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Characterizations of Tracial Functionals on C^*-Algebras

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Abstract:	We establish several characterizations of tracial functionals φ on the finite C^*-algebra $\mathbb{M}_n$ (that is, $\varphi =k\, \text{\rm tr}$ for some number $k>0$) via any one of the inequalities $\varphi(A^{\alpha}B^{1-\alpha}) \leq \alpha \,\varphi(A)+ (1-\alpha)\,\varphi(B)$ and $\varphi(\|e^A\|)\leq\varphi(e^{\text{\rm Re}A})$, which are well-known when $\varphi=\tr$. In addition, we characterize the trace on $\mathbb{M}_n$ among all positive linear functionals φ with $\varphi(I)=n$ through an inequality for determinant. We also establish that such a functional is equal to the usual trace if $\varphi \left(\left(Y^{\frac{1}{2}}XY^{\frac{1}{2}}\right)^m\right) \leq \varphi(X)^m\varphi(Y)^m$ for some positive integer m and all $X,Y\in \mathbb{M}_n^+$. Furthermore, we show that there is no state φ on $\mathbb{M}_n, \,\, n\geq 2$ such that $\varphi(B^{1/2}AB^{1/2})\leq \varphi(A)\varphi(B)$ for all $A, B\in \mathbb{M}_n^+$. Finally, we establish that for a positive linear functional φ on a C^*-algebra $\mathcal{A}$, the following conditions are equivalent: (i) φ is tracial; (ii) $\varphi(e^{AB}-I) \geq 0 $ for all $A, B\in \mathcal{A}^+$. A new criterion for the commutativity of C^*-algebras is also provided.

CHARACTERIZATIONS OF TRACIAL FUNCTIONALS ON C^*-ALGEBRAS

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ABSTRACT. We establish several characterizations of tracial functionals φ on the finite C^*-algebra $\mathbb{M}_n$ (that is, $\varphi = k\,\mathrm{tr}$ for some number $k > 0$) via any one of the inequalities $\varphi(A^\alpha B^{1-\alpha}) \leq \alpha\,\varphi(A) + (1-\alpha)\,\varphi(B)$ and $\varphi(|e^A|) \leq \varphi(e^{\mathrm{Re}A})$, which are well-known when $\varphi = \mathrm{tr}$.

In addition, we characterize the trace on $\mathbb{M}_n$ among all positive linear functionals φ with $\varphi(I) = n$ through an inequality for determinant. We also establish that such a functional is equal to the usual trace if $\varphi\left(\left(Y^{\frac{1}{2}}XY^{\frac{1}{2}}\right)^m\right) \leq \varphi(X)^m\varphi(Y)^m$ for some positive integer m and all $X, Y \in \mathbb{M}_n^+$. Furthermore, we show that there is no state φ on $\mathbb{M}_n$, $n \geq 2$ such that $\varphi(B^{1/2}AB^{1/2}) \leq \varphi(A)\varphi(B)$ for all $A, B \in \mathbb{M}_n^+$.

Finally, we establish that for a positive linear functional φ on a C^*-algebra $\mathcal{A}$, the following conditions are equivalent: (i) φ is tracial; (ii) $\varphi(e^{AB} - I) \geq 0$ for all $A, B \in \mathcal{A}^+$. A new criterion for the commutativity of C^*-algebras is also provided.

1. INTRODUCTION

Let $\mathcal{B}(\mathcal{H})$ denote the C^*-algebra of all bounded linear operators on a complex Hilbert space $\mathcal{H}$. It can be identified with the full matrix algebra $\mathbb{M}_n$ of all complex matrices when $\dim\mathcal{H} = n$. For a C^*-algebra $\mathcal{A}$, let $\mathcal{A}^{\mathrm{sa}}$, $\mathcal{A}^+$, and $\mathcal{A}^{\mathrm{pr}}$ denote its self-adjoint part, positive cone and lattice of projections, respectively. If $Z \in \mathcal{A}$, then we denote the modulus of Z by $|Z| := (Z^*Z)^{\frac{1}{2}}$. For $P, Q \in \mathcal{A}^{\mathrm{pr}}$, we write $P \sim Q$ (the Murray–von Neumann equivalence) if $P = U^*U$ and $Q = UU^*$ for some $U \in \mathcal{A}$. If P is a projection in a von Neumann algebra $\mathcal{M}$, then we denote $\mathcal{M}_P = \{PX|_{P\mathcal{H}} :\ X \in \mathcal{M}\}$ as the reduced von Neumann algebra.

A linear functional φ on $\mathcal{A}$ is said to be positive if $\varphi(A) \geq 0$ for all $A \in \mathcal{A}^+$. It is referred to as tracial if $\varphi(AB) = \varphi(BA)$ for all $A, B \in \mathcal{A}$. A positive linear functional φ on a von Neumann algebra $\mathcal{M}$ is said to be normal if $\varphi(\sup_i X_i) = \sup_i \varphi(X_i)$ for any bounded increasing net (X_i) in $\mathcal{M}^+$.

By the universal representation of a C^*-algebra $\mathcal{A}$, one means the pair

$$\{\pi, \mathfrak{H}\} = \sum_{\varphi \in S(\mathcal{A})}^{\oplus} \{\pi_\varphi, \mathfrak{H}_\varphi\},$$

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where $S(\mathcal{A})$ is the set of all states on $\mathcal{A}$ and $\{\pi_\varphi, \mathfrak{H}_\varphi\}$ is the Gel'fand–Naimark–Segal representation of the C^*-algebra $\mathcal{A}$ associated with φ. In this case, the von Neumann algebra $\mathcal{M} = \pi(\mathcal{A})''$ generated by $\pi(\mathcal{A})$ is referred to as the universal enveloping von Neumann algebra of the C^*-algebra $\mathcal{A}$ [22, Chap. III, Definition 2.3].

Let φ be a positive linear functional on a C^*-algebra $\mathcal{A}$, and let π be the universal representation of $\mathcal{A}$. It follows from the construction of π that an arbitrary state on $\mathcal{A}$ becomes a vector state on $\pi(\mathcal{A})$, and therefore can be extended to a normal state on the universal enveloping algebra $\mathcal{M} = \pi(\mathcal{A})''$. Therefore, for φ, there is a positive normal functional $\widehat{\varphi}$ on the universal enveloping von Neumann algebra such that

$$\widehat{\varphi}(\pi(A)) = \varphi(A), \quad A \in \mathcal{A}^+.$$

Weights and traces on C^*-algebras are fundamental tools in operator theory and its applications. Therefore, the characterization of traces in various classes of weights on C^*-algebras is important and attracts the attention of mathematicians; see [9, 11, 15, 16, 17, 21].

While there is only one faithful tracial state on $\mathbb{M}_n$, which is the usual trace $\mathrm{tr}(\cdot)$, there are C^*-algebras with no tracial state. An example is the Cuntz algebra $\mathcal{O}_n$ generated by $n \geq 2$ isometries $U_1, \dots, U_n$ such that $\sum_{i=1}^n U_i U_i^* = I$. If φ is a tracial state on $\mathcal{O}_n$, then $1 = \varphi(I) = \varphi(U_i^* U_i) = \varphi(U_i U_i^*)$ for all $1 \leq i \leq n$. Therefore, $1 = \varphi(I) = \varphi\left(\sum_{i=1}^n U_i U_i^*\right) = \sum_{i=1}^n \varphi(U_i U_i^*) = n$, which leads to a contradiction.

If φ is a tracial normal positive linear functional on a von Neumann algebra $\mathcal{A}$, and p and q are positive numbers such that $1/p + 1/q = 1$, then the following holds:

- Hölder inequality [22, Chapter IX, Theorem 2.13] and [18, Theorem 5]:
$$\varphi(|XY|) \leq \varphi(X^p)^{1/p} \varphi(Y^q)^{1/q} \ \text{ for all } \ X, Y \in \mathcal{A}^+;$$
- Golden–Thompson inequality [19, Theorem 4]:
$$\varphi(e^{X+Y}) \leq \varphi(e^{X/2} e^Y e^{X/2}) \ \text{ for all } \ X, Y \in \mathcal{A}^{\mathrm{sa}};$$
- Peierls–Bogoliubov inequality [19, Theorem 7] (see also [5]):
$$\varphi(e^X) \exp \frac{\varphi(e^{X/2} Y e^{X/2})}{\varphi(e^X)} \leq \varphi(e^{X+Y}) \ \text{ for all } \ X, Y \in \mathcal{A}^+.$$

Araki [2] proved the following inequality:

$$\mathrm{tr}((X^{1/2} Y X^{1/2})^{rp}) \leq \mathrm{tr}((X^{r/2} Y^r X^{r/2})^p), \quad r \geq 1, \ p > 0, \ X, Y \in \mathcal{A}^+.$$

The former inequality generalizes the Lieb and Thirring inequalities, and is closely related to the Golden–Thompson inequality (see [20, §8]).

It is interesting to investigate the inequalities that characterize the trace among positive linear functionals. It is well-known that any one of the following inequalities: Hölder, Cauchy–Schwarz–Buniakowski, Golden–Thompson, Peierls–Bogoliubov, Araki–Lieb–Thirring, limited only to projections, characterizes the tracial functionals among all normal positive linear functionals on a von Neumann algebra $\mathcal{A}$; see [5], [8] and [12].

It is not true that any inequality involving the usual trace characterizes it. For example,

$$\mathrm{tr}(A+B) - \mathrm{tr}\,|A-B| \le 2\ \mathrm{tr}(A^\alpha B^{1-\alpha}) \le \alpha\ \mathrm{tr}(A) + (1-\alpha)\ \mathrm{tr}(B)$$

for all $0 \le \alpha \le 1$ and all $A, B \in \mathbb{M}_n^+$. The first inequality is known as the Powers–Størmer inequality (see [1]) and the second one is proved in Lemma 2.2. Note that the left-hand side of this inequality is independent of the parameter α and $\inf\{\alpha\ \mathrm{tr}(A) + (1-\alpha)\ \mathrm{tr}(B) :\ 0 \le \alpha \le 1\} = \min\{\mathrm{tr}(A), \mathrm{tr}(B)\}$. Therefore,

$$\mathrm{tr}(A+B) - \mathrm{tr}\,|A-B| \le 2\min\{\mathrm{tr}(A), \mathrm{tr}(B)\}.$$

Now, let φ be a (not necessarily tracial) positive linear functional on a C^*-algebra $\mathcal{A}$. We show that

$$\varphi(A+B) - \varphi(|A-B|) \le 2\min\{\varphi(A), \varphi(B)\} \tag{1.1}$$

for all $A, B \in \mathcal{A}^{\mathrm{sa}}$.

To prove this, we consider the Jordan decomposition of the operator $A - B \in \mathcal{A}^{\mathrm{sa}}$ into its positive and negative parts $A - B = (A-B)_+ - (A-B)_-$ and $|A-B| = (A-B)_+ + (A-B)_-$. Then,

$$\frac{1}{2}(\varphi(A) + \varphi(B) - \varphi(|A-B|)) = \varphi(A) - \varphi((A-B)_+)$$

and so, (1.1) is equivalent to the inequality

$$\varphi(A) - \min\{\varphi(A), \varphi(B)\} \le \varphi((A-B)_+). \tag{1.2}$$

Since $B + (A-B)_+ = A + (A-B)_- \ge A$, we have

$$\varphi(A) - \varphi(B) \le \varphi((A-B)_+),$$

which leads us to (1.2).

Another motivation for studying the tracial property of a positive linear functional on the full matrix algebra $\mathbb{M}_n$ is that if a von Neumann algebra $\mathcal{A}$ is noncommutative, then there exist nonzero mutually orthogonal equivalent projections $P = U^*U$ and $Q = UU^*$ in $\mathcal{A}$ for some partial isometry U. Then, the $*$-subalgebra of $(P+Q)\mathcal{A}(P+Q)$ generated by U is $*$-isomorphic to $\mathbb{M}_2$.

In this paper, we characterize the tracial functionals on the full matrix algebra $\mathbb{M}_n$ (that is, $\varphi = k\ \mathrm{tr}$ for some number $k > 0$).

In Section 2, we characterize tracial functionals φ on the full matrix algebra $\mathbb{M}_n$ via the inequalities $\varphi(A^\alpha B^{1-\alpha}) \le \alpha\,\varphi(A) + (1-\alpha)\,\varphi(B)$ (see Theorem 2.3) and $\varphi(|e^A|) \le \varphi(e^{\mathrm{Re}A})$ (see Theorem 2.4). Moreover, using an inequality for determinant, we identify the trace on the algebra $\mathbb{M}_n$ among all positive linear functionals φ with $\varphi(I) = n$ (see Theorem 2.5). We also establish that such a functional is equal to the usual trace if $\varphi\left(\left(Y^{\frac{1}{2}}XY^{\frac{1}{2}}\right)^m\right) \le \varphi(X)^m\varphi(Y)^m$ for some positive integer m and all $X, Y \in \mathbb{M}_n^+$ (see Theorem 2.6).

In Section 3, we establish that for a positive linear functional φ on a C^*-algebra $\mathcal{A}$, the following conditions are equivalent: (i) φ is tracial; (ii) $\varphi(e^{AB} - I) \ge 0$ for all $A, B \in \mathcal{A}^+$ (see Theorem 3.1). A new criterion for the commutativity of C^*-algebras is proved as well (see Corollary 3.2).

For undefined notations and terminologies, readers are referred to [22] for the theory of operator algebras and to [4] for the theory of matrices.

2. Characterizations of the trace functional on $\mathbb{M}_n$

In this section, we present several characterizations of the standard trace on the full matrix algebra $\mathbb{M}_n$.

2.1. **Characterizations of usual trace concerning the arithmetic geometric inequality.** To achieve the next result, let us recall Taylor's formula with Peano's remainder.

Lemma 2.1. *It is held that*

$$\log(1+x) = x - \frac{x^2}{2} + \cdots + (-1)^{n-1}\frac{x^n}{n} + o(x^n) \quad as \quad x \to 0.$$

If $b \in \mathbb{R}$, then

$$(1+x)^b = 1 + bx + \frac{1}{2!}b(b-1)x^2 + \ldots + \frac{1}{n!}b(b-1)\cdots(b-n+1)x^n + o(x^n) \quad as \quad x \to 0.$$

The next lemma can be found in the forthcoming monograph [10, Theorem 2.1.8]. For convenience, we provide a proof for it.

Lemma 2.2. *The inequality*

$$\operatorname{tr}(A^\alpha B^{1-\alpha}) \le \alpha \operatorname{tr}(A) + (1-\alpha) \operatorname{tr}(B)$$

holds for all $A, B \in \mathbb{M}_n^+$ and all $0 \le \alpha \le 1$.

Proof. Let $A = UCU^*$ and $B = VDV^*$ be the spectral decompositions of A and B, respectively, where $C = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$ and $D = \operatorname{diag}(\mu_1, \ldots, \mu_n)$ with $\lambda_i \ge 0$, $\mu_i \ge 0$, and $1 \le i \le n$. Since $U^*V = [u_{ij}]$ is a unitary matrix, it follows that

$$\begin{aligned}
\operatorname{tr}(A^\alpha B^{1-\alpha}) &= \operatorname{tr}(UC^\alpha U^*VD^{1-\alpha}V^*) = \operatorname{tr}((U^*V)^*C^\alpha(U^*V)D^{1-\alpha}) \\
&= \sum_{i,j=1}^n |u_{ij}|^2\lambda_i^\alpha\mu_j^{1-\alpha} \le \sum_{i,j=1}^n |u_{ij}|^2(\alpha\lambda_i + (1-\alpha)\mu_j) \\
&= \alpha\sum_{i=1}^n \lambda_i + (1-\alpha)\sum_{j=1}^n \mu_j = \alpha \operatorname{tr} A + (1-\alpha)\operatorname{tr} B,
\end{aligned}$$

where the inequality follows from the weighted arithmetic-geometric mean inequality. □

Our first characterization of the trace functionals is as follows. The inequality in (ii) is a type of arithmetic-geometric inequality for tracial functionals.

Theorem 2.3. *For a positive linear functional φ on $\mathbb{M}_n$, the following conditions are equivalent:*

(i) *φ is tracial, or equivalently, $\varphi = k$ tr for some number $k > 0$;*

(ii) *$\varphi(A^{\alpha/2}B^{1-\alpha}A^{\alpha/2}) \le \alpha\,\varphi(A) + (1-\alpha)\,\varphi(B)$ for all $A, B \in \mathbb{M}_n^+$ and $0 \le \alpha \le 1$;*

Proof. (i)⇒(ii). It is concluded from Lemma 2.2 by noting that $\mathrm{tr}(A^{\alpha}B^{1-\alpha}) = \mathrm{tr}(A^{\alpha/2}B^{1-\alpha}A^{\alpha/2})$ for all $A, B \in \mathbb{M}_n^+$ and $0 \le \alpha \le 1$.

(ii)⇒(i). The inner product $\langle X, Y\rangle = \mathrm{Tr}(XY^*)$ makes $\mathbb{M}_n$ a Hilbert space. Using the Riesz representation theorem, we can conclude that the positive linear functional φ on $\mathbb{M}_n$ is of the form $\varphi(X) = \mathrm{Tr}(S_\varphi X)$ for some positive semidefinite matrix S_φ. By employing the matrix Schur decomposition of S_φ, the tracial property of the trace functional, and replacing φ by $\varphi \circ \mathrm{Ad}_U$ for some unitary U, we can assume that S_φ is a diagonal matrix $\mathrm{diag}(s_1, s_2, \ldots, s_n)$. Our goal is to prove that $s_1 = s_2 = \cdots = s_n$. It is enough to prove $s_1 = s_2$. Therefore, without loss of generality, we can assume that $n = 2$ and φ is of norm one, and hence we can write

$$S_\varphi = \mathrm{diag}\Big(\frac{1}{2} - s, \frac{1}{2} + s\Big) \tag{2.1}$$

for some $0 \le s \le \frac{1}{2}$. The two-by-two matrix S_φ is called the density matrix of φ.

Thus, $\varphi(X)$ equals $(1/2 - s)x_{11} + (1/2 + s)x_{22}$ for $X = [x_{ij}]_{i,j=1}^2$ in $\mathbb{M}_2$. Let $\delta \in \mathbb{C}$ with $|\delta| = 1$ and $0 \le t \le 1$. By $R^{(t,\delta)}$, we denote the following projection in $\mathbb{M}_2$

$$R^{(t,\delta)} = \begin{bmatrix} t & \delta\sqrt{t - t^2} \\ \bar{\delta}\sqrt{t - t^2} & 1 - t \end{bmatrix}. \tag{2.2}$$

For $0 < \varepsilon < 1/2$, let

$$A := (1 + s\varepsilon)R^{(1/2-\varepsilon,1)} \quad \text{and} \quad B := (1 - s\varepsilon)R^{(1/2+\varepsilon,1)}.$$

Consider S_φ as in (2.1). Then,

$$\varphi(A) = \frac{1}{2} + \frac{5}{2}s\varepsilon + 2s^2\varepsilon^2 \quad \text{and} \quad \varphi(B) = \frac{1}{2} - \frac{5}{2}s\varepsilon + 2s^2\varepsilon^2.$$

Put $\alpha = 1/2$. Then, $\alpha\,\varphi(A) + (1-\alpha)\,\varphi(B) = 1/2 + 2s^2\varepsilon^2$ and

$$A^{\alpha/2}B^{1-\alpha}A^{\alpha/2} = A^{1/4}B^{1/2}A^{1/4} = (1 - s^2\varepsilon^2)^{1/2}(1 - 4\varepsilon^2)R^{(1/2-\varepsilon,1)}.$$

It follows from Lemma 2.1 that

$$(1 - s^2\varepsilon^2)^{1/2} = 1 - \frac{1}{2}s^2\varepsilon^2 + o(\varepsilon^2) \quad \text{as} \quad \varepsilon \to 0^+.$$

Hence,

$$\begin{aligned}\varphi(A^{1/4}B^{1/2}A^{1/4}) &= (1-s^2\varepsilon^2)^{1/2}(1-4\varepsilon^2)\Big(\frac{1}{2}+2s\varepsilon\Big)\\ &= \Big(1-\frac{1}{2}s^2\varepsilon^2+o(\varepsilon^2)\Big)(1-4\varepsilon^2)\Big(\frac{1}{2}+2s\varepsilon\Big)\\ &= \frac{1}{2}+2s\varepsilon-2\varepsilon^2-\frac{s^2}{4}\varepsilon^2+o(\varepsilon^2) \quad \text{as} \quad \varepsilon\to 0^+.\end{aligned}$$

Rewrite inequality (ii) as

$$\frac{1}{2}+2s\varepsilon-2\varepsilon^2-\frac{s^2}{4}\varepsilon^2+o(\varepsilon^2)\le \frac{1}{2}+2s^2\varepsilon^2 \quad \text{as} \quad \varepsilon\to 0^+.$$

This inequality is true for all $0<\varepsilon<1/2$ only when $s=0$. □

2.2. **Characterizations of usual trace concerning the exponential function.** The following theorem provides a characterization of the standard trace in relation to an exponential inequality.

Theorem 2.4. *For a positive linear functional φ on $\mathbb{M}_n$ the following conditions are equivalent:*

(i) *φ is tracial;*

(ii) *$\varphi(|e^A|)\le\varphi(e^{\mathrm{Re}A})$ for all $A\in\mathbb{M}_n$.*

Proof. (i)⇒(ii). The mapping

$$X\mapsto\varphi(|X|),\quad X\in\mathbb{M}_n,$$

defines a unitarily invariant norm on the full matrix algebra $\mathbb{M}_n$. To obtain the result, it is enough to apply [4, Theorem IX.3.1].

(ii)⇒(i). *Step 1.* Assume that $n=2$.

Define a nilpotent matrix as follows:

$$A=\begin{bmatrix}0 & 2\varepsilon\\ 0 & 0\end{bmatrix}$$

for $0<\varepsilon<1$. Since $A^n=0$ for all $n\ge 2$ we have $e^A=I+A$. The matrix

$$|e^A|^2=\begin{bmatrix}1 & 2\varepsilon\\ 2\varepsilon & 1+4\varepsilon^2\end{bmatrix}$$

has the characteristic equation $\lambda^2-2(1+2\varepsilon^2)\lambda+1=0$. Hence, its characteristic roots are $\lambda_i=1+2\varepsilon^2\pm 2\varepsilon\sqrt{1+\varepsilon^2}$ for $i=1,2$. According to Lemma 2.1, we have

$$\sqrt{1+\varepsilon^2}=1+\frac{\varepsilon^2}{2}+o(\varepsilon^3)\ \text{ as }\ \varepsilon\to 0^+.$$

Therefore,

$$\lambda_1=1+2\varepsilon+2\varepsilon^2+o(\varepsilon^3),\quad \lambda_2=1-2\varepsilon+2\varepsilon^2+o(\varepsilon^3)\quad \text{as }\ \varepsilon\to 0^+.$$

Let us represent B using the finite-dimensional spectral theorem as

$$B = |e^A|^2 = \lambda_1 R^{(t,1)} + \lambda_2 R^{(t,1)\perp} = \lambda_1 R^{(t,1)} + \lambda_2 R^{(1-t,-1)}, \tag{2.3}$$

where we use the notation (2.2). We can find the parameter $t \in [0,1]$ from the equation

$$b_{11} = 1 = \lambda_1 t + \lambda_2(1-t),$$

whence

$$0 = -2\varepsilon + 4\varepsilon t + 2\varepsilon^2 + o(\varepsilon^3) \quad \text{as} \ \ \varepsilon \to 0^+.$$

Hence,

$$t = \frac{1-\varepsilon}{2} + o(\varepsilon^2) \quad \text{as} \ \ \varepsilon \to 0^+.$$

From (2.3) and Lemma 2.1, we get $|e^A| = \sqrt{\lambda_1} R^{(t,1)} + \sqrt{\lambda_2} R^{(1-t,-1)}$, where

$$\sqrt{\lambda_1} = 1 + \varepsilon + \frac{\varepsilon^2}{2} + o(\varepsilon^2) \quad \text{and} \quad \sqrt{\lambda_2} = 1 - \varepsilon + \frac{\varepsilon^2}{2} + o(\varepsilon^2) \quad \text{as} \ \ \varepsilon \to 0^+.$$

Now,

$$\begin{aligned}
\varphi(|e^A|) &= \Big(1 + \varepsilon + \frac{\varepsilon^2}{2} + o(\varepsilon^2)\Big)\Big(\Big(\frac{1-\varepsilon}{2} + o(\varepsilon^2)\Big)\Big(\frac{1}{2} - s\Big) + \Big(\frac{1+\varepsilon}{2} + o(\varepsilon^2)\Big)\Big(\frac{1}{2} + s\Big)\Big) \\
&\quad + \Big(1 - \varepsilon + \frac{\varepsilon^2}{2} + o(\varepsilon^2)\Big)\Big(\Big(\frac{1+\varepsilon}{2} + o(\varepsilon^2)\Big)\Big(\frac{1}{2} - s\Big) + \Big(\frac{1-\varepsilon}{2} + o(\varepsilon^2)\Big)\Big(\frac{1}{2} + s\Big)\Big) \\
&= \Big(1 + \varepsilon + \frac{\varepsilon^2}{2} + o(\varepsilon^2)\Big)\Big(\frac{1}{2} + s\varepsilon + o(\varepsilon^2)\Big) + \Big(1 - \varepsilon + \frac{\varepsilon^2}{2} + o(\varepsilon^2)\Big)\Big(\frac{1}{2} - s\varepsilon + o(\varepsilon^2)\Big) \\
&= 1 + \frac{\varepsilon^2}{2} + 2s\varepsilon^2 + o(\varepsilon^2) \quad \text{as} \quad \varepsilon \to 0^+.
\end{aligned}$$

We have

$$\text{Re}A = \varepsilon \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \varepsilon T$$

with the Hermitian symmetry T, meaning $T = T^*$ and $T^2 = I$. Then,

$$\begin{aligned}
e^{\text{Re}A} &= I + \varepsilon T + \frac{\varepsilon^2}{2!} I + \frac{\varepsilon^3}{3!} T + \frac{\varepsilon^4}{4!} I + \ldots = \Big(1 + \frac{\varepsilon^2}{2!} + \frac{\varepsilon^4}{4!} + \ldots\Big) I + \Big(\varepsilon + \frac{\varepsilon^3}{3!} + \frac{\varepsilon^5}{5!} + \ldots\Big) T \\
&= \text{cosh}\varepsilon \cdot I + \text{sinh}\varepsilon \cdot T = \begin{bmatrix} \text{cosh}\varepsilon & \text{sinh}\varepsilon \\ \text{sinh}\varepsilon & \text{cosh}\varepsilon \end{bmatrix} \in \mathbb{M}_2^+.
\end{aligned}$$

Therefore, according to Taylor's formula, we have

$$\varphi(e^{\text{Re}A}) = \text{cosh}\varepsilon = 1 + \frac{\varepsilon^2}{2!} + o(\varepsilon^2) \quad \text{as} \quad \varepsilon \to 0^+.$$

Let us rewrite the inequality $\varphi(|e^A|) \le \varphi(e^{\text{Re}A})$ as

$$1 + \frac{\varepsilon^2}{2} + 2s\varepsilon^2 + o(\varepsilon^2) \le 1 + \frac{\varepsilon^2}{2} + o(\varepsilon^2) \quad \text{as} \quad \varepsilon \to 0^+.$$

This inequality holds true for all $0 < \varepsilon < 1$ only when $s = 0$.

Step 2. Let us prove the assertion for the general case of $n \in \mathbb{N}$. Choose an orthonormal basis $\xi_1, \ldots, \xi_n \in \mathbb{C}^n$ such that the density matrix of φ has the form $S_\varphi = \mathrm{diag}(s_1, s_2, \ldots, s_n)$ with $0 \le s_1 \le s_2 \le \ldots \le s_n$. Thus, $\varphi(X)$ equals $s_1 x_{11} + s_2 x_{22} + \cdots + s_n x_{nn}$ for $X = [x_{ij}]_{i,j=1}^n$ in $\mathbb{M}_n$. Assume that φ is nontracial and $s_1 < s_2$. To conclude the result, it is enough to consider the matrix A from Step 1 and put $B = \mathrm{diag}\Big(A, \underbrace{0, \ldots, 0}_{n-2}\Big)$. □

2.3. **Characterizations of usual trace related to the determinant.** There are several characterizations of the trace involving determinant in the literature; see [7, 10] and the references mentioned therein. For example, if φ is a positive linear functional on $\mathbb{M}_n$ with $\varphi(I) = n$, then the following conditions are equivalent:

- $\varphi = \mathrm{tr}$;
- $\det(A)^{1/n} \le \frac{1}{n}\varphi(A)$ for all matrices $A \in \mathbb{M}_n^+$;
- $\det(\exp(A)) \ge \exp(\varphi(A))$ for all matrices $A \in \mathbb{M}_n^+$.
- $\det(I + \varepsilon A) = 1 + \varepsilon\varphi(A) + o(\varepsilon)$ as $\varepsilon \to 0^+$ for all matrices $A \in \mathbb{M}_n^+$.

The next result provide another characterization of the trace functional.

Theorem 2.5. *For a positive linear functional φ on $\mathbb{M}_n$ with $\varphi(I) = n$, the following conditions are equivalent:*

(i) $\varphi = \mathrm{tr}$*;*

(ii) $\log(\det A) \le \varphi(A) - n$ *for all positive definite* $A \in \mathbb{M}_n$*.*

Proof. For (i)⇒(ii) see [10, Proposition 2.1.6].

(ii)⇒(i). Let the density matrix of φ be of a diagonal matrix with entries $s_1 \ge s_2 \ge \ldots \ge s_n \ge 0$ and $s_1 + s_2 + \cdots + s_n = n$. Suppose that $\varphi \ne \mathrm{tr}$. Then, $s_1 > 1 > s_n$. Put in this basis, $A = \mathrm{diag}\Big(1 - \varepsilon, \underbrace{1, \ldots, 1}_{n-1}\Big)$ for $0 < \varepsilon < 1$. Then, we have $\det(A) = 1 - \varepsilon$ and $\varphi(A) - n = n - s_1\varepsilon - n = -s_1\varepsilon$. The Taylor formula with Peano remainder (see Lemma 2.1) yields that

$$\log(1 - \varepsilon) = -\varepsilon + o(\varepsilon) \quad \text{as} \quad \varepsilon \to 0^+.$$

Now, (ii) takes the form

$$-\varepsilon + o(\varepsilon) \le -s_1\varepsilon \quad \text{as} \quad \varepsilon \to 0^+.$$

Consequently, $s_1 \le 1$; which yields a contradiction. □

2.4. **Chracterization of the usual trace involving a multiplicative inequality.** There has been substantial interest in trace inequalities involving the product of certain types of matrices; see the survey [14]. For example, Bellman [3] discussed the inequality $\operatorname{tr}((XY)^2) \leq \operatorname{tr}(X)^2 \operatorname{tr}(Y)^2$ for $X, Y \in \mathbb{M}_n^+$. Yang [24] extended this to

$$\operatorname{tr}((XY)^m)) \leq \operatorname{tr}(X)^m \operatorname{tr}(Y)^m. \tag{2.4}$$

for $m \in \mathbb{N}$. To establish the main result of this section, we require the well-known fact that if $X, Y \in \mathbb{M}_n^+$, then

$$\operatorname{tr}(XY) \leq \operatorname{tr}(X) \operatorname{tr}(Y); \tag{2.5}$$

as shown in [10, Corollary 2.1.10].

Theorem 2.6. *For a positive linear functional φ on $\mathbb{M}_n$ with $\varphi(I) = n$, the following conditions are equivalent:*

(i) $\varphi = \operatorname{tr}$;
(ii) $\varphi\left(\left(Y^{\frac{1}{2}}XY^{\frac{1}{2}}\right)^m\right) \leq \varphi(X)^m\varphi(Y)^m$ *for all positive integers m and all matrices $X, Y \in \mathbb{M}_n^+$;*
(iii) $\varphi\left(\left(Y^{\frac{1}{2}}XY^{\frac{1}{2}}\right)^m\right) \leq \varphi(X)^m\varphi(Y)^m$ *for some positive integer m and all matrices $X, Y \in \mathbb{M}_n^+$.*

Proof. (i) $\Longrightarrow$ (ii). By iteratively using (2.5) and the tracial property of $\operatorname{tr}(\cdot)$ we get the following inequalities:

$$\begin{aligned}
\operatorname{tr}\left(\left(Y^{\frac{1}{2}}XY^{\frac{1}{2}}\right)^m\right) &= \operatorname{tr}\left(\left(Y^{1/2}XY^{1/2}\right)\left(Y^{1/2}XY^{1/2}\right)^{m-1}\right)\\
&\leq \operatorname{tr}\left(Y^{1/2}XY^{1/2}\right)\operatorname{tr}\left(\left(Y^{1/2}XY^{1/2}\right)^{m-1}\right)\\
&\leq \operatorname{tr}\left(Y^{1/2}XY^{1/2}\right)^2\operatorname{tr}\left(\left(Y^{1/2}XY^{1/2}\right)^{m-2}\right)\\
&\leq \cdots\\
&\leq \operatorname{tr}\left(Y^{1/2}XY^{1/2}\right)^m\\
&= \operatorname{tr}(XY)^m\\
&\leq \operatorname{tr}(X)^m \operatorname{tr}(Y)^m.
\end{aligned}$$

Note that due to

$$\begin{aligned}
\operatorname{tr}\left(\left(Y^{\frac{1}{2}}XY^{\frac{1}{2}}\right)^m\right) &= \operatorname{tr}\left(Y^{\frac{1}{2}}\cdot XY^{\frac{1}{2}}\left(Y^{\frac{1}{2}}XY^{\frac{1}{2}}\right)\cdots\left(Y^{\frac{1}{2}}XY^{\frac{1}{2}}\right)\right)\\
&= \operatorname{tr}\left(XY^{\frac{1}{2}}\left(Y^{\frac{1}{2}}XY^{\frac{1}{2}}\right)\cdots\left(Y^{\frac{1}{2}}XY^{\frac{1}{2}}\cdot\right)Y^{\frac{1}{2}}\right)\\
&= \operatorname{tr}\left(\left(XY^{\frac{1}{2}}Y^{\frac{1}{2}}\right)\left(XY^{\frac{1}{2}}Y^{\frac{1}{2}}\right)\cdots\left(XY^{\frac{1}{2}}Y^{\frac{1}{2}}\right)\right)\\
&= \operatorname{tr}\left((XY)^m\right),
\end{aligned} \tag{2.6}$$

we reach a simple proof for the main result (2.4) from [24].

(ii) $\Longrightarrow$ (iii). It is obvious.

(iii) $\Longrightarrow$ (i). As before, we assume that $n = 2$ and $\varphi(X) = \operatorname{tr}(DX)$ for some diagonal matrix D (the density matrix of φ) with $\operatorname{tr}(D) = 2$. Therefore, we can set

$D = \operatorname{diag}(1+d, 1-d)$ for some $0 \le d \le 1$. We shall show that $d = 0$. Consider the matrices

$$X = \begin{bmatrix} 1 & p \\ p & p^2 \end{bmatrix} \quad \text{and} \quad Y = \begin{bmatrix} p^2 & p \\ p & 1 \end{bmatrix},$$

where $p > 1$. Simple computation shows that

$$X^m = (p^2+1)^m X, \quad Y^m = (p^2+1)^m Y,$$

$$Y^{\frac{1}{2}} = \frac{1}{\sqrt{p^2+1}} Y, \quad \text{and} \quad Y^{\frac{1}{2}} X Y^{\frac{1}{2}} = \left(\frac{4p^2}{p^2+1}\right)^m (p^2+1)^{m-1} Y.$$

Assumption (iii) therefore leads to:

$$\left(\frac{4p^2}{p^2+1}\right)^m (p^2+1)^{m-1} \left((1+d)p^2 + (1-d)1\right)$$
$$\le (p^2+1)^{2(m-1)} \left((1+d)1 + (1-d)p^2\right)\left((1+d)p^2 + (1-d)1\right).$$

Hence,

$$\left(\frac{4p^2}{p^2+1}\right)^m \le (p^2+1)^{(m-1)} \left((1+p^2) + d(1-p^2)\right).$$

Therefore,

$$d \cdot \frac{p^2-1}{p^2+1} \le 1 - \left(\frac{4p^2}{p^2+1}\right)^m = \left(1 - \frac{4p^2}{p^2+1}\right)\left(1 + \frac{4p^2}{p^2+1} + \cdots + \left(\frac{4p^2}{p^2+1}\right)^{m-1}\right),$$

whence

$$d \le \frac{p^2-1}{p^2+1}\left(1 + \frac{4p^2}{p^2+1} + \cdots + \left(\frac{4p^2}{p^2+1}\right)^{m-1}\right).$$

Letting $p \to 1$, we get $d \le 0$. Thus, $d = 0$. □

Remark 2.7. We cannot replace the condition $\varphi(I) = n$ in Theorem 2.6 with $\varphi(I) = 1$ when $n \ge 2$. In other words, there is no state φ on $\mathbb{M}_n$ such that

$$\varphi\left(Y^{\frac{1}{2}} X Y^{\frac{1}{2}}\right) \le \varphi(X)\varphi(Y)$$

for all $X, Y \in \mathbb{M}_n^+$. If there were such a state and A is a positive semidefinite matrix, then with $X = Y = A$, we get $\varphi(A^2) \le \varphi(A)^2$.

According to the Kadison inequality, it holds that $\varphi(A)^2 \le \varphi(A^2)$. Thus, $\varphi(A^2) = \varphi(A)^2$. It follows from [13, Theorem 3.1] that

$$\varphi(AB) = \varphi(A)\varphi(B) \tag{2.7}$$

for all $B \in \mathbb{M}_n$. Since the positive semidefinite matrices linearly span the full matrix algebra $\mathbb{M}_n$, we conclude that (2.7) holds for all matrices A and B. Therefore, $\ker\varphi$ is a two-sided ideal of the simple algebra $\mathbb{M}_n$. Therefore, $\ker\varphi = \{0\}$, and so $\varphi : \mathbb{M}_n \to \mathbb{C}$ is a one-to-one $*$-homomorphism. Therefore, $n = 1$, which leads to a contradiction.

3. Characterization of tracial functionals on C^*-algebras

Theorem 3.1. *For a positive linear functional φ on a unital C^*-algebra $\mathcal{A}$ the following conditions are equivalent:*
(i) *φ is tracial;*
(ii) *$\varphi(e^{AB} - I) \geq 0$ for all $A, B \in \mathcal{A}^+$.*

Proof. (i)⇒(ii). Inequality (2.6) still holds with a tracial positive linear functional ϕ on $\mathcal{A}$ rather than tr. Therefore,

$$\varphi(e^{AB} - I) = \varphi\left(\sum_{m=1}^{\infty} \frac{(AB)^m}{m!}\right) = \sum_{m=1}^{\infty} \frac{\varphi((AB)^m)}{m!} = \sum_{m=1}^{\infty} \frac{\varphi\left(\left(B^{\frac{1}{2}}AB^{\frac{1}{2}}\right)^m\right)}{m!} \geq 0,$$

since $\left(B^{\frac{1}{2}}AB^{\frac{1}{2}}\right)^m \geq 0$.

(ii)⇒(i). **Step 1.** We demonstrate that, for a positive linear functional φ on any von Neumann algebra, the proof of the inverse implication can be reduced to the case of the algebra $\mathbb{M}_2$ similar to other cases (refer to [15] and [23]). It is a well known fact [15] that a positive linear functional φ on a von Neumann algebra $\mathcal{M}$ is tracial if and only if $\varphi(P) = \varphi(Q)$ for all $P, Q \in \mathcal{M}^{\text{pr}}$ with $PQ = 0$ and $P \sim Q$ (also see [23, Lemma 2]). Let us assume that the $*$-algebra $\mathcal{N}$ in the reduced algebra $\mathcal{M}_{P+Q}$ is induced by a partial isometry $V \in \mathcal{M}$ realizing the equivalence of P and Q. In this case, $\mathcal{N}$ is $*$-isomorphic to $\mathbb{M}_2$, and the inequality in (ii) remains valid for operators of $\mathcal{N}$ and the restriction of the functional $\varphi|\mathcal{N}$.

For $0 \leq s \leq 1/2$ and $0 < \varepsilon \leq 1/2$, put $S_\varphi = \operatorname{diag}\left(\frac{1}{2} + s, \frac{1}{2} - s\right)$ and

$$A = R^{(1/2-\varepsilon,1)}, \quad B = R^{(1/2-\varepsilon,-1)},$$

see (2.2). Then $ABA = 4\varepsilon^2 A$, $BAB = 4\varepsilon^2 B$ and

$$e^{AB} - I = AB + \frac{4\varepsilon^2}{2!}AB + \cdots + \frac{(4\varepsilon^2)^{n-1}}{n!}AB + \cdots,$$

thus,

$$\begin{aligned}
\varphi(e^{AB} - I) &= \varphi\Big(\Big(1 + \frac{4\varepsilon^2}{2!} + \cdots + \frac{(4\varepsilon^2)^{n-1}}{n!} + \cdots\Big)AB\Big) \\
&= \varphi\Big(\frac{1}{4\varepsilon^2}\Big(4\varepsilon^2 + \frac{(4\varepsilon^2)^2}{2!} + \cdots + \frac{(4\varepsilon^2)^n}{n!} + \cdots\Big)AB\Big) \\
&= \varphi\Big(\frac{1}{4\varepsilon^2}(e^{4\varepsilon^2} - 1)AB\Big) \\
&= \frac{e^{4\varepsilon^2} - 1}{4\varepsilon^2}\varphi(AB).
\end{aligned}$$

Since

$$AB = \varepsilon \begin{bmatrix} 2\varepsilon - 1 & 2(1/4 - \varepsilon^2)^{1/2} \\ -2(1/4 - \varepsilon^2)^{1/2} & 2\varepsilon + 1 \end{bmatrix},$$

we have

$$\varphi(AB) = \varepsilon\Big((2\varepsilon - 1)\Big(\frac{1}{2} + s\Big) + (2\varepsilon + 1)\Big(\frac{1}{2} - s\Big)\Big) = 2\varepsilon(\varepsilon - s) < 0,$$

for all $0 < \varepsilon < s$. Hence, $s = 0$. It tells us that such a restriction is a tracial functional on $\mathcal{N}$; therefore, $\varphi(P) = \varphi(Q)$.

Step 2. We move on the universal enveloping von Neumann algebra of the C^*-algebra $\mathcal{A}$ [22, III.2]. Let π be the corresponding universal representation of the C^*-algebra $\mathcal{A}$, and let $\widehat{\varphi}$ be a positive normal functional on $\mathcal{M} = \pi(\mathcal{A})''$ such that $\widehat{\varphi}(\pi(T)) = \varphi(T)$ for $T \in \mathcal{A}$. Let $\widehat{A}, \widehat{B} \in \mathcal{M}^+$. By the Kaplansky density theorem, there exist bounded nets $\{X_i\}$ and $\{Y_i\}$ in $\pi(\mathcal{A})^+$ that converge to $\widehat{A}$ and $\widehat{B}$, respectively, in the strong operator topology. Let $\{A_i\}$ and $\{B_i\}$ be such that

$$X_i = \pi(A_i) \quad \text{and} \quad Y_i = \pi(B_i).$$

We may assume that $A_i, B_i \in \mathcal{A}^+$, as shown in the proof of Theorem 2 in [6]. Thus, there exist bounded nets $\{A_i\}$ and $\{B_i\}$ in $\mathcal{A}^+$ such that $\pi(A_i) \to \widehat{A}$ and $\pi(B_i) \to \widehat{B}$ in the strong operator topology. By (ii), we have

$$\widehat{\varphi}(e^{\pi(A_i)\pi(B_i)} - \pi(I)) = \widehat{\varphi}(e^{\pi(A_iB_i)} - \pi(I)) = \widehat{\varphi}(\pi(e^{A_iB_i} - I)) = \varphi(e^{A_iB_i} - I) \geq 0.$$

Taking into account the strong continuity of the functional calculus and making the passage to the limit in the strong operator topology in the last inequality, we obtain $\widehat{\varphi}(e^{\widehat{A}\widehat{B}} - I) \geq 0$.

By Step 1, $\widehat{\varphi}$ is a tracial functional on $\mathcal{M}$. Now, for every $X, Y \in \mathcal{A}$, we have

$$\varphi(XY) = \widehat{\varphi}(\pi(XY)) = \widehat{\varphi}(\pi(X)\pi(Y)) = \widehat{\varphi}(\pi(Y)\pi(X)) = \widehat{\varphi}(\pi(YX)) = \varphi(YX).$$

Thus, φ is a tracial functional on $\mathcal{A}$. This completes the proof of the theorem. □

Corollary 3.2. *The following conditions are equivalent for every unital C^*-algebra $\mathcal{A}$:*

(i) $\mathcal{A}$ *is commutative;*

(ii) $e^{AB} \geq I$ *for all* $A, B \in \mathcal{A}^+$.

Proof. Let us prove the implication (ii)⇒(i). For every positive functional φ on $\mathcal{A}$, the inequality in part (ii) of Theorem 3.1 holds. This implies that every positive functional on $\mathcal{A}$ is tracial, i.e., $\varphi(XY) = \varphi(YX)$ for every elements $X, Y \in \mathcal{A}$. Since the set of positive functionals separates the points of the algebra $\mathcal{A}$, it follows from the last condition that $XY = YX$ $(X, Y \in \mathcal{A})$, and thus the C^*-algebra $\mathcal{A}$ is commutative. □

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