Peculiarities of Elastic Wave Refraction from the Layer with Fractal Distribution of Density

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Abstract

In this study, we underline the peculiarities of the refraction problem of elastic waves from a layer with a fractal density distribution. The refraction problem is reduced to the system of ordinary differential equations with linear coefficients. Analytic solutions for each of its equations are found. The case for the layer with fractal density distribution is investigated numerically. Characteristic maxima of the reflected wave

energy are outlined. Graphs illustrate the dependence of the reflected energy from self-similar properties of the fractal curve.

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1 Statement of the problem

Let harmonic elastic wave of the form $u_0(x)\exp\{i\omega t\}$, in which

$$u_0(x) = A_0 e^{-ik_1 x}, \qquad k_1 = \frac{\omega}{v_1},$$

fall from domain $\{x < 0\}$ onto the layer of thickness L(medium denoted as 2 $\{0 < x < L\}$, with density $\rho_2(x)$ and velocity v_2). As a consequence of diffraction, a portion of $u_1(x)$ of the incident wave is reflected, part of the $u_3(x)$ passes to medium 3 $\{x > L\}$. We need to find the completed diffracted field.

For harmonic oscillations of a homogeneous and isotropic elastic medium, displacement satisfies the wave equation

$$u''(x) + k^2 u(x) = 0, \tag{1}$$

where $k = \omega/v$ is wave number of the medium. The solution of Eq. (1) has the form

$$u(x) = Ae^{-ikx} + Be^{ikx},$$

with stress

$$\sigma(x) = i\omega\rho v(-Ae^{-ikx} + Be^{ikx}).$$

In the general case [25], oscillations of elastic waves are described by equation

$$\rho(x)\frac{\partial^2 u}{\partial t^2} = \frac{\partial \sigma}{\partial x}, \tag{2}$$

where stress associated with displacement via the following relation:

$$\sigma(x) = \rho(x)v^2 u'(x).$$

Here $\rho(x)v^2$ is the modulus $G(x)$. Then Eq. (2) for harmonic waves takes the form

$$\frac{\partial}{\partial x}(\rho(x)v^2\frac{\partial u}{\partial x}) + \rho(x)\omega^2 u = 0.$$

Reflected and transmitted waves are solutions of Eq. (1). They will be sought in the form $u_1(x) = B_1 e^{ik_1 x}$ and $u_3(x) = A_3 e^{-ik_3(x-L)}$.

At the interface, displacement and stress must be continuous. Then for $x = 0$ we have

$$A_0 + B_1 = u_2(0), \tag{3}$$

$$i\omega\rho_1 v_1(-A_0 + B_1) = \rho_2(0)v_2^2 u_2'(0), \tag{4}$$

for $x = L$:

$$u_2(L) = A_3, \tag{5}$$

$$\rho_2(L)v_2^2 u_2'(L) = -i\omega\rho_3 v_3 A_3. \tag{6}$$

Thus, by eliminating the unknowns B_1 and A_3 of Eqs. (3)–(6), we reduce the solution of the diffraction problem to the determination of displacement $u_2(x)$, which satisfies the equation

$$(\rho_2(x)v_2^2 u_2'(x))' + \rho_2(x)\omega^2 u_2(x) = 0 \tag{7}$$

with the boundary conditions of the third type

$$\rho_2(0)v_2^2 u_2'(0) - i\omega\rho_1 v_1 u_2(0) = -i2\omega\rho_1 v_1 A_0,$$
$$\rho_2(L)v_2^2 u_2'(L) + i\omega\rho_3 v_3 u_2(L) = 0. \tag{8}$$

We consider the case where $v_1 = v_2 = v_3 = v$, $\rho_1 = \rho_3 = \rho$, and $\rho_2(x)$ is the fractal interpolation function.

2   Solution of the problem

We consider the set of N adjacent sectors located close to each other instead of the layer $(0 < x < L)$. In each of the layer, the density distribution is given by $\rho_n(x) = k_n x + b_n$, and v is velocity, which is constant. Moreover, $\rho_n(x)v^2$ is a continuous function all ever. Then the wave equation for each layer $(x_{n-1} < x < x_n)$ takes the form

$$((k_n x + b_n)v^2 u_n'(x))' + (k_n x + b_n)\omega^2 u_n(x) = 0. \tag{9}$$

We make the substitution

$$x = \frac{v}{\omega}z - \frac{b_n}{k_n}$$

and consider the following function

$$y_n(z) = u_n\left(\frac{v}{\omega}z - \frac{b_n}{k_n}\right).$$

Equation (2) becomes the Bessel equation

$$zy_n''(z) + y_n'(z) + zy_n(z) = 0.$$

The sign of z for each interval $(x_{n-1} < x < x_n)$ in the obtained equation remains constant and depends on the sign of k_n. After combining the solutions for positive and negative z, the general solution can be written as

$$y_n(z) = A_n J_0(|z|) + B_n Y_0(|z|).$$

Thus, the solution of Eq. (2) will be of the following form:

$$u_n(x) = A_n J_0\left(\left|\frac{\omega}{v}x + \frac{\omega b_n}{vk_n}\right|\right) + B_n Y_0\left(\left|\frac{\omega}{v}x + \frac{\omega b_n}{vk_n}\right|\right). \tag{10}$$

The following conditions must be satisfied at the junction of the layers

$$u_n(x_n) = u_{n+1}(x_n), \quad u'_n(x_n) = u'_{n+1}(x_n), \quad n = 1, \ldots, N-1,$$

which leads to the following equations:

$$A_n J_0(|\xi_n|) + B_n Y_0(|\xi_n|) = A_{n+1} J_0(|\zeta_n|) + B_{n+1} Y_0(|\zeta_n|),$$

$$A_n J_1(|\xi_n|) + B_n Y_1(|\xi_n|) = \frac{\text{sign}\zeta_n}{\text{sign}\xi_n}(A_{n+1} J_1(|\zeta_n|) + B_{n+1} Y_1(|\zeta_n|)),$$

where

$$\xi_n = \frac{x_n k_n + b_n}{vk_n}\omega, \quad \zeta_{n-1} = \frac{x_{n-1} k_n + b_n}{vk_n}\omega, \quad n = 1, \ldots, N.$$

Conditions at boundaries of layers with half-planes are

$$B_1 vu'(0) - i\omega\rho u(0) = -i2\omega\rho A_0,$$

$$(k_N L + b_N)vu'(L) + i\omega\rho u(L) = 0.$$

The conditions define relationships between coefficients

$$A_1[-b_1 \text{sign}\zeta_0 J_1(|\zeta_0|) - i\rho J_0(|\zeta_0|)]+$$

$$+B_1[-b_1 \text{sign}\zeta_0 Y_1(|\zeta_0|) - i\rho Y_0(|\zeta_0|)] = -i2\rho A_0,$$

$$A_N[-(k_N L + b_N)\text{sign}\xi_N J_1(|\xi_N|) + i\rho J_0(|\xi_N|)]+$$

$$+B_N[-(k_N L + b_N)\text{sign}\xi_N Y_1(|\xi_N|) + i\rho Y_0(|\xi_N|)] = 0.$$

The above equations for coefficients A_n and B_n can be written in the matrix form via $\boldsymbol{\Omega} \cdot \mathbf{c} = \mathbf{f}$, where $\mathbf{c} = \{A_1, B_1, A_2, B_2, \ldots, A_N, B_N\}$ and $\mathbf{f} = \{-i2A_0\rho/b_1, 0, \ldots, 0\}$.

The matrix $\boldsymbol{\Omega}$ is a five-diagonal matrix

$$
\begin{pmatrix}
\Omega_{11} & \Omega_{12} & 0 & 0 \\
J_0(|\xi_1|) & Y_0(|\xi_1|) & -J_0(|\zeta_1|) & -Y_0(|\zeta_1|) \\
J_1(|\xi_1|) & Y_1(|\xi_1|) & s_1 J_1(|\zeta_1|) & s_1 Y_1(|\zeta_1|) \\
0 & 0 & J_0(|\xi_2|) & Y_0(|\xi_2|) \\
0 & 0 & J_1(|\xi_2|) & Y_1(|\xi_2|) \\
& & \ddots & \ddots \\
& & 0 & J_0(|\xi_{N-1}|) \\
& & 0 & J_1(|\xi_{N-1}|) \\
& & & 0
\end{pmatrix} \rightarrow
$$

$$
\rightarrow
\begin{pmatrix}
0 & 0 \\
0 & 0 \\
-J_0(|\zeta_2|) & -Y_0(|\zeta_2|) \\
s_2 J_1(|\zeta_2|) & s_2 Y_1(|\zeta_2|) \\
\ddots & \ddots & 0 \\
Y_0(|\xi_{N-1}|) & -J_0(|\zeta_{N-1}|) & -Y_0(|\zeta_{N-1}|) \\
Y_1(|\xi_{N-1}|) & s_{N-1} J_1(|\zeta_{N-1}|) & S_{N-1} Y_1(|\zeta_{N-1}|) \\
0 & \Omega_{NN-1} & \Omega_{NN}
\end{pmatrix},
$$

where $s_n = -\mathrm{sign}\zeta_n / \mathrm{sign}\xi_n$,

$$
\Omega_{11} = -\mathrm{sign}\zeta_0 J_1(|\zeta_0|) - i\frac{\rho}{b_1} J_0(|\zeta_0|),
$$

$$
\Omega_{12} = -\mathrm{sign}\zeta_0 Y_1(|\zeta_0|) - i\frac{\rho}{b_1} Y_0(|\zeta_0|),
$$

$$
\Omega_{NN-1} = -\mathrm{sign}\xi_N J_1(|\xi_N|) + i\frac{\rho}{k_N x_N + b_N} J_0(|\xi_N|),
$$

$$
\Omega_{NN} = -\mathrm{sign}\xi_N Y_1(|\xi_N|) + i\frac{\rho}{k_N x_N + b_N} Y_0(|\xi_N|).
$$

The resulting linear algebraic equation is solved using the Thomas algorithm [26].

3   Fractal interpolation functions

There exist two ways of constructing fractal interpolation functions. Such functions are presented in [1] in the form of attractors of iterated function systems of a special form. In this study, we adopt a more general approach developed by P. Massopust [27].

Let $[a, b] \subset \mathbb{R}$ be a non-empty interval, $1 < N \in \mathbb{N}$ and $\{(x_i, y_i) \in [a, b] \times \mathbb{R} \mid a = x_0 < x_1 < \cdots < x_{N-1} < x_N = b\}$ be points of interpolation. We consider the affine transformation of the plane for each $i = \overline{1, N}$

$$A_i : \mathbb{R}^2 \to \mathbb{R}^2, \quad A_i \begin{pmatrix} x \\ y \end{pmatrix} := \begin{pmatrix} a_i & 0 \\ c_i & \lambda_i \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} \alpha_i \\ \beta_i \end{pmatrix}.$$

We require that for all $i = \overline{1, N}$, the following two conditions be

$$A_i(x_0, y_0) = (x_{i-1}, y_{i-1}), \quad A_i(x_N, y_N) = (x_i, y_i).$$

In that case,

$$a_i = \frac{x_i - x_{i-1}}{b - a}, \quad c_i = \frac{y_i - y_{i-1} - \lambda_i(y_N - y_0)}{b - a},$$

$$\alpha_i = \frac{b x_{i-1} - a x_i}{b - a}, \quad \beta_i = \frac{b y_{i-1} - a y_i - \lambda_i(b y_0 - a y_N)}{b - a},$$

and λ_i, $i = \overline{1, N}$ is considered as a family of parameters. Note that under this definition, the operators A_i transform a given line segment connecting points (x_0, y_0) and (x_N, y_N) to a zigzag-shaped line, by sequentially connecting the points of interpolation to each other.

For each $i = \overline{1, N}$, we denote

$$u_i : [a, b] \to [x_{i-1}, x_i], \quad u_i(x) := a_i x + \alpha_i,$$

$$p_i : [a, b] \to \mathbb{R}, \quad p_i(x) := c_i x + \beta_i,$$

$$p(x) := \sum_{i=1}^{N} (p_i \circ u_i^{-1})(x) \chi_{[x_{i-1}, x_i]}(x),$$

where χ_S is the characteristic function of S. In [27], it is shown that the functional operator T, which make use of the following rule,

$$(Tg)(x) := p(x) + \sum_{i=1}^{N} \lambda_i (g \circ u_i^{-1})(x) \chi_{[x_{i-1}, x_i]}(x),$$

maps continuous functions into some other continuous functions. Moreover, if $|\lambda_i| < 1$ for all $i = \overline{1, N}$, then T is a contraction operator on the Banach space $(C[a, b], \| \ \|_\infty)$ with a contraction ratio $\lambda := \max\{|\lambda_i| \mid i = \overline{1, N}\}$.

According to the fixed point theorem, there exists a unique contraction mapping function $\mathfrak{f} \in C[a, b]$ such that $T\mathfrak{f} = \mathfrak{f}$. Moreover, for any $f \in C[a, b]$, we have

$$\lim_{n \to \infty} \|T^n(f) - \mathfrak{f}\|_\infty = 0.$$

The function $\mathfrak{f}$ is called the fractal interpolation function. It is easy to see that if $f \in C[a, b]$, $f(x_0) = y_0$ and $f(x_N) = y_N$, then $T(f)$ passes through the points of interpolation. In this case, the function $T^n(f)$ is called an interpolation pre-fractal function of order n.

In Fig. 1, the fractal interpolation function constructed from the interpolation points $(0, 1000)$, $(50, 1500)$ and $(100, 1000)$ is shown for values $\lambda_1 = \lambda_2 = 0.5$.
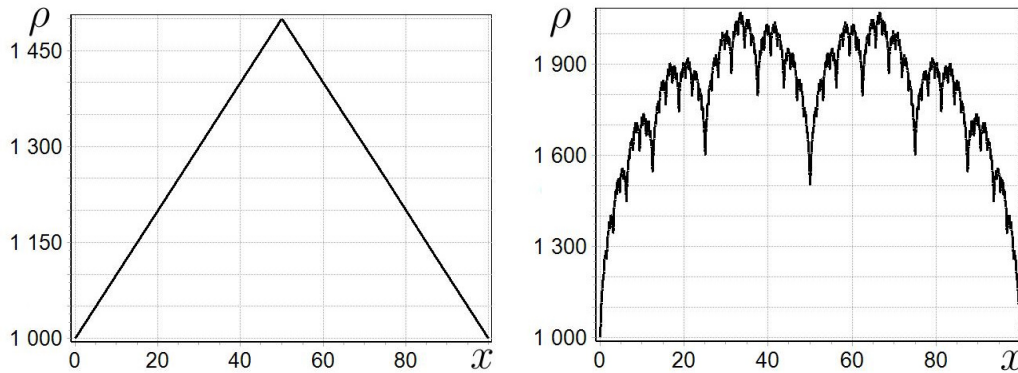

Figure 1: Graphs of pre-fractals, left one is for pre-fractal of the first level, right one is for pre-fractal of the ninth level.

4 Numerical results

We consider the differential Eq. (7) with boundary conditions (8). We seek dependence of solutions $u_2(0)$ (reflected energy) on density $\rho(x)$ and frequency ω. In other words, if $\rho : [0, L] \rightarrow \mathbb{R}_+$ is density of the layer, then we seek the function

$$E\rho : \mathbb{R}_+ \rightarrow [0, 1], \quad E\rho(\omega) = u_2(0).$$

We investigate the behavior of the function $E\rho$ when ρ is a pre-fractal interpolation function. Let ρ be a fractal interpolation function constructed from interpolation points $(0, 1000)$, $(50, 1500)$ and $(100, 1000)$ with parameters $\lambda_1 = \lambda_2 = 0.7$. We denote $\rho^0 \equiv 1000$; ρ^n is a pre-fractal interpolation function of order n. Note that ρ^n is a continuous piecewise linear function consisting of 2^n line segments, passing through points (x_i^n, y_i^n), $i = \overline{0, 2^n}$. From the construction, it follows that $x_i^n = 100 \cdot i \cdot 2^{-n}$. In addition, the set of break points $A^n := \{(x_i^n, y_i^n)\}_{i=0}^{2^n}$ is a subset of A^{n+1}.

The following conclusions are inferred through comparing the graphs for $E\rho^8$ and $E\rho^9$ (Fig. 2). First, the graphs are nearly identical except for the observed $E\rho^9$ maximum at frequencies close to 9700. This effect is due to the fact that the graph of ρ^9 is obtained from ρ^8 by replacing each line segment
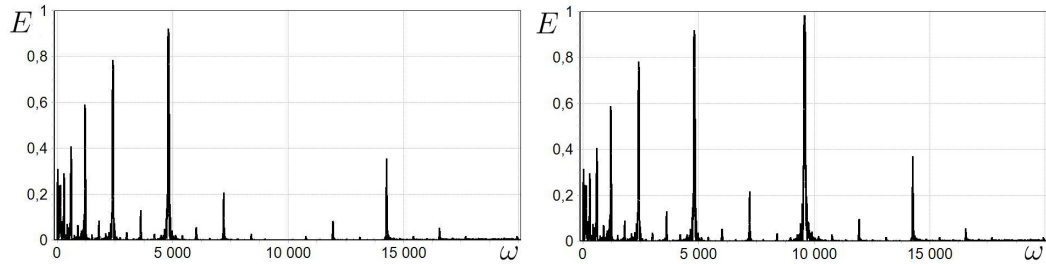
Figure 2: Graphs of functions: left one is for $E\rho^8$, right one is for $E\rho^9$, where ρ^n is pre-fractal interpolation curve constructed by interpolating points $(0, 1000)$, $(50, 1500)$ and $(100, 1000)$ with parameters having values $\lambda_1 = \lambda_2 = 0.7$.

with a zigzag line comprising of two parts. Second, the function $E\rho^n$ inherits self-similar properties of the function ρ^n in the sense that the peaks of the function $E\rho^n$ are the points $C \cdot 2^k$, where C is a constant and $k = \overline{1, n}$. This is due to the fact that the ratio of lengths of (x_i^n, x_{i+1}^n) and $(x_j^{n+1}, x_{j+1}^{n+1})$ equals 2. Third, the presence of distinct peaks in the function $E\rho^n$ is due to the fact that projections onto the axis X of all line segments that make up the graph ρ^n have the same length equal to $100 \cdot 2^{-n}$. In other words, $x_{i+1}^n - x_i^n = 100 \cdot 2^{-n}$ for any $i = \overline{0, 2^n - 1}$.

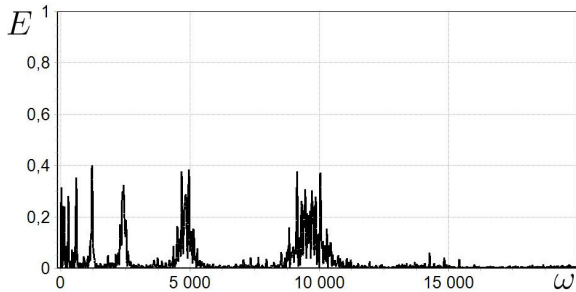
Figure 3: The graph of $E\rho^9$, where ρ^9 is a pre-fractal built by using interpolation points $(0, 1000)$, $(51, 1500)$ and $(100, 1000)$ with parameters having values $\lambda_1 = \lambda_2 = 0.7$.

We displace one point of interpolation by one unit to the right on axis X to verify the last conclusions. Now let ρ be a fractal interpolation function constructed from the interpolation points $(0, 1000)$, $(51, 1500)$ and $(100, 1000)$ for the parameters $\lambda_1 = \lambda_2 = 0.7$. As in the previous case, pre-fractal ρ^n is a continuous piecewise linear function and consist of 2^n line segments passing through the point (x_i^n, y_i^n), $i = \overline{0, 2^n}$. However, $x_i^n \neq 100 \cdot i \cdot 2^{-n}$. It is easy to verify that the number of segments of length $x_{i+1}^n - x_i^n$ is distributed according to the binomial law, i.e. for any $k = \overline{0, n}$ has exactly C_n^k intervals of length $100 \cdot 0.51^k \cdot 0.49^{n-k}$. Comparison of Figs. 3 and 2 shows that the presence

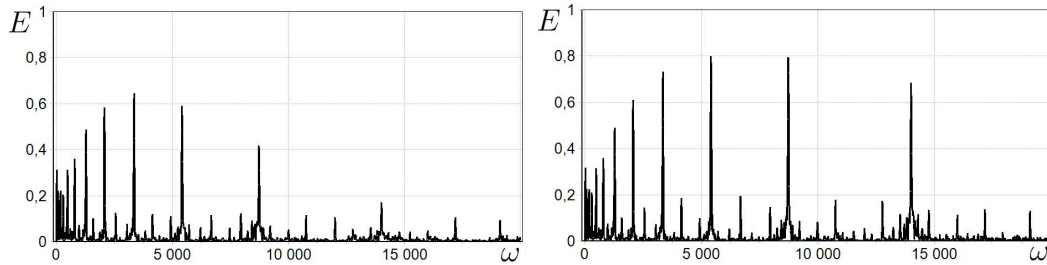of distinct peaks of the function $E\rho^n$ depends on the partitioning of interval $[0, 100]$ by points x_i^n.

Figure 4: Graphs of functions $E\rho^8$ and $E\rho^9$, where ρ^n is a pre-fractal, constructed by interpolating points $(0, 1000)$ $(61.8, 1500)$ and $(100, 1000)$ with parameters having values $\lambda_1 = \lambda_2 = 0.7$.

Some more interesting results are also illustrated in Fig. 4. Selection of the second point of interpolation $(61.8, 1500)$ is not accidental. Let $\alpha \approx 0.618$ be the golden ratio, i.e. the smallest root of the equation $x^2 = 1 - x$. Among the intervals $x_{i+1}^n - x_i^n$ is the C_n^k intervals of length $100\alpha^k(1 - \alpha)^{n-k}$. Since $\alpha^2 = 1 - \alpha$, we find that among the intervals of the $(n+1)$-th level of $x_{i+1}^{n+1} - x_i^{n+1}$ are intervals of lengths exactly equal to the lengths of intervals of the n-th level. Thus, the peaks corresponding to different pre-fractals are superimposed onto each other.

5   Conclusions

Through exploring behavior of the reflected energy $E\rho^n$ for the case, when ρ^n is a pre-fractal, the following conclusions can be made.

Fine bursts of reflected energy show up only in those fractals whose line segments in graph $\rho(x)$ have projections onto axis X of the same length. Moreover, increase in the number of segments of equal length increases the value of reflected energy. These self-similar structures include pre-fractals derived from a pre-fractal of a smaller level by means of dividing the segments into equal parts, or dividing the segments using the golden ratio.

If the fractal is built via the principle of dividing into equal parts, the increase in order does not alter pre-fractal peaks of the reflected energy, but only adds new peaks at higher frequencies.

In case, when the fractal is constructed by dividing the segments in the golden ratio, increase in the order increases pre-fractal peaks of the reflected energy as well as adds new ones at higher frequencies.

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