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STRUCTURAL MECHANICS AND STRENGTH OF FLIGHT VEHICLES

Stability of a Cylindrical Shell under Axial Compression

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Abstract—A new approach to solving the task of stability for a circular cylindrical shell under axial compression is presented, taking into account the dynamic buckling. The modification of the algorithm for solving the known problem by the Ritz method is also proposed that allows investigating the effect of geometrical factors on the buckling load.

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INTRODUCTION

Shell structures are widely used in contemporary aeronautical engineering. An important practical problem is to study the stability of a closed circular cylindrical shell that is compressed along the generatrix by the loads p uniformly distributed along the arc edges. Despite the large number of solutions of this problem presented in the literature (the first studies refer to the beginning of the 20th century), some questions still remain open. As noted in [1], this case of loading is the most indicative of the inconsistency of the theoretical solutions for most experiments. In particular, the problem of the influence of geometrical parameters on the critical load has still remained unsolved.

In this paper, two approaches to this problem are proposed. The first approach is based on the known solution of nonlinear equations by the Ritz method. Modification of the algorithm makes it possible to take into account the effect of geometric factors. The second approach is based on analyzing the shell stability with taking into account the dynamic factors succeeding the process of buckling. It brings the theoretical setting of the problem to pilot studies.

INFLUENCE OF GEOMETRICAL FACTORS ON THE BUCKLING LOAD

The problem considered in the nonlinear formulation is reduced to solving a system of equations [2]

$$\frac{1}{E}\nabla^4\Phi+\frac{1}{2}L(w,w)+\frac{1}{R}\frac{\partial^2 w}{\partial x^2}=0,\quad D\nabla^4 w-hL(w,\Phi)-\frac{h}{R}\frac{\partial^2\Phi}{\partial x^2}=0, \tag{1}$$

where w, Φ are the functions of deflection and forces to be determined; h, R are the shell radius and thickness; $D=Eh^3/12(1-\nu^2)$ is the flexural rigidity; E, ν are the elastic modulus and Poisson's ratio; $\nabla^4=\frac{\partial^4}{\partial x^4}+\ +2\frac{\partial^4}{\partial x^2\partial y^2}+\frac{\partial^4}{\partial y^4}$; $L(w,\Phi)=\frac{\partial^2 w}{\partial x^2}\frac{\partial^2\Phi}{\partial y^2}-2\frac{\partial^2 w}{\partial x\partial y}\frac{\partial^2\Phi}{\partial x\partial y}+\frac{\partial^2\Phi}{\partial x^2}\frac{\partial^2 w}{\partial y^2}$.

To integrate system (1) we use the Ritz method, choosing the approximating function of deflection in the form

$$w(x,y)=f_1\sin\frac{m\pi x}{L}\sin\frac{ny}{R}+f_2\sin^2\frac{m\pi x}{L}+f_0\,, \tag{2}$$

where m, n are the wave numbers; L is the generatrix length. The first term in (2) corresponds to the solution of the linearized problem. The second term reflects the asymmetry of the deflection with respect to the middle surface of the predominant direction of the center of curvature; f_0 is the term corresponding to the radial displacements of the end points of the cross sections associated with the amplitudes f_1 and f_2 by the formula $\frac{f_0}{R} = -\frac{\nu p}{E} + f_1^2 \frac{n^2}{8R^2} - \frac{f_2}{2R}$ derived in [2] from the periodicity condition of the arc displacement v. Experiments show that the choice of function (2) describes rather reliably the form of dents observed after buckling.

With regard to (2) the formula for the total energy of the system is written as

$$\tilde{\exists} = -\tilde{p}^2 - \tilde{p}\left(b_1\zeta_1^2 + b_2\zeta_2^2\right) + b_3\zeta_1^2 + b_4\zeta_2^2 + b_5\zeta_1^4 + b_6\zeta_2\zeta_1^2 + b_7\zeta_1^2\zeta_2^2, \quad (3)$$

where we introduce the following notation

$$b_1 = \frac{\eta\theta^2}{4};\ b_2 = \frac{\eta\theta^2}{2};\ b_3 = \frac{\eta^2 s_1^2}{48\left(1-\nu^2\right)} + \frac{\theta^4}{4s_1^2};\ b_4 = \frac{1}{8} + \frac{\eta^2\theta^4}{6\left(1-\nu^2\right)};\ b_5 = \frac{\eta^2\left(1+\theta^4\right)}{128};$$

$$b_6 = -\frac{\eta}{16}\left(1 + \frac{8\theta^4}{4s_1^2}\right);\ b_7 = \frac{\eta^2\theta^4}{4}\left(\frac{1}{s_1^2} + \frac{1}{s_2^2}\right);\ \theta = \frac{m\pi R}{nL};\ \eta = \frac{hn^2}{R};$$

$$s_1 = 1+\theta^2;\ s_2 = 1+9\theta^2;\ \exists = \tilde{\exists}\frac{Eh^3L\pi}{R};\ \tilde{p} = \frac{pR}{Eh};\ \zeta_1 = \frac{f_1}{h};\ \zeta_2 = \frac{f_2}{h}.$$

The Ritz method leads to a system of equations

$$\frac{\partial\tilde{\exists}}{\partial\zeta_1} = 0;\ \frac{\partial\tilde{\exists}}{\partial\zeta_2} = 0. \quad (4)$$

From the first equation (4) we find the dimensionless load, and from the second—the expression for the square of the amplitude ζ_1: $\tilde{p} = \frac{b_3 + 2b_5\zeta_1^2 + b_6\zeta_2 + b_7\zeta_2^2}{b_1}$, $\zeta_1^2 = \frac{2\left(\tilde{p}b_2 - b_4\right)\zeta_2}{b_6 + 2b_7\zeta_2}$, whence

$$\tilde{p} = \frac{b_3b_6 + \left(2b_3b_7 + b_6^2 - 4b_4b_5\right)\zeta_2 + 3b_6b_7\zeta_2{}^2 + 2b_7{}^2\zeta_2^3}{b_1b_6 + \left(2b_1b_7 - 4b_2b_5\right)\zeta_2}. \quad (5)$$

By adding the condition of minimum load to Eq. (5), we obtain the cubic equation with respect to the amplitude ζ_2, by solving which and substituting the roots found into formula (5), we find the dimensionless critical load $\tilde{p}_{cr}$. Calculations show that this quantity is essentially dependent on the wave parameters η and θ entering the coefficients of Eq. (3). The known results [1, 2] were obtained by minimizing the critical load with respect to two parameters mentioned that leads to the values of $\tilde{p}_{cr}$, which not depend on the shell geometry. This is in disagreement with the experimental data.

The expressions, derived by several authors [1] based on the processing results of extensive testing, show that the critical load value depends on both the relative thickness of the shell (parameter R/h) and on its relative length (parameter L/R). For example, the following expression is proposed in [1]

$$\tilde{p}_k = \frac{1}{\sqrt{3\left(1-\nu^2\right)}}\left[3.87\sqrt{h/R} + 10^{-3}\sqrt{R/h} + 1.46\left(R/L\right)^2\left(h/R\right)\right]. \quad (6)$$

As is noted formula (6) as well as other similar empirical correlations is approximate in nature and it is desirable to update it using the more accurate theoretical calculations.

Note that the parameters η and θ are not independent quantities, since they contain the same wave number n. Consequently, it is impossible to minimize it with respect to these two parameters. The authors suggest the following calculation algorithm. By fixing the wave number m, we express the wave number

n through the parameter η and substitute this value into θ. The load $\tilde{p}$ is minimized with respect to the parameter η. Carrying out calculations for different values of m, we choose such a value that corresponds to the lowest critical load. Figure 1 shows the graphs obtained by analyzing a shell with the parameters $R/h = 400$, $L/R = 3$. It can be seen that in this case, the minimum load is attained at $m = 4$.

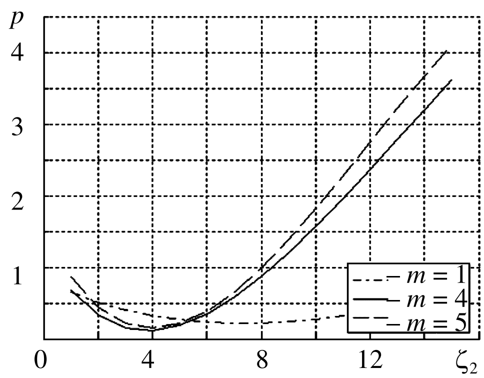

Fig. 1. Load versus deflection amplitude.

These results obtained show that the value $\tilde{p}_k$ depends on the shell geometry. This conclusion agrees well with the experimental data and formula (6). Table presents the results of calculations for different values of geometric parameters. The table shows that the influence of the shell geometry on the value of $\tilde{p}_k$ decreases with increase of relations R/h, L/h and tends to the value $\tilde{p}_k = 0.1143$, which is obtained by minimizing the load with respect to two parameters η and θ.

Table

L/R	Critical parameters	R/h			
		100	150	400	1000
1	m	1	1	1	4
	n	4	4	7	13
	$\tilde{p}_k$	0.2125	0.1439	0.1179	0.1143
2	m	1	1	2	4
	n	4	4	7	13
	$\tilde{p}_k$	0.1378	0.1297	0.1179	0.1143
3	m	2	2	4	6
	n	4	5	8	13
	$\tilde{p}_k$	0.1156	0.1155	0.1154	0.1143
5	m	3	4	6	10
	n	4	8	8	13
	$\tilde{p}_k$	0.1149	0.1148	0.1143	0.1143

SOLUTION OF THE PROBLEM WITH TAKING INTO ACCOUNT THE DYNAMIC PROCESSES

Analysis of supercritical strains of the shell [2] shows that the process of transition from one to the other equilibrium form is not static, namely, the shell moves from the initial shape to the curved dynamically stable one by a snap. It usually entails the emergence of large plastic strains and leads to the loss of load-carrying capacity of the shell. In view of this, let us propose the problem solution with taking into account the dynamic processes.

To obtain the equation of motion of the shell, let us use the Ostrogradskii–Hamilton principle [3]

$$\delta\int_0^t L\,dt = 0, \tag{7}$$

where the Lagrange function is $L = K - \exists$, K is the kinetic energy; $\exists$ is the potential energy, for which expression (3) is obtained; t is the time.

The value K will be found by the following formula

$$K = \frac{\rho h}{2}\iint\left(\frac{\partial w}{\partial t}\right)^2 dxdy.$$

Taking into account expressions (2) and designations accepted earlier we can obtain

$$\tilde{K} = \frac{1}{4}\left(\frac{R}{V}\right)^2\left(\dot{\zeta}_1^2 + \frac{1}{2}\dot{\zeta}_2^2 + \frac{\eta^2}{4}\dot{\zeta}_1^2\zeta_1^2\right), \tag{8}$$

where $K = \tilde{K}\dfrac{\pi L E h^3}{R}$; V is the velocity of sound in the shell material.

From Eq. (7) we obtain a system of Lagrange equations [4]

$$\frac{d}{dt}\left(\frac{\partial\tilde{K}}{\partial\dot{\zeta}_1}\right) - \frac{\partial\tilde{K}}{\partial\zeta_1} + \frac{\partial\tilde{\exists}}{\partial\zeta_1} = 0;\quad \frac{d}{dt}\left(\frac{\partial\tilde{K}}{\partial\dot{\zeta}_2}\right) - \frac{\partial\tilde{K}}{\partial\zeta_2} + \frac{\partial\tilde{\exists}}{\partial\zeta_2} = 0, \tag{9}$$

and the conditions at the initial moment of time

$$\zeta_1(0) = \zeta_{10};\ \zeta_2(0) = \zeta_{20};\ \dot{\zeta}_1(0) = \dot{\zeta}_2(0) = 0 \tag{10}$$

and at the moment of buckling $t = t_k$

$$\dot{\zeta}_1(t_k) = \dot{\zeta}_2(t_k) = 0. \tag{11}$$

Conditions (11) correspond to the dynamic stability criterion proposed by A.V. Sachenkov [5].

Equations (9), taking into account Eqs. (3) and (8), take the following form

$$\begin{gathered}\frac{d^2\zeta_1}{d\tau^2} + \frac{16\zeta_1}{4+\eta^2\zeta_1^2}\left(-\tilde{p}b_1 + b_3 + b_6\zeta_2 + 2b_5\zeta_1^2 + b_7\zeta_2^2 + \frac{\eta^2}{16}\dot{\zeta}_1^2\right) = 0;\\ \frac{d^2\zeta_2}{d\tau^2} + 4\left(-2\tilde{p}b_2\zeta_2 + 2b_4\zeta_2 + b_6\zeta_1^2 + 2b_7\zeta_1^2\zeta_2\right) = 0,\end{gathered} \tag{12}$$

where $\tau = tV/R$ is the dimensionless time parameter.

Numerical solution of (12) with the initial conditions $\zeta_{10} = 0.01$; $\zeta_{20} = 0$ shows that at low loads ($\tilde{p} < \tilde{p}_k$) the shell is equilibrium around the initial position (see Fig. 2a).

When the load increases to a value of $\tilde{p}_k$, there is a sharp increase in the amplitude of deflection (there is the stability loss of motion according to A.M. Lyapunov) (see Fig. 2b).

The calculations show that the value of the critical load $\tilde{p}_k$ essentially depends on the choice of the initial value ζ_{10} that complicates the quantitative analysis of the results obtained. The method being proposed for solving system (12) allows obtaining the value of $\tilde{p}_k$ regardless of the initial deflection.

From the static analogue [3] of the first equation of system (12) we find

$$\zeta_1^2 = \frac{\tilde{p}b_1 - b_3 - b_6\zeta_2 - b_7\zeta_2^2}{2b_5}.$$

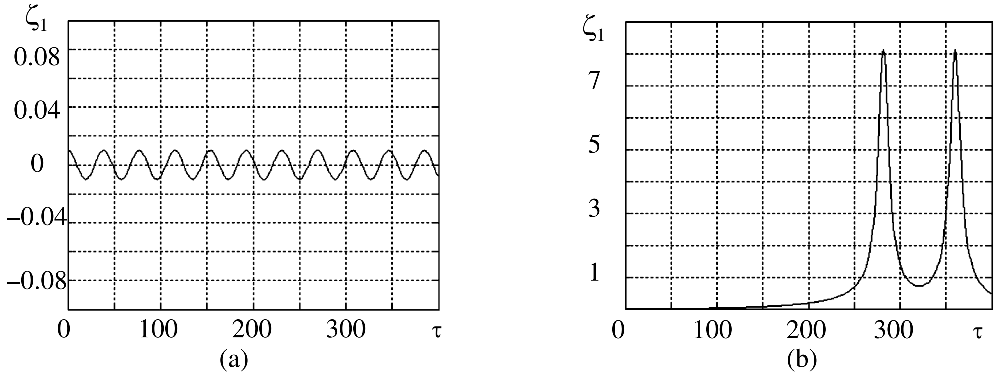

Fig. 2. The deflection amplitude at low loads (a) and at critical load (b).

By substituting the value found into the second equation, we obtain

$$\frac{d^2\zeta_2}{d\tau^2} = A_0 + A_1\zeta_2 + A_2\zeta_2^2 + A_3\zeta_2^3, \tag{13}$$

where the coefficients A_k depend on the parameters of the load $\tilde{p}$ and parameters b_i.

By supposing that $\zeta_{20} = 0$, we multiply Eq. (13) by $\frac{d\zeta_2}{d\tau}$ and integrate both sides of the equation from 0 to τ_k:

$$\dot{\zeta}_2^2(\tau_k) - \dot{\zeta}_2^2(0) = 2A_0\zeta_2 + A_1\zeta_2^2 + 2A_2\frac{\zeta_2^3}{3} + A_3\frac{\zeta_2^4}{2}$$

According to conditions (10)–(11) the left-hand side of the equation is zero. We obtain a cubic equation for the function $\zeta_2(\tilde{p})$. By adding to it the condition for the minimum load ζ_2, we find the solution for a fixed value of the wave parameter η. Figure 3 presents the dependence of $\tilde{p}$ on η for the shell with the parameters *R*/*h* = 150, *L*/R = 1, *m* = 1. The minimum value of $\tilde{p}_k = 0.26$ is reached at η = 0.29.

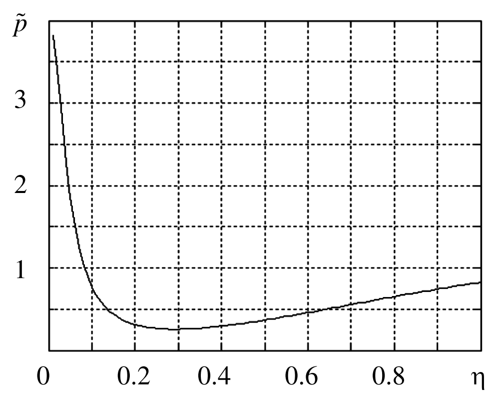

Fig. 3. Critical load versus the wave parameter.

Thus, accounting for the dynamics of buckling, yields results in good agreement with the experimental data (according to experiments [2], the load value $\tilde{p}_k$ belongs to an interval (0.23, 0.35)). In addition, it is possible to trace the pattern of stability loss (see Fig. 2).

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