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Block Projection Operators in Normed Solid Spaces of Measurable Operators

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Abstract—We prove a Hermitian analog of the well-known operator triangle inequality for von Neumann algebras. In the semifinite case we show that a block projection operator is a linear positive contraction on a wide class of solid spaces of Segal measurable operators. We describe some applications of the results.

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INTRODUCTION

In this paper we obtain inequalities which improve the operator "triangle inequalities" in the case of Hermitian operators and are their analogs for Jordan algebras. Our results enable one to obtain new proofs of known inequalities for the Hardy—Littlewood—Pólya weak spectral order. We show that formula (3) defines a linear positive contraction $\Phi : \mathcal{X} \to \mathcal{X}$ for normed ideal spaces $\mathcal{X} \subset S(\mathcal{M})$ satisfying conditions (A) and (B). The proof of this fact is based on a new combinatorial lemma (Lemma 2). We improve one result obtained by us in 1998 and generalize the result of F. Hiai and H. Kosaki (1999) for an important particular case of the operator Y in a special form (see Corollary 5).

1. AN INEQUALITY FOR HERMITIAN OPERATORS

Let $\mathcal{M}$ be a von Neumann algebra of operators in a Hilbert space $\mathcal{H}$. We denote its subsets of Hermitian elements, positive elements, and projection operators by $\mathcal{M}^{\mathrm{sa}}$, $\mathcal{M}^{+}$, and $\mathcal{M}^{\mathrm{pr}}$, respectively. Let $|X| = (X^*X)^{1/2}$ for $X \in \mathcal{M}$, let $\mathbb{I}$ be the unit of $\mathcal{M}$, and let $P^{\perp} = \mathbb{I} - P$ for $P \in \mathcal{M}^{\mathrm{pr}}$.

Theorem 1. *Let $\mathcal{M}$ be a von Neumann algebra and let $X \in \mathcal{M}^{\mathrm{sa}}$ and $Y \in \mathcal{M}^{+}$ be such that $-Y \le X \le Y$. Then $2|X| \le Y + UYU$ for some unitary $U \in \mathcal{M}^{\mathrm{sa}}$.*

Sketch of Proof. Let $X \in \mathcal{M}^{\mathrm{sa}}$ and let $X = X_{+} - X_{-}$ be its Jordan decomposition into positive and negative parts. Then $X_{+}, X_{-} \in \mathcal{M}^{+}$, $P := s_r(X_{+}) + s_r(X)^{\perp} \in \mathcal{M}^{\mathrm{pr}}$, and $PX_{-} = P^{\perp}X_{+} = 0$, $P^{\perp}X_{-} = X_{-}$. Here $s_r(Z)$ is the support of the operator $Z \in \mathcal{M}^{\mathrm{sa}}$. For the pair $\{-X, Y\}$ we have $-Y \le -X \le Y$, hence

$$-PYP \le PXP \le PYP, \quad -P^{\perp}YP^{\perp} \le -P^{\perp}XP^{\perp} \le P^{\perp}YP^{\perp}. \tag{1}$$

Since $U = P - P^{\perp} \in \mathcal{M}^{\mathrm{sa}}$ is unitary and $|X| = UX = XU = PXP - P^{\perp}XP^{\perp}$, summing up the inequalities in (1) termwise, we get

$$|X| \le PYP + P^{\perp}YP^{\perp} = \frac{Y + UYU}{2}.$$

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Corollary 1. If $\mathcal{M}$ is a von Neumann algebra, then for any $A, B \in \mathcal{M}^{\rm sa}$ there exists a unitary operator $U \in \mathcal{M}^{\rm sa}$ such that

$$|A+B| \leq \frac{|A|+|B|+U(|A|+|B|)U}{2}. \tag{2}$$

Sketch of Proof. The inequalities $-|A| \leq A \leq |A|$ and $-|B| \leq B \leq |B|$ give

$$-|A|-|B| \leq A+B \leq |A|+|B|.$$

Applying Theorem 1 to the operators

$$X = A+B \in \mathcal{M}^{\rm sa}, \quad Y = |A|+|B| \in \mathcal{M}^{+},$$

we obtain formula (2). □

Corollary 2. Let τ be an arbitrary trace on a von Neumann algebra $\mathcal{M}$ and $X, Y \in \mathcal{M}^{\rm sa}$. Then $\tau(|X+Y|) \leq \tau(|X|) + \tau(|Y|)$. If $Y \in \mathcal{M}^{+}$ and $-Y \leq X \leq Y$, then there exists $Z \in \mathcal{M}^{+}$ such that $|X| \leq Z$ and $\tau(Y) = \tau(Z)$.

Remark 1. An analog of formula (2) holds true for any finite collection of operators from $\mathcal{M}^{\rm sa}$. Inequality (2) is valid for arbitrary (not necessarily Hermitian) operators if and only if the algebra $\mathcal{M}$ is commutative.

2. IDEAL SPACES ON SEMIFINITE VON NEUMANN ALGEBRAS

Let τ be a faithful normal semifinite trace on $\mathcal{M}$. The set $S(\mathcal{M})$ of all measurable operators is an $*$-algebra with respect to transition to the adjoint operator, multiplication by a scalar value, and operations of strong addition and multiplication obtained as closures of usual ones [1]. We denote the subset of positive operators and that of Hermitian ones of a family $\mathcal{K} \subset S(\mathcal{M})$ by $\mathcal{K}^{+}$ and $\mathcal{K}^{\rm sa}$, respectively. We denote the partial order in $S(\mathcal{M})^{\rm sa}$ generated by the proper cone $S(\mathcal{M})^{+}$ as $\leq$; the expression $X_i \uparrow X$ means that $X_i \leq X_j$ for $i \leq j$ and $X = \sup\limits_i X_i$. If $X \in S(\mathcal{M})$, then $|X| = (X^*X)^{1/2} \in S(\mathcal{M})^{+}$. Let $L_1(\mathcal{M}, \tau)$ be the Banach space of all τ-integrable operators [1] from $S(\mathcal{M})$ with the norm $\|X\|_1 = \tau(|X|)$.

One can extend statements of Theorem 1 and Corollaries 1 and 2 (with analogous proofs) to the $*$-algebra $S(\mathcal{M})$ of measurable operators. In particular, our results enable one to obtain new proofs of the inequalities known earlier for the Hardy–Littlewood–Pólya weak spectral order (see lemma 1 in [2] and proposition 2.1 in [3]). Theorem 1 and Corollaries 1 and 2 were announced in [4]. They improve the operator "triangle inequalities" [5, 6] in case of Hermitian operators and are their analogs for Jordan algebras. The above said implies the following assertion.

Corollary 3 (cf. [5], [6]). For each finite collection $\{X_k\}_{k=1}^{n}$ in $S(\mathcal{M})^{\rm sa}$ there exists a unitary operator $U \in \mathcal{M}^{\rm sa}$ such that

$$|X_1 + \cdots + X_n| \leq \frac{|X_1| + \cdots + |X_n| + U(|X_1| + \cdots + |X_n|)U}{2}.$$

Definition. A normed space $\mathcal{E} \subset S(\mathcal{M})$ is called a *normed solid space* (NSS) on $(\mathcal{M}, \tau)$, if

(i) $X \in \mathcal{E} \Rightarrow X^* \in \mathcal{E}$ and $\|X^*\|_{\mathcal{E}} = \|X\|_{\mathcal{E}}$;

(ii) $X \in S(\mathcal{M})$, $Y \in \mathcal{E}$, while $|X| \leq |Y| \Rightarrow X \in \mathcal{E}$ and $\|X\|_{\mathcal{E}} \leq \|Y\|_{\mathcal{E}}$.

Corollary 4. Let $\mathcal{M}$ be a von Neumann algebra with a faithful normal semifinite trace τ, let $\mathcal{E}$ be an NSS on $(\mathcal{M}, \tau)$, and $A, B, C, X, Y \in S(\mathcal{M})^{\rm sa}$. If $A, C \in \mathcal{E}$ and $A \leq B \leq C$, then $B \in \mathcal{E}$ and $\|B\|_{\mathcal{E}} \leq \big\| \, |A| + |C| \, \big\|_{\mathcal{E}}$. If $Y \in \mathcal{E}^{+}$ and $-Y \leq X \leq Y$, then $X \in \mathcal{E}$ and $\|X\|_{\mathcal{E}} \leq \|Y\|_{\mathcal{E}}$.

Sketch of Proof. We have $-|A| \le A \le B \le C \le |C|$, therefore,

$$-|A| - |C| \le B \le |A| + |C|.$$

There exists (cf. Theorem 1) a unitary operator $U \in \mathcal{M}^{\mathrm{sa}}$ such that

$$|B| \le \frac{|A| + |C| + U(|A| + |C|)U}{2}.$$

Consequently,

$$2\|B\|_{\mathcal{E}} \le \big\| \, |A| + |C| + U(|A| + |C|)U \big\|_{\mathcal{E}} \le 2\big\| \, |A| + |C| \, \big\|_{\mathcal{E}}.$$

In [7] with the help of one nontrivial inequality from [8] one shows that the block projection operator

$$\Phi(X) = \sum_{k=1}^{\infty} P_k X P_k, \quad \{P_k\}_{k=1}^{\infty} \subset \mathcal{M}^{\mathrm{pr}}, \quad P_k P_m = 0 \ (k \ne m), \tag{3}$$

is a positive linear contraction in $\mathcal{M}$ and $L_1(\mathcal{M}, \tau)$. This fact was used for the description of extreme points of convex completely symmetric subsets in $L_1(\mathcal{M}, \tau) + \mathcal{M}$. From the theory of interpolation of linear operators it follows that formula (3) defines a positive linear continuous operator $\Phi : \mathcal{X} \to \mathcal{X}$ for all Banach symmetric subspaces $\mathcal{X} \subset S(\mathcal{M})$ which are interpolation spaces between $L_1(\mathcal{M}, \tau)$ and $\mathcal{M}$.

In this paper we show that formula (3) also defines a linear positive contraction $\Phi : \mathcal{X} \to \mathcal{X}$ for a wide class of normed subspaces $\mathcal{X} \subset S(\mathcal{M})$. The method is new even for the algebra $\mathcal{M} = \mathcal{B}(\mathcal{H})$ equipped with the canonical trace $\tau = \mathrm{tr}$. In this case the operator Φ was studied in [9] (Chap. II, § 5; Chap. III, theorem 4.2; § 7, 6°; theorem 8.7).

We say ([10], Chap. IV, § 3) that in an NSS $\mathcal{E}$

— the norm is *orderly continuous* or condition (A) is fulfilled, if $X_n \downarrow 0 \ \Rightarrow \ \|X_n\|_{\mathcal{E}} \to 0$;

— the norm is *monotonically complete* or condition (B) is fulfilled, if $0 \le X_n \uparrow$, $X_n \in \mathcal{E}$ $(n \in \mathbb{N})$, $\sup\limits_n \|X_n\|_{\mathcal{E}} < \infty \ \Rightarrow \ \exists X \in \mathcal{E} : X_n \uparrow X$.

Theorem 2. *Let $\langle \mathcal{E}, \| \cdot \|_{\mathcal{E}} \rangle$ be an NSS on $(\mathcal{M}, \tau)$ satisfying conditions* (A) *and* (B). *Then formula* (3) *defines a linear positive continuous operator $\Phi : \mathcal{E} \to \mathcal{E}$ with $\|\Phi\|_{\mathcal{E} \to \mathcal{E}} \le 1$.*

The proof of this theorem is based on two lemmas. It is easy to prove the next assertion.

Lemma 1. *Let $\langle \mathcal{E}, \| \cdot \|_{\mathcal{E}} \rangle$ be an NSS on $(\mathcal{M}, \tau)$, $X \in \mathcal{E}$, and $V, W \in \mathcal{M}$. Then $VXW \in \mathcal{E}$ and $\|VXW\|_{\mathcal{E}} \le \|V\| \, \|W\| \, \|X\|_{\mathcal{E}}$.*

Lemma 2. *Let $\mathcal{A}$ be an arbitrary algebra and $X, V_k, Y_k \in \mathcal{A}$, $k \in \mathbb{N}$. Let*

$$S_n = \sum_{k=1}^{n} V_k X Y_k, \ \ n \in \mathbb{N}.$$

Then for each $n \in \mathbb{N}$ there exist 2^{n-1} collections $t_j^{(n)} = \{t_{jk}^{(n)}\}_{k=1}^{n}$ with $t_{jk}^{(n)} \in \{-1, 1\}$, $k = \overline{1, n}$, $j = \overline{1, 2^{n-1}}$, such that

$$2^{n-1} S_n = \sum_{j=1}^{2^{n-1}} H_j^{(n)} X Z_j^{(n)}, \quad H_j^{(n)} \equiv \sum_{k=1}^{n} t_{jk}^{(n)} V_k, \quad Z_j^{(n)} \equiv \sum_{k=1}^{n} t_{jk}^{(n)} Y_k. \tag{4}$$

Proof. Let us prove the lemma by induction. For $n = 1$ we have $t_1^{(1)} = \{1\}$. Let 2^{n-1} collections $t_j^{(n)}$ of the length n satisfy condition (4). The step of induction consists of obtaining two new collections of the length $n + 1$ from each $t_j^{(n)}$ by attaching -1 and 1, respectively, at the $(n + 1)$st position and using the equality $2^n S_{n+1} = 2 \cdot 2^{n-1} S_n + 2^n V_{n+1} X Y_{n+1}$. The summands in the form $V_{n+1} X Y_k$ or $V_k X Y_{n+1}$, $k = \overline{1, n}$, emerge in pairs and with the opposite signs. ∎

Sketch of Proof of Theorem 2. Let $X \in \mathcal{E}$, then $X^* \in \mathcal{E}$ in view of item (i) in the definition of an NSS. Thus, $\mathrm{Re}\, X, \mathrm{Im}\, X \in \mathcal{E}^{\mathrm{sa}}$. If $Z \in \mathcal{E}^{\mathrm{sa}}$ and $Z = Z_+ - Z_-$, then $|Z| = Z_+ + Z_-$ and $Z_+, Z_- \in \mathcal{E}^+$ in view of item (ii). Thus, the following representation takes place:

$$X = X_1 - X_2 + iX_3 - iX_4, \quad X_m \in \mathcal{E}^+, \ m = \overline{1,4}, \ i \in \mathbb{C}, \ i^2 = -1.$$

It suffices to prove the $\|\cdot\|_{\mathcal{E}}$-convergence of the series from (3) for $X \in \mathcal{E}^+$. By Lemma 2 with $V_k = Y_k = P_k$ for each $n \in \mathbb{N}$ there exist 2^{n-1} collections $t_j^{(n)} = \{t_{jk}^{(n)}\}_{k=1}^n$ with $t_{jk}^{(n)} \in \{-1, 1\}$, $k = \overline{1,n}$, $j = \overline{1, 2^{n-1}}$, satisfying condition (4). Since $|Z_j^{(n)}| = \sum\limits_{k=1}^n P_k$, we have $\|Z_j^{(n)}\| \le 1$ and

$$\|2^{n-1} S_n\|_{\mathcal{E}} \le \sum_{j=1}^{2^{n-1}} \|Z_j^{(n)} X Z_j^{(n)}\|_{\mathcal{E}} \le \sum_{j=1}^{2^{n-1}} \|X\|_{\mathcal{E}} = 2^{n-1} \|X\|_{\mathcal{E}} \tag{5}$$

by the triangle inequality and Lemma 1. Dividing (5) by 2^{n-1}, we get $\|S_n\|_{\mathcal{E}} \le \|X\|_{\mathcal{E}}$, $n \in \mathbb{N}$. Obviously, $0 \le S_n \uparrow$. Since the norm in the NSS X is monotonically complete, there exists $S \in \mathcal{E}$ such that $S_n \uparrow S$ $(= \Phi(X))$. Since $S - S_n \downarrow 0$, in view of condition (A) we get $\|S - S_n\|_{\mathcal{E}} \to 0$ as $n \to \infty$. Thus, for $X \in \mathcal{E}^+$ we have $\Phi(X) \in \mathcal{E}^+$ and

$$\|\Phi(X)\|_{\mathcal{E}} \le \|X\|_{\mathcal{E}}. \tag{6}$$

Since correlation (5) is valid for arbitrary $X \in \mathcal{E}$, by the above reasoning inequality (6) is also valid for such $X \in \mathcal{E}$. Thus, $\|\Phi\|_{\mathcal{E} \to \mathcal{E}} \le 1$.

Corollary 5. Let τ be a faithful normal semifinite trace on a von Neumann algebra $\mathcal{M}$. Assume that $X \in S(\mathcal{M})^+$, $Y = \sum\limits_{k=1}^{\infty} b_k P_k \in S(\mathcal{M})^{\mathrm{sa}}$, $b_k \in \mathbb{R}$, and the series converges τ-almost everywhere. If $\mathrm{Re}\,(XY) \in L_1(\mathcal{M}, \tau)$, then $X^{1/2} Y X^{1/2} \in L_1(\mathcal{M}, \tau)$ and $\|X^{1/2} Y X^{1/2}\|_1 \le \|\mathrm{Re}\,(XY)\|_1$.

Sketch of Proof. We have $\tau(P_k X P_k) = \tau(X^{1/2} P_k X^{1/2})$ for all $k \in \mathbb{N}$,

$$\Phi(\mathrm{Re}\,(XY)) = \sum_{k=1}^{\infty} b_k\, P_k X P_k, \quad |\Phi(\mathrm{Re}\,(XY))| = \sum_{k=1}^{\infty} |b_k|\, P_k X P_k,$$

$X^{1/2} Y X^{1/2} = \sum\limits_{k=1}^{\infty} b_k X^{1/2} P_k X^{1/2}$, and the series $\|\cdot\|_1$-converge. Therefore,

$$\begin{aligned}\|X^{1/2} Y X^{1/2}\|_1 &= \lim_{n\to\infty} \left\| \sum_{k=1}^{n} b_k\, X^{1/2} P_k X^{1/2} \right\|_1 \le \lim_{n\to\infty} \left\| \sum_{k=1}^{n} |b_k|\, X^{1/2} P_k X^{1/2} \right\|_1 \\ &= \lim_{n\to\infty} \left\| \sum_{k=1}^{n} |b_k|\, P_k X P_k \right\|_1 = \|\Phi(\mathrm{Re}\,(XY))\|_1 \le \|\mathrm{Re}\,(XY)\|_1.\end{aligned}$$

Remark 2. Corollary 5 improves theorem 3 in [11] and generalizes the result obtained in [12] for an important particular case of the operator Y (not necessarily bounded) in the indicated form. Formula (3) defines a linear positive contraction $\Phi : \mathcal{Z} \to \mathcal{Z}$ for all Banach subspaces $\mathcal{Z} \subset S(\mathcal{M})$ which are the interpolation spaces between two NSS $\mathcal{X}$ and $\mathcal{Y}$ on $(\mathcal{M}, \tau)$ satisfying conditions (A) and (B).

Remark 3. Let $\mathcal{M}$ be an arbitrary von Neumann algebra, $\{P_k\}_{k=1}^n \subset \mathcal{M}^{\mathrm{pr}}$ with $\sum\limits_{k=1}^n P_k = \mathbb{I}$. Then

$$\mathcal{N} = \left\{ \sum_{k=1}^{n} P_k X P_k : X \in \mathcal{M} \right\}$$

is a von Neumann subalgebra on $\mathcal{M}$ and the expression

$$\Phi(X) = \sum_{k=1}^{n} P_k X P_k \ \ (X \in \mathcal{M})$$

defines the conditional expectation of $\Phi : \mathcal{M} \to \mathcal{N}$.

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