MATHEMATICAL ANALYSIS OF THE GUIDED MODES OF AN INTEGRATED OPTICAL GUIDE

Evgueni Kartchevski and George Hanson*

Department of Applied Mathematics, Kazan State University,
E-mail: Evgenii.Karchevskii@ksu.ru

*Department of Electrical Engineering, University of Wisconsin-Milwaukee,
E-mail: george@uwm.edu

ABSTRACT

The eigenvalue problem for guided modes of an integrated optical guide is reduced to a strongly-singular domain integral equation. It is proved that the operator of the domain integral equation is a Fredholm operator with zero index. It is also proved that the spectrum of the original problem can only be a set of isolated points.

INTRODUCTION

In this work we study the natural modes of an optical fiber integrated into a three-layer planar medium, which is representative of typical optical circuits. In the absence of a planar background, the basic properties of optical fibers are described in [1]. More recently, rigorous mathematical methods have been applied to the analysis of the modes of optical fibers, see, e.g., [2]-[4]. For the integrated optical guide, rigorous mathematical analysis has been presented for the guided modes in [5]-[7]. Due to the complexity of the integrated optical structure, domain integral equations utilizing appropriate Green's functions (to account for the background media) are a popular practical approach for computing the natural fiber modes [8]-[10]. In this work a rigorous mathematical analysis of the guided modes of an integrated optical fiber is presented based upon a strongly-singular domain integral equation.

STATEMENT OF THE PROBLEM

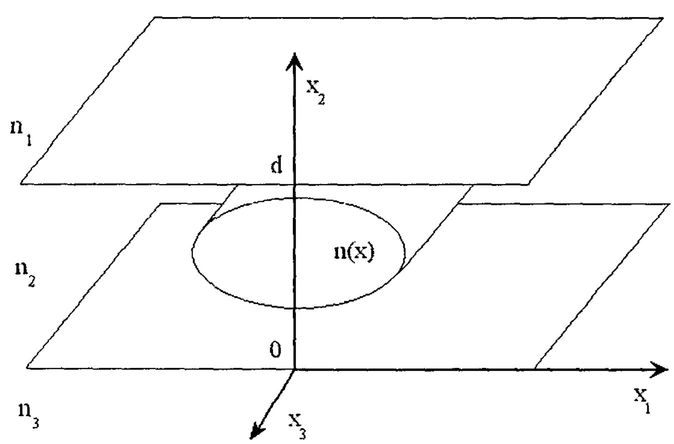

Fig.1. An integrated optical guide.

We consider the guided modes of the integrated optical guide (see Fig. 1). Let the three-dimensional space be occupied by an isotropic source-free medium, and let the refractive index

be prescribed as a positive real-valued function $n = n(x_1, x_2)$ independent of the longitudinal coordinate x_3. We assume that there exists a bounded domain Ω on the plane $R^2 = \{(x_1, x_2): -\infty < x_1, x_2 < \infty\}$ such that $n = n_\infty(x_2)$, $x = (x_1, x_2) \in \Omega_\infty = R^2 \setminus \overline{\Omega}$, where $n_\infty(x_2)$ depends only on the x_2 variable. It is a piecewise-constant function represents the refractive index of so-called associated planar waveguide. For simplicity, we take $n_\infty(x_2) = \{n_1 \text{ if } x_2 > d,\ n_2 \text{ if } 0 < x_2 < d,\ n_3 \text{ if } x_2 < 0\}$. We assume without loss of generality that $n_2 \geq n_3 \geq n_1$. Denote by n_+ the maximum of the function n in the domain Ω. We assume that $\Omega \subset \Omega_2 = \{(x_1, x_2): -\infty < x_1 < \infty, 0 < x_2 < d\}$, $n_+ > n_2$, and also that function n is a continuous function in Ω_2, i.e., that the guide does not have a sharp boundary.
The modal problem can be formulated as a vector eigenvalue problem for the set of differential equations (we use notations [2] for differential operators)

$$\mathrm{Rot}_\beta\, \mathbf{E} = i\omega\mu_0 \mathbf{H}, \quad \mathrm{Rot}_\beta\, \mathbf{H} = -i\omega\varepsilon_0 n^2 \mathbf{E}. \tag{1}$$

Here ε_0, μ_0 are the free-space dielectric and magnetic constants, respectively. We consider the propagation constant β as an unknown complex parameter and radian frequency $\omega > 0$ as a given parameter. We seek non-zero solutions $[\mathbf{E}, \mathbf{H}]$ of set (1) in the space $\left(L_2\left(R^2\right)\right)^6$.

Denote by $\Lambda^{(1)}$ the sheet of the Riemann surface of the function $\sqrt{k^2 n_2^2 - \beta^2}$, where $k^2 = \omega^2 \varepsilon_0 \mu_0$, which is specified by the condition $\mathrm{Im}\sqrt{k^2 n_2^2 - \beta^2} \geq 0$. Denote by β_j the propagation constants of TE and TM modes of the associated planar waveguide [1]. It is well known that there exist no more than a finite number of values β_j. All of the values β_j belong to domain $\{\beta \in \Lambda^{(1)}: \mathrm{Im}\,\beta = 0, kn_3 < |\beta| < kn_2\}$. In a similar way to [7] we can see that the domain $D = \{\beta \in \Lambda^{(1)}: \mathrm{Re}\,\beta = 0\} \cup \{\beta \in \Lambda^{(1)}: \mathrm{Im}\,\beta = 0, |\beta| < \gamma\}$, where $\gamma = \max\limits_j \beta_j$, corresponds to the continuum of propagation constants of radiation modes that do not belong to $\left(L_2\left(R^2\right)\right)^6$. Therefore we do not investigate the values $\beta \in D$.

Definition 1. *A nonzero vector* $[\mathbf{E}, \mathbf{H}] \in \left(L_2\left(R^2\right)\right)^6$ *is referred to as an eigenvector of problem (1) corresponding to an eigenvalue* $\beta \in \Lambda = \Lambda^{(1)} \setminus D$ *if relation (1) is valid. The set of all eigenvalues of problem (1) is called the spectrum of this problem.*

MAIN RESULTS

Theorem 1. *The set* $\{\beta \in \Lambda^{(1)}: \mathrm{Im}\,\beta = 0, |\beta| \geq kn_+\}$ *is free of the eigenvalues of problem (1).*

This theorem was proved in [1] for the case $n_2 = n_3 = n_1$. For the general case the proof is analogous.

If $[\mathbf{E}, \mathbf{H}]$ is an eigenvector of problem (1) corresponding to an eigenvalue $\beta \in \Lambda$, then

$$\mathbf{E}(x) = \left(k^2 n_\infty^2 + \mathrm{Grad}_\beta\, \mathrm{Div}_\beta\right) \frac{1}{n_\infty^2} \int_\Omega \left(n^2(y) - n_\infty^2\right) G(\beta; x, y) \mathbf{E}(y) dy, \tag{2}$$

$$\mathbf{H}(x) = -i\omega\varepsilon_0 \operatorname{Rot}_\beta \int_\Omega \left(n^2(y) - n_\infty^2\right) G(\beta; x, y)\mathbf{E}(y)dy, \quad x \notin \partial\Omega_2, \tag{3}$$

where function G is the well known tensor Green function [9]. For any $(x,y) \in \Omega^2$ the function G is analytic for $\beta \in \Lambda$. Passing the operator $\operatorname{Grad}_\beta \operatorname{Div}_\beta$ under the integral in relation (2), and using the differentiation rule [11] for weakly singular integrals we obtain a nonlinear spectral problem for a strongly-singular domain integral equation

$$A(\beta)\mathbf{E} = 0, x \in \Omega; \quad A : \left(L_2(\Omega)\right)^3 \to \left(L_2(\Omega)\right)^3. \tag{4}$$

Theorem 2. *For all* $\beta \in \Lambda$ *the operator* $A(\beta)$ *is Fredholm with zero index.*

This theorem is proved by general results of the theory of singular integral operators.

Definition 2. *A nonzero vector* $\mathbf{E} \in \left(L_2(\Omega)\right)^3$ *is called an eigenvector of the operator-valued function* $A(\beta)$ *corresponding to an eigenvalue* $\beta \in \Lambda$ *if relation (4) is valid.*

Theorem 3. *Suppose* $[\mathbf{E},\mathbf{H}] \in \left(L_2(R^2)\right)^6$ *is an eigenvector of the problem (1) corresponding to an eigenvalue* $\beta \in \Lambda$. *Then* $\mathbf{E} \in \left(L_2(\Omega)\right)^3$ *is the eigenvector of the operator-valued function* $A(\beta)$ *corresponding to the same eigenvalue* β. *Suppose* $\mathbf{E} \in \left(L_2(\Omega)\right)^3$ *is an eigenvector of the operator-valued function* $A(\beta)$ *corresponding to an eigenvalue* $\beta \in \Lambda$ *and also let vector* $[\mathbf{E},\mathbf{H}]$ *is defined by (3), (4) on* R^2 *Then* $[\mathbf{E},\mathbf{H}] \in \left(L_2(R^2)\right)^6$ *and* $[\mathbf{E},\mathbf{H}]$ *is the eigenvector of the problem (1) corresponding to the same eigenvalue* β.

This theorem is proved by direct calculations.

Theorem 4. *The spectrum of problem (1) can be only a set of isolated points on* Λ.

This theorem is followed from theorems 1-3 and general results of the theory of operator-valued functions [12].

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