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An Extension of the Krein–Smulian and Lozanovskii Theorems to Metrizable Spaces with a Cone

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Abstract—The Krein–Smulian theorem that a closed generating cone in a Banach space is nonflattened and a theorem of Lozanovskii about the automatic continuity of linear positive operators are generalized to completely metrizable topological vector spaces.

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The purpose of this paper is to generalize the classical Krein–Smulian theorem that a closed generating cone in a Banach space is nonflattened to the case of completely metrizable topological vector spaces and transfer a result of G.Ya. Lozanovskii about the automatic continuity of linear positive operators to this case. In [1], Zabreiko obtained a generalization of the Krein–Smulian theorem to metrizable spaces; however, our approach differs that of [1] both in the method of proof and in the form of the result. Moreover, we somewhat relax the condition on the cone under consideration; namely, we replace the usual closedness requirement by a condition which we call *fullness*. Thus, already in the case of Banach spaces, we obtain a certain strengthening of results in the direction under consideration.

It seems natural to formulate some auxiliary assertions for metrizable topological Abelian groups.

Definition 1. Let G be an Abelian group under addition. A function $||\cdot||\colon G \to \mathbb{R}^+$ is called a *g-norm* if it satisfies the following conditions:

(a) $||x|| = 0 \iff x = 0 \quad (x \in G)$;

(b) $||x + y|| \le ||x|| + ||y|| \quad (x, y \in G)$;

(c) $||-x|| = ||x|| \quad (x \in G)$.

Note that the relation $\rho(x, y) = ||x - y||$ determines a one-to-one correspondence between translation-invariant metrics ρ and g-norms $||\cdot||$ on G; when considering convergence, completeness, etc. with respect to a g-norm, we mean the fulfillment of the corresponding conditions with respect to the corresponding invariant metric. Two g-norms on the same Abelian group are said to be *equivalent* if the corresponding invariant metrics are equivalent, that is, generate the same topology.

Proposition 1. *For a subsemigroup S of an Abelian group G with g-norm $||\cdot||$, the following conditions are equivalent:*

(i) *any fundamental sequence $\{x_n\}$ of elements of S such that $x_{n+1} - x_n \in S$ for all $n \in \mathbb{N}$ converges to some $x \in S$;*

(ii) *if $x_n \in S$ and $\sum_{n=1}^{\infty} ||x_n|| < \infty$, then the series $\sum_{n=1}^{\infty} x_n$ converges in g-norm to some $x \in S$.*

Proof. Note at once that the proof of this proposition virtually coincides with the proof of the well-known completeness criterion for a normed space (see, e.g., [2, Section 148.4]).

(i) $\Longrightarrow$ (ii) This implication directly follows from that if the series $\sum_{n=1}^{\infty} ||x_n||$ converges, then the sequence $\left\{\sum_{i=1}^{n} x_i\right\}$ of partial sums is fundamental.

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(ii) $\Longrightarrow$ (i). Suppose that condition (ii) holds and let $\{x_n\}$ be a fundamental sequence of elements of S such that $x_{n+1} - x_n \in S$ for all $n \in \mathbb{N}$. Passing to a subsequence if necessary, we can assume without loss of generality that $\sum_{n=1}^{\infty} ||x_{n+1} - x_n|| < \infty$. Then the series $\sum_{n=1}^{\infty}(x_{n+1} - x_n)$ converges to some $x \in S$, and therefore, $x_n \to x_1 + x \in S$. □

Remark 1. Taking for S the group G itself in Proposition 1 we obtain a completeness criterion for an Abelian group with a g-norm.

Definition 2. We say that a subsemigroup S of an Abelian group G with g-norm $||\cdot||$ is *filled* if it satisfies one of the equivalent conditions in Proposition 1.

Remark 2. It is easy to show by using condition (i) that, for a subsemigroup, the property of being filled is invariant under the passage to an equivalent g-norm.

Definition 3. Let $(G, ||\cdot||)$ be an Abelian group with a g-norm. Its subsemigroup S is said to be *generating* if any $x \in G$ can be is represented in the form $x = x' - x''$, where $x', x'' \in S$. If, for some $\lambda > 0$, such x' and x'' can always be chosen so that $||x'|| + ||x''|| \leq \lambda ||x||$, then we say that S is λ-*generating*. In this case, we also say that S λ-*generates* G.

Lemma 1. *If a subsemigroup S of an Abelian group G with g-norm* $||\cdot||$ *is filled and* λ-*generating for some* $\lambda > 0$, *then G is complete as a metric space.*

Proof. We apply Remark 1. Let x_n be elements of G such that $\sum_{n=1}^{\infty} ||x_n|| < \infty$. For each n, take $x'_n, x''_n \in S$ such that $x_n = x'_n - x''_n$ and $||x'_n|| + ||x''_n|| \leq \lambda ||x_n||$. We have $\sum_{n=1}^{\infty} ||x'_n|| < \infty$ and $\sum_{n=1}^{\infty} ||x''_n|| < \infty$; therefore, the series $\sum_{n=1}^{\infty} x'_n$ converges and $\sum_{n=1}^{\infty} x''_n$; hence, $\sum_{n=1}^{\infty} x_n$ converges as well. □

Lemma 2. *Suppose that S is a filled subsemigroup of an Abelian group G with g-norm* $||\cdot||_G$, *S contains* 0, *and E is the subgroup generated by S. The formula*

$$||x||_E = \inf\{||x'||_G + ||x''||_G \mid x', x'' \in S,\ x = x' - x''\}$$

defines a g-norm on E, with respect to which the space E is complete as a metric space. For any $x \in E$, $||x||_E \geq ||x||_G$, *and for* $x \in S$, $||x||_E = ||x||_G$. *The semigroup S is filled and* λ-*generating for any* $\lambda > 1$ *in the group E with g-norm* $||\cdot||_E$. *If the semigroup S closed in* $(G, ||\cdot||_G)$, *then it is closed in* $(E, ||\cdot||_E)$.

Proof. First, consider the relationship between the functions $||\cdot||_E$ and $||\cdot||_G$ on E and on S. For any representation of an element $x \in E$ as the difference $x = x' - x''$ of elements of S, the norms of these elements satisfy the inequality $||x||_G = ||x' - x''||_G \leq ||x'||_G + ||x''||_G$; therefore,

$$||x||_G \leq \inf\{||x'||_G + ||x''||_G \mid x', x'' \in K, x = x' - x''\} = ||x||_E.$$

If $x \in S$, then $x = x - 0$ is one of the representations of x as a difference of elements of S; therefore,

$$||x||_E = \inf\{||x'||_G + ||x''||_G \mid x', x'' \in K, x = x' - x''\} \leq ||x||_G + ||0||_G = ||x||_G.$$

Now, let us verify that the function $||\cdot||_E$ on E satisfies the g-norm axioms. The nonnegativity of the function $||\cdot||_E$ is obvious. Its faithfulness follows from the inequality $||x||_G \leq ||x||_E$. Property (c), that is, the relation $||-x||_E = ||x||_E$, holds trivially. Let us prove the triangle inequality. Suppose that $x = x_1 + x_2$ (where $x_1, x_2 \in E$). For any $\varepsilon > 0$ and $i = 1, 2$, we choose $x'_i, x''_i \in S$ so that $x_i = x'_i - x''_i$ and $||x'_i||_G + ||x''_i||_G \leq ||x_i||_E + \varepsilon/2$. We have $x = (x'_1 + x'_2) - (x''_1 + x''_2)$ and

$$||x||_E \leq ||x'_1 + x'_2||_G + ||x''_1 + x''_2||_G \leq ||x'_1||_G + ||x''_1||_G + ||x'_2||_G + ||x''_2||_G \leq ||x_1||_E + ||x_2||_E + \varepsilon.$$

These inequalities and the arbitrariness of ε imply $||x||_E \leq ||x_1||_E + ||x_2||_E$.

Let us prove that S is filled in $(E, ||\cdot||_E)$. Take $x_n \in S$ such that $x_{n+1} - x_n \in S$ for all $n \in \mathbb{N}$ and the sequence $\{x_n\}$ is fundamental in the g-norm $||\cdot||_E$. The sequence $\{x_n\}$ remains fundamental in the g-norm $||\cdot||_G$; hence, $||x - x_n||_G \to 0$ for some $x \in S$. However, $x - x_n = \lim_{m\to\infty}(x_m - x_n)$ in the g-norm $||\cdot||_G$ for each $n \in \mathbb{N}$; therefore, $x - x_n \in S$, whence $||x - x_n||_E \to 0$.

It is clear from the definition of $||\cdot||_E$ and the coincidence of $||\cdot||_E$ with $||\cdot||_G$ on S that S λ-generates E for any $\lambda > 1$. Lemma 1 implies that the space E is complete as a metric space with respect to $||\cdot||_E$.

If the semigroup S is closed in $(G, ||\cdot||_G)$, then the closedness of S in $(E, ||\cdot||_E)$ follows directly from the inequality $||x||_G \leq ||x||_E \quad (x \in E)$. □

Now, suppose that X is a vector space over the field $\mathbb{R}$. Recall that its subset K is called a *wedge* if $K + K \subset K$ and $\alpha K \subset K$ for any $\alpha \in \mathbb{R}^+$. If, in addition, $K \bigcap (-K) = \{0\}$, then K is called a *cone*.

Any vector space is an Abelian group under addition, and any wedge in it is a subsemigroup. This allows us to consider g-norms on vector spaces and filled wedges in vector spaces with a g-norm.

A g-norm $||\cdot||$ on a vector space X is called an *F-norm* if the multiplication $(\alpha, x) \mapsto \alpha x$ $(\alpha \in \mathbb{R},\ x \in X)$ is jointly continuous. An F-norm $||\cdot||$ is said to be *nondecreasing* if $||\alpha x|| \le ||x||$ for any $x \in X$ and $0 \le \alpha \le 1$. A complete space with an F-norm is called an *F-space*.

Lemma 3. *Suppose that $||\cdot||_X$ is a nondecreasing F-norm on X and K is a filled wedge in the space $(X, ||\cdot||_X)$. Let $(E, ||\cdot||_E)$ be defined as in Lemma 2 with X and K taken for G and S, respectively, that is, as $E = K - K$ and $||x||_E = \inf\{||x'||_X + ||x''||_X \mid x', x'' \in K,\ x = x' - x''\}$. Then $(E, ||\cdot||_E)$ is an F-space.*

Proof. It is easy to show that E is a linear subspace of X. It remains to prove the joint continuity in $(E, ||\cdot||_E)$ of multiplication by a scalar. According to Theorem II.1.12 in [3], for this purpose, it suffices to verify its separate continuity, that is, show that

(1) $||\alpha x_n||_E \to 0$ as $||x_n||_E \to 0$ $(\alpha \in \mathbb{R})$;

(2) $||\alpha_n x||_E \to 0$ as $\alpha_n \to 0$ $(x \in E)$.

Let us prove (1). Without loss of generality, we assume that $\alpha > 0$ and let $[\alpha]$ denote the integer part of α. For each positive integer n, we choose $x'_n, x''_n \in K$ so that $x_n = x'_n - x''_n$ and $||x'_n||_X + ||x''_n||_X \to 0$. We have

$$||\alpha x_n||_E \le ||\alpha x'_n||_X + ||\alpha x''_n||_X \le ||[\alpha]x'_n||_X + ||(\alpha - [\alpha])x'_n||_X + ||[\alpha]x''_n||_X + ||(\alpha - [\alpha])x''_n||_X$$
$$\le ([\alpha] + 1)(||x'_n||_X + ||x''_n||_X) \to 0.$$

Let us prove (2). Take $x', x'' \in K$ such that $x = x' - x''$. We have

$$||\alpha_n x||_E = |||\alpha_n| x||_E \le |||\alpha_n| x'||_X + |||\alpha_n| x''||_X \to 0,$$

because $||\cdot||_X$ is an F-norm and $|\alpha_n| \to 0$. □

Theorem 1 [an analog of the Krein–Smulian theorem] *Let K be a generating filled wedge in an F-space $(X, ||\cdot||)$. Then there exists an equivalent F-norm in which the wedge K is λ-generating for any $\lambda > 1$.*

Proof. According to Theorem I.2.2 in [4], the formula

$$||x||_0 = \sup_{0 \le \beta \le 1} ||\beta x||, \quad x \in X,$$

defines a nondecreasing F-norm $||\cdot||_0$ equivalent to the initial F-norm $||\cdot||$. Let us define a norm $||\cdot||_1$ by analogy with Lemma 2 as

$$||x||_1 = \inf\{||x'||_0 + ||x''||_0 \mid x', x'' \in K,\ x = x' - x''\}, \quad x \in X.$$

According to Lemmas 2 and 3, the space $(X, ||\cdot||_1)$ is an F-space, and the wedge K in it is λ-generating for any $\lambda > 1$. Since $||x||_1 \le ||x||_0$ for all $x \in X$, it follows that the F-norm $||\cdot||_1$ is equivalent to $||\cdot||_0$ (see Theorem II.2.5 in [3]) and, therefore, to $||\cdot||$. □

In [5], a series of corollaries of Theorem 1 similar to the corollaries of the Krein–Smulian theorem given in [6, 7] were proved. Here, we prove only one of them. Note that the assertion of Remark 2 allows us to consider filled wedges in metrizable topological vector spaces.

Corollary 1. Suppose that X_1 and X_2 are completely metrizable topological vector spaces, K_1 is a generating filled wedge in X_1, and K_2 is a closed cone in X_2. Then any linear operator from X_1 to X_2 mapping K_1 to K_2 is continuous.

Proof. According to Theorem 1, we can assume that X_1 is an F-space with an F-norm $||\cdot||_1$ in which the wedge K_1 is filled and λ-generating for some $\lambda > 1$.

According to the closed graph theorem (see, e.g., Theorem II.2.4 in [3]), to prove the continuity of the operator T, it suffices to prove its closedness. Let $\{x_n\}$ be a sequence in X_1 such that $||x_n||_1 \to 0$ and $Tx_n \to y$ in the space X_2. We must show that $y = 0$. Without loss of generality, we assume that

$\sum_{n=1}^{\infty} n||x_n||_1 < \infty$. For each x_n, take $x'_n, x''_n \in K_1$ such that $x_n = x'_n - x''_n$ and $||x'_n||_1 + ||x''_n||_1 \leq \lambda||x_n||_1$ and let $u_n = x'_n + x''_n$. Since

$$\sum_{n=1}^{\infty} ||nu_n||_1 \leq \sum_{n=1}^{\infty} n||u_n||_1 \leq \sum_{n=1}^{\infty} n(||x'_n||_1 + ||x''_n||_1) \leq \lambda \sum_{n=1}^{\infty} n||x_n||_1 < +\infty,$$

it follows that the series $\sum_{n=1}^{\infty} nu_n$ converges to some $w \in K_1$. Using the notation $s \leq t$ for $t - s \in K_i$ $(s, t \in X_i)$, we obtain

$$-w \leq -nu_n \leq nx_n \leq nu_n \leq w,$$

whence

$$-\frac{1}{n}Tw \leq Tx_n \leq \frac{1}{n}Tw.$$

Since the cone K_2 closed, we can pass to the limits in these inequalities and obtain $0 \leq y \leq 0$, that is, $y = 0$. □

Corollary 1 generalizes a result of Lozanovskii (apparently, first published in [8]) about the automatic continuity of positive operators on ordered Banach spaces, which, as mentioned in [6], was the strongest result obtained in this direction at that time. Note that the proof given above is the proof of Theorem 2.1 in [9] adapted to the situation under consideration.

In conclusion, we discuss the property of being filled introduced in Definition 2. Clearly, this property is inherent in closed subgroups of an Abelian group with a g-norm, provided that this group is complete as a metric space. However, there exist examples of different type. It can be verified trivially that a wedge K in a normed space is filled if and only if it ideally convex in the sense of Lifshits [10[(see also Section 1.6 in [9]), that is, for any bounded sequence $\{x_n\} \subset K$ and any sequence $\{\alpha_n\} \subset \mathbb{R}^+$ such that $\sum_{n=1}^{\infty} \alpha_n = 1$, the series $\sum_{n=1}^{\infty} \alpha_n x_n$ converges to an element of K. Exercise 1.14 in [9] implies that any wedge in a finite-dimensional space is filled. Another instructive example can be given in the framework of Sherstnev's approach to the theory of integration with respect to a faithful normal semifinite weight φ on a von Neumann algebra M (see [11, 12]): in the Banach space $L_1^h(\varphi)$ of Hermitian integrable bilinear forms, the cone of closable positive integrable bilinear forms is not generally closed, but it generates $L_1^h(\varphi)$ and is filled.

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