

# ASSESSMENT OF THE LEVEL OF STUDENTS' PRODUCTIVE ACTIVITY IN SOLVING MATHEMATICAL TASKS

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## Abstract

One of the major problems of mathematical education is teaching students the ability to solve mathematical tasks. This explains the relevance of intrinsic understanding of math problem characteristics by a teacher. The key characteristic of a mathematical problem is the level of difficulty, in which it is possible to select the objective (structural and content-related features) and subjective (associated with the experience of a particular individual) components. The account of these components determines the choice of the classification of mathematical tasks used in teaching. From the point of view of subjectivity, the classification of tasks is based, firstly, on the requirements for activities to be carried out by a student in the process of its solution, and, secondly, on his experience. The classification of tasks according to requirements for activities provides reproductive and productive levels of learning. For the most efficient and effective teaching of mathematics it is necessary to develop clear criteria for the selection of tasks for each of these levels and for the evaluation of solved task by students.

68 10-grade students (16-17 years old) from one of the lyceum schools in Kazan (Russia) took part in an empirical study to identify the dependence of the quality of the solution of plane geometry tasks on the level of their difficulty. The students had taken the course of plane geometry, and they were preparing for the unified state exam in mathematics after grade 11. In accordance with the training program and the requirements for the mathematical preparation of students on this stage of learning, the system of tasks on the theme of "Circles and polygons" of reproductive ("student" and the model) and productive (creative and non-typical) levels of learning activities (according to the classification by V.P. Bepalko) was designed. This topic is one of the most difficult in the school course of plane geometry, as for solving such problems there is the necessity for systemic understanding of the construction "circle – polygon". The experiment was conducted for two days with a one week break.

The analysis of the solved problems has shown that 72% of the students have a sufficient level of reproductive activity and 97% – unsatisfactory level of productive activity in solving problems. Obviously, the vast majority of students has no experience in productive activities when solving plane geometry problems. The qualitative analysis of the results has revealed a number of deficiencies in the ability to work with a geometric task: to carry out the analysis of the content correctly, construct a geometrical drawing, find the relationship between elements, etc. The result of this research is to develop criteria for the selection of plane geometry tasks aimed at the development of the productive level of learning activities. Using that by a teacher will enhance the efficiency of teaching mathematics and raise the level of students' mathematical preparation.

Keywords: teaching mathematics, school mathematical task, reproductive activity, productive activity.

## 1 INTRODUCTION

The search for effective methods of teaching students the ability to solve mathematical problems has been an important issue of the methodology of mathematics for many years. Student's satisfaction and joy in the process of solving mathematical problems, stimulating his interest in learning mathematics, depend on the level of its novelty and capability of the learner to overcome the existing difficulty in it. When solving mathematical problems one can create such a problematic situation having a clear and reasonable idea about the level of difficulty for each individual learner. The concept of "level of difficulty of math problems" covers a set of objective and subjective components [1]. The objective (invariant) component of the level of difficulty of a mathematical problem relates to its content. The subjective component allows a task to be different for different groups of people who solve it. The objective component of math problems contains an external (informational) and internal structure. The informational structure includes data, sought solutions of a task, relationships between them as well as a theoretical basis and a method of its solution. The internal structure is determined by the number of elements, relations and types of relations between them [2]. For example, the task

about the property of geometric locus (hereinafter GL), equidistant from the ends of the interval (of a perpendicular bisector) has the same ideas in its informational structure as the task of the geometric locus equidistant from three points (the center of the circumcircle of the triangle). Adding a number of similar chains of reasoning to the structure of the problem increases the difficulty by several orders of magnitude. Therefore, we can conclude about the deep interplay between external and internal structures of the objective element of difficulty of a mathematical task. Multiple use of the same fact in the same task enriches the external structure of the task. The previous example shows that the repetition of the same fact three times leads to new properties of GL and the relationship with other geometric concepts and their properties.

The subjective component of the level of difficulty is determined by: 1) the level of «knowness» of a task for a student, 2) his cognitive abilities; 3) conditions in which he is proposed to solve a math problem (in a classroom, at home, at a competition, etc.). That is why the learning process is organized so as to reverse the subjective component of the level of difficulty. For this purpose, there is the separation of children by age (level of cognitive development in accordance with age) and by classes where children are taught together for several years (creation of approximately the same educational experience; training in the same conditions). At school the subjective component of the level of difficulty of the mathematical task is defined, first and foremost, by the individual level of cognitive development of a child, his hereditary characteristics, and educational environment (education in the family, additional math lessons, etc.).

## 2 METHODOLOGY

Consideration of the concept "level of difficulty of a math problem", stages of solving a problem was the subject of several works of the authors of articles ([1], [3], [4], [5]). The results of the researches allow to conclude that to improve the efficiency of solution of mathematical tasks it is necessary to develop the ability to diagnose their level of difficulty not only by teachers, but also by [1]. For this purpose, it is essential to use tasks of a productive level of learning. The importance of using a chosen system of mathematical problems in a special way aimed at the development of cognitive and mental activity of students has been noted by many researchers ([6], [7], [8], [9], [10], [11]). They analyzed the contents of used tasks in the learning process from the perspective of: (a) characteristics of the task (strategies of solutions, the number and the type of representation), (b) cognitive demands (memorization, typical, non-typical, research).

The productivity of cognitive activity in a problem-solving process is influenced by internal (procedural) and external (directed to self-esteem) components of the evaluation. Stressful experiences of failure in solving mathematical problems are more associated with external social conditions, not personal characteristics of students; moreover, depending on the success of solving tasks students have different opinions of the difficulty of proposed mathematical tasks [12].

Arbaugh and Brown (2005) examined the levels of cognitive demand when solving mathematical problems. They demonstrated the growth in knowledge of pedagogical content (thinking about a mathematical task) and the change in practice (choosing mathematical tasks). According to them, "engaging teachers in learning to examine mathematical tasks using the LCD (Level of Cognitive Demand) criteria supports both the growth in knowledge of pedagogical content (ways of thinking about mathematical tasks) and the change in practice (choosing mathematical tasks)" [8].

Understanding of the level of difficulty of tasks allows to present their classification in various ways ([2], [6], [13]). We used the classification of mathematical tasks according to the level of mastering of activities by V. P. Bepalko (1989) [14] which allows to take into account the objective and subjective components of the difficulty level of mathematical problems. First, Bepalko distinguishes two types of learning activities: reproductive and productive, which are related to the degree of «knowness» of a task to a student. The reproductive activity is implemented in the course of solving tasks of a student's level of learning (tasks for identification, distinction, classification of the studied objects) and algorithmic level of learning (tests-substitution, constructive and routine tasks). The productive activity is implemented in the course of solving tasks of the heuristic level of mastering (a non-typical task), and creative level of learning (tasks-problems). Such differentiation of task types allows to consider the objective characteristics of the level of difficulty of the task. Bepalko in his classification takes into account the coefficient of mastering  $K_o$ , which indicates the quality of learning [14]. It is equal to the ratio of the number of operations done by students when solving a problem to the number of operations required for the correct solution of the problem. This coefficient must be found separately for all types of tasks of reproductive and productive levels.

The selection of mathematical tasks of various difficulty levels for teaching students allows to assess the level of preparation of students working with certain concepts.

### 3 RESULTS

In 2016, there has been conducted an empirical study aimed at identifying the dependence of the quality of solution of mathematical tasks on their level of difficulty. It was attended by 68 10-grade students (16-17 years) from one of the lyceums of Kazan (Russia). In grade 9 they had taken the course of plane geometry. During the studies the students were preparing actively for the unified state examination in mathematics after the final 11th grade. Students at various levels of training in mathematics have been selected: 10A – with the focus on physics and mathematics; 10B – chemical-biological; 10B – humanitarian. In accordance with the training program and the requirements for the mathematical preparation of students of this stage of training, there has been designed the system of tasks on the theme of "Circles and polygons" of different levels of mastery. For the solution of tasks of both levels of learning a student should be familiar with the following concepts: a circle, inscribed and central angles, a secant and a chord of a circle, a trapezoid, an inscribed trapezoid, an inscribed triangle, the location of circles relative to each other. The need for a systemic understanding of the construction "circle – polygon" by students makes this topic one of the most difficult in the course of plane geometry. When solving tasks on this subject, it is necessary to consider the properties of each shape (circles, polygons, intervals and angles related to circles) as well as the resulting new properties of the construction as a whole. If we turn to the internal structure (complexity) of tasks on incircle and circumcircle, i.e. the number of elements, relations and types of relations between them [2], even simple tasks on this subject will refer to the most challenging tasks of the school course of geometry. Here are some examples of tasks at different levels of learning, which were offered to students.

**Task (student level of mastering).** The distance between the center of the circle and the line is 5.4 cm, Determine the mutual position of the circle and the straight line if the radius of the circle is 4.5 cm

*(Put a tick next to the correct answers).*

- a) do not intersect ,
- b) relate ,
- c) intersect at one point ,
- d) intersect at two points .

**Task (creative level of mastering).** There have been drawn numerous lines through the point M, on which this circle with the central point O truncates intervals, being its chords. Find a multitude of centers of such chords, if the point M is: a) outside the circle; b) inside the circle; C) on the circle. *(Write the solution and reasoning of the problem in detail).*

The experiment was conducted for two days with a one-week break. Each day 4 math tasks were offered: 3 tasks on a reproductive level of mastering and 1 task on a productive one. The students were given worksheets in which the statements of the tasks were presented; after each statement the space for the solution of the corresponding problem was given. Time to solve four tasks was 45 minutes, but since students participated voluntarily (without receiving any marks for their solution of the tasks from the teacher), individual students handed their work in 20 minutes after the start of work. The motivation for the solution of geometric problems was low, but there was interest in participating in experimental work.

Some study results are presented in table 1, which shows that 94% of the students master a student's level of learning, 21% – a typical level, 2% (one student) can speculate productively on solving geometric problems on a given topic.

**Table 1.** Results of the study of levels of mastering of 10-grade students on the theme "Circles and polygons".

No	Level of mastering		Range K (min – 0; max – 1)	Number of students	Average value
1	Reproductive	Student	0	0	0,72
			0,25 – 0,5	4	
			0,51 – 0,75	2	
			0,76 – 0,99	19	
			1	41	
		Typical	0 – 0,24	10	
			0,25 – 0,5	22	
			0,51 – 0,75	20	
			0,76 – 0,99	4	
			1	10	
2	Productive	Non-typical	0	55	0,024
			0,17	1	
			0,83	1	
		Creative	0	45	
			0,2 – 0,4	3	
			1	1	

It should be noted that the vast majority of the students trying to solve problems of non-typical and creative levels of learning were unable to correctly construct a geometric drawing to the task and to show various properties of geometric shapes. So, out of 17 attempts to build a drawing to Task No. 8 of a creative level, only two were correct. In the solutions of typical tasks, students did not specify a theorem which was used in the solution, confused elements in ratios (for example, the product of the secant intervals).

#### 4 CONCLUSIONS

The results of the study show the necessity of introducing a larger number of tasks aimed at productive activities of students into the content of geometry course. As noted by Smith & Stein (1998), "when classifying a mathematical task as "good" that is, as having the potential to engage students in high level thinking, we first consider the students - their age, grade level, prior knowledge and experiences - and the norms and expectations for work in their classroom" [7]. For the formation of students' thinking it is necessary to provide students with the opportunity to solve problems of high level of learning (heuristic and creative), not just on memorization or reproduction. Emphasizing the importance of development of cognitive activity of students in solving mathematical problems, Tekkumru-Kisa, M., Stein, M. K., Schunn, C. (2015) consider tasks that teach students to think: problems with scientific content and practice-oriented [11]. They performed the analysis of the learning tasks in two dimensions: cognitive demand and the integration of scientific content and practices. Simon, M. A. & Tzur, R. (2004) said that "Indeed if more time were spent in classrooms with students engaged in working on cognitively demanding non-routine tasks, as opposed to exercises in which a known procedure is practiced, students' opportunities for thinking and learning would likely be enhanced" [15]. The use of cognitive demanding tasks in the solving process when students as the whole class practice discussion and debate allows to achieve higher opportunities to obtain high-quality knowledge [16]. Productive small-group discussions of problems of high level of cognitive demand, in which participants have the opportunity to contribute with different roles and focus on conceptual understanding, allow students to feel comfortable. This practice enables them to maintain a high level of education [17].

It is worth mentioning that one of the difficult aspects of our study was that the students were offered the tasks of plane geometry. Geometry has always been a "special" branch of mathematics. Broad

discussions about the differences between geometric problems and their particular impact on the development of schoolchildren have been conducted over several years. The subject of a large-scale study of spouses Pierre & Dina Hiele-Geldolf [18] has become the place and role of visualization in learning geometry. They built a model of learning geometry, according to which there is a certain correlation between the level of geometry teaching and developmental levels of geometric thinking of students.

A number of studies have been dedicated to the search for effective methods of teaching students to solve planimetric tasks. Duyen et al. (2016) proposed a new approach to design an optimal system that helps to solve planimetric tasks. The intellectually developed method is intended to provide a solver with technical support. The strategy of teaching using this innovative method is aimed at developing students' abilities. [19]. Didactic activities which focussed on the use of a series of orientations of heuristic essence allow the students to be protagonists in the building of their own knowledge. According to Santos Loo et al. (2017), the activities designed refer to a logical sequence in the application of the heuristic instruction for the exercising of this cognitive domain, reflected in the quality of learning and the stimuli to the students to express their ideas and judgements related to motivation towards learning Plane Geometry [20]. The research by Uyangor (2012) was dedicated to the study of the effect of project-based learning (student-driven, teacher-facilitated approach to learning) on the performance of the task solution in Plane Geometry. It was concluded that there was a significant difference between pre-and posttests of students' achievements and attitudes [21]. Falileeva M. (2013) conducted a qualitative analysis of the students' solved tasks of plane geometry. The study showed that it is necessary to pay close attention to the production of methods of geometric drawing and its use in the analysis of the problem [22]. The findings by Delahunty et al. (2013) have illustrated a possible conditioned approach to building and representing visual-mental images. This conditioning may have the overall effect of militating against students' ability to synthesise, manipulate and communicate both perceptual and conceptual graphical information. These findings raise a number of questions relating to the style of cognitive activity that students are engaging in as part of contemporary graphical education [23].

The important issue is the establishment of effective private methods and techniques of work with constructions which include several concepts (in particular, on the theme «Circles, and polygons»). The combination of several concepts within one math problem increases its objective component; in this case, the nature and level of complexity of the tasks correlate with the number of new properties arising in this combination. The selection of this type of planimetric problems in teaching contributes to a productive learning level of students. The study showed that the use of such tasks by a teacher will strengthen the effectiveness of geometry teaching and improve students' overall level of mathematical preparation.

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