditions on the contour Γ and, if $M \in S_2$, the partial condition. We denote the interior problems by $D_1^{(j)}$ and the exterior problems by $D_2^{(j)}$. The sets of $\beta \in \Lambda$ for which the problems $D_i^{(j)}$ have nontrivial solutions are denoted by $\sigma\left(D_i^{(j)}\right)$, $i, j = 1, 2$. It is known that the sets $\sigma\left(D_1^{(j)}\right)$ consist only of isolated points lying on the imaginary axis and a closed interval $(-k_0 n_j, k_0 n_j)$ of the real axis. It follows from [8, 9] that the sets $\sigma\left(D_2^{(j)}\right)$, $j = 1, 2$, consist only of isolated points. Moreover, the points of the spectrum $\sigma\left(D_2^{(j)}\right)$ in Λ_0 can lie only on Λ_j^-, $j = 1, 2$. Following [8], we can readily show that if $\beta \in \Lambda \backslash \left(\bigcup_{i=1,2} \sigma\left(D_i^{(j)}\right)\right)$, then an arbitrary solution u, v of problem (1)–(3) can be represented in the domain S_j, $j = 1, 2$, by single layer potentials with kernel Φ_j and densities $\varphi_j, \psi_j \in C^{0,\alpha}$, respectively; furthermore, if a single layer potential vanishes on S_j for some $\beta \in \Lambda \backslash \sigma\left(D_{3-j}^{(j)}\right)$, then its density identically vanishes on Γ. Hence if problem (4) has a nontrivial solution for some $\beta \in \Lambda \backslash \left(\sigma\left(D_2^{(1)}\right) \cup \sigma\left(D_1^{(2)}\right)\right)$, then problem (1)–(3) has a nontrivial solution for the same β. Conversely, if problem (1)–(3) has a nontrivial solution for some $\beta \in \Lambda \backslash \left(\bigcup_{i,j=1,2} \sigma\left(D_i^{(j)}\right)\right)$, then problem (4) has a nontrivial solution for the same β. Therefore, problems (1)–(3) and (4) are equivalent everywhere in Λ except for a discrete set of points. Now it follows from the second assertion of the theorem and the Fredholm property of the operator function $A(\beta)$ that $A(\beta)$ is invertible for $\beta \in \Lambda_0 \backslash \left(G \cup \Lambda_2^- \cup \Lambda_2^+ \cup \sigma\left(D_2^{(1)}\right) \cup \sigma\left(D_1^{(2)}\right)\right)$. Consequently [10], the spectrum of problems (4) and (1)–(3) can consist only of isolated points. The proof of the theorem is complete.

REFERENCES

1. Karchevskii, E.M., *Zh. Vychislit. Mat. Mat. Fiz.*, 1998, vol. 38, no. 1, pp. 136–140.
2. Voitovich, N.N., Katsenelenbaum, B.Z., Sivov, A.N., and Shatrov, A.D., *Radiotekhnika i Elektronika*, 1979, vol. 24, no. 7, pp. 1245–1263.
3. Sveshnikov, A.G., in *Vychislit. metody i programmirovanie* (Numerical Methods and Programming), Moscow, 1969, issue 13, pp. 145–151.
4. Shestopalov, V.P., *Spektral'naya teoriya i vozbuzhdenie otkrytykh struktur* (Spectral Theory and Excitation of Open Structures), Kiev, 1987.
5. Gabdulkhaev, B.G., *Pryamye metody resheniya singulyarnykh integral'nykh uravnenii pervogo roda* (Direct Methods for Singular Integral Equations of the First Kind), Kazan, 1994.
6. Muskhelishvili, N.I., *Singulyarnye integral'nye uravneniya* (Singular Integral Equations), Moscow, 1968.
7. Il'inskii, A.S. and Shestopalov, Yu.V., *Primenenie metodov spektral'noi teorii v zadachakh rasprostraneniya voln* (Application of Methods of Spectral Theory to Wave Propagation Problems), Moscow, 1989.
8. Vekua, I.N., *Tr. Tbil. Mat. In-ta*, 1943, vol. 12, pp. 105–174.
9. Muravei, L.A., *Mat. Sb.*, 1978, vol. 105, no. 1, pp. 63–108.
10. Gokhberg, I.Ts. and Krein, M.G., *Uspekhi Mat. Nauk*, 1957, vol. 12, no. 2, pp. 44–118.