Integral Equation Methods in Optical Waveguide Theory

Alexander Frolov and Evgeny Kartchevskiy

Abstract Optical waveguides are regular dielectric rods having various cross-sectional shapes where generally the permittivity may vary in the waveguide's cross section. The permittivity of the surrounding medium may be a step-index function of coordinates. The eigenvalue problems for natural modes (surface and leaky eigenmodes) of inhomogeneous optical waveguides in the weakly guiding approximation formulated as problems for Helmholtz equations with partial radiation conditions at infinity in the cross-sectional plane. The original problems are reduced with the aid of the integral equation method (using appropriate Green functions) to nonlinear spectral problems with Fredholm integral operators. Theorems on the spectrum localization are proved. It is shown that the sets of all eigenvalues of the original problems may consist of isolated points on the Riemann surface and each eigenvalue depends continuously on the frequency and permittivity and can appear or disappear only at the boundary of the Riemann surface. The original problems for surface waves are reduced to linear eigenvalue problems for integral operators with real-valued symmetric polar kernels. The existence, localization, and dependence on parameters of the spectrum are investigated. The collocation method for numerical calculations of the natural modes is proposed, the convergence of the method is proved, and some results of numerical experiments are discussed.

1 Introduction

Optical fibers are dielectric waveguides (DWs), i.e., regular dielectric rods, having various cross-sectional shapes, and where generally the refractive index of the dielectric may vary in the waveguide's cross section [10]. Historically, the first DWs

A. Frolov • E. Kartchevskiy (✉)
Kazan (Volga Region) Federal University, Kremlevskaya 18, Kazan, Russia
e-mail: Alexander_ksu@mail.ru; ekarchev@yandex.ru

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to be studied were step-index waveguides with circular cross section; interior to the waveguide, the refractive index was either homogeneous or coaxial-layered. In these cases, by using separation of variables, modal eigenvalue problems are easily reduced to families of transcendental dispersion equations associated with the azimuthal indices (see, e.g., [6, 10]). The study of the source-free electromagnetic fields, called natural modes, that can propagate on DWs necessitates that longitudinally the rod extend to infinity. Since often DWs are not shielded, the medium surrounding the waveguide transversely forms an unbounded domain, typically taken to be free space. This fact plays an extremely important role in the mathematical analysis of natural waveguide modes and brings into consideration a variety of possible formulations. They differ in the form of the condition imposed at infinity in the cross-sectional plane, and hence in the functional class of the natural-mode field. This also restricts the localization of the eigenvalues in the complex plane of the eigenparameter [3]. During recent years partial condition has been widely used for statements of various wave propagation problems [9]. All of the known natural-mode solutions (i.e., guided modes, leaky modes, and complex modes) satisfy partial condition at infinity. The wavenumbers β may be generally considered on the appropriate logarithmic Riemann surface. For real wavenumbers on the principal ("proper") sheet of this Riemann surface, one can reduce partial condition to either the Sommerfeld radiation condition or to the condition of exponential decay. Partial condition may be considered as a generalization of the Sommerfeld radiation condition and can be applied for complex wavenumbers. This condition may also be considered as the continuation of the Sommerfeld radiation condition from a part of the real axis of the complex parameter β to the appropriate logarithmic Riemann surface.

In this paper we consider the problem of determination of eigenwaves propagating along inhomogeneous optical waveguides with piecewise continuous permittivity in the weakly guiding approximation when the hybrid-mode character of the normal waves is neglected. In this approximation, the considered problem on eigenwaves is virtually equivalent to the determination of eigenoscillations of the cylindrical resonators having the same cross section as the guides under study [4].

We reduce the analysis to the boundary eigenvalue problems for Helmholtz equations with partial radiation conditions at infinity in the cross-sectional plane and latter to finding characteristic numbers of integral equations.

The methods of the spectral theory of integral operator-valued functions were applied to the study of the oscillations in slotted resonators [8] and to the study of the normal waves of slotted waveguides [7].

2 Statement of the Problem

We consider the natural modes of an inhomogeneous optical fiber. Let the three-dimensional space be occupied by an isotropic source-free medium, and let the refractive index be prescribed as a positive real-valued function $n = n(x_1, x_2)$

independent of the longitudinal coordinate x_3 and equal to a constant n_∞ outside a cylinder. The axis of the cylinder is parallel to the x_3–axis, and its cross section is a bounded domain Ω on the plane $R^2 = \{(x_1, x_2) : -\infty < x_1, x_2 < \infty\}$ with a boundary Γ belongs to the class $C^{1,\alpha}$. Denote by Ω_∞ the unbounded domain $\Omega_\infty = R^2 \backslash \overline{\Omega}$, and denote by n_+ the maximum of the function n in the domain Ω, where $n_+ > n_\infty$. Let the function n belong to the space of real-valued twice continuously differentiable in Ω functions. It is supposed that U is the space of twice continuously differentiable in Ω and Ω_∞, continuous and continuously differentiable in $\bar{\Omega}$ and $\bar{\Omega}_\infty$ real-valued functions. The modal problem for the weakly guiding optical fiber can be formulated [2] as an eigenvalue problem for a Helmholtz equation:

$$\left[\Delta + \left(k^2 n^2 - \beta^2\right)\right] u = 0, \quad x \in R^2 \backslash \Gamma. \tag{1}$$

Here $k^2 = \omega^2 \varepsilon_0 \mu_0$; ε_0, μ_0 are the free-space dielectric and magnetic constants, respectively. We consider the propagation constant β as a complex parameter and radian frequency ω as a positive parameter. We seek nonzero solutions u of equation (1) in the space U. Functions u have to satisfy the conjugation conditions:

$$u^+ = u^-, \quad \frac{\partial u^+}{\partial \nu} = \frac{\partial u^-}{\partial \nu}, \quad x \in \Gamma. \tag{2}$$

Here ν is the normal vector. We say that function u satisfies partial condition if u can be represented for all $|x| > R$ as

$$u = \sum_{l=-\infty}^{\infty} a_l H_l^{(1)}(\chi r) \exp(il\varphi), \tag{3}$$

where $H_l^{(1)}$ is the Hankel function of the first kind and index l, (r, φ) are the polar coordinates of the point x and $\chi(\beta) = \sqrt{k^2 n_\infty^2 - \beta^2}$. The series in (3) should converge uniformly and absolutely.

The Hankel functions $H_l^{(1)}(\chi(\beta) r)$ are many-valued functions of the variable β. If we want to consider these functions as holomorphic functions, it is seen that β should be considered on the set Λ, which is the Riemann surface of the function $\ln(\chi(\beta) r)$. This is due to the fact that Hankel functions can be represented as

$$H_l^{(1)}(\chi(\beta) r) = c_l^{(1)}(\chi r) \ln(\chi r) + R_l^{(1)}(\chi r), \tag{4}$$

where $c_l^{(1)}(\chi r)$ and $R_l^{(1)}(\chi r)$ are holomorphic single-valued functions [1]. The Riemann surface Λ is infinitely sheeted, with each sheet having two branch points, $\beta = \pm k n_\infty$. More precisely, due to the branching of $\chi(\beta)$ itself, we consider an infinite number of logarithmic branches Λ_m, $m = 0, \pm 1, \ldots$, each consisting of two square-root sheets of the complex variable β: $\Lambda_m^{(1)}$ and $\Lambda_m^{(2)}$. By $\Lambda_0^{(1)}$ denote the principal ("proper") sheet of Λ, which is specified by the following conditions:

$$-\pi/2 < \arg \chi(\beta) < 3\pi/2, \quad \mathrm{Im}\left(\chi(\beta)\right) \geq 0, \quad \beta \in \Lambda_0^{(1)}. \tag{5}$$

The "improper" sheet $\Lambda_0^{(2)}$ is specified by the conditions

$$-\pi/2 < \arg\chi(\beta) < 3\pi/2, \quad \mathrm{Im}(\chi(\beta)) < 0, \quad \beta \in \Lambda_0^{(2)}. \tag{6}$$

Denote also the whole real axis of $\Lambda_0^{(1)}$ as $R_0^{(1)}$ and that of $\Lambda_0^{(2)}$ as $R_0^{(2)}$. All of the other pairs of sheets $\Lambda_{m\neq 0}^{(1),(2)}$ differ from $\Lambda_0^{(1),(2)}$ by shift in $\arg\chi(\beta)$ equal to $2\pi m$ and satisfy the conditions

$$-\pi/2 < \arg\chi(\beta) < 3\pi/2, \quad \mathrm{Im}(\chi(\beta)) \geq 0, \quad \beta \in \Lambda_m^{(1)}, \tag{7}$$

$$-\pi/2 < \arg\chi(\beta) < 3\pi/2, \quad \mathrm{Im}(\chi(\beta)) < 0, \quad \beta \in \Lambda_m^{(2)}. \tag{8}$$

Hence on $\Lambda_0^{(1)}$ there is only a pair of branch-cuts dividing it from $\Lambda_0^{(2)}$; they run along the real axis at $|\beta| < kn_\infty$ and along the imaginary axis. On $\Lambda_0^{(2)}$, additionally, there is a pair of branch-cuts dividing it from $\Lambda_{\pm 1}^{(1)}$; they run along the real axis at $|\beta| > kn_\infty$.

Definition 1. A nonzero function $u \in U$ is referred to as an eigenfunction (generalized mode) of the problem (1)–(3) corresponding to some eigenvalues $\beta \in \Lambda$ and $\omega > 0$ if the relations of problem (1)–(3) are valid. The set of all eigenvalues of the problem (1)–(3) is called the spectrum of this problem (Fig. 1).

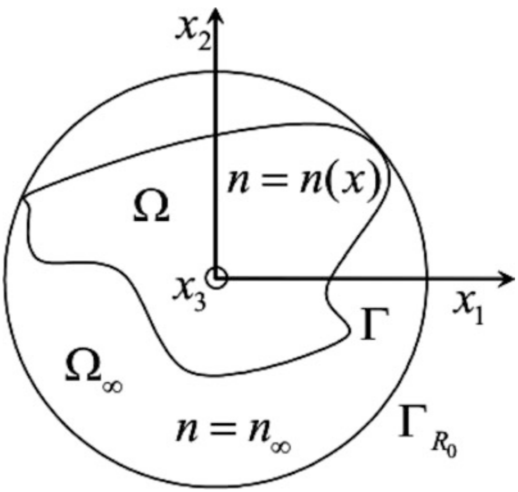

Fig. 1 Cross section of the waveguide which is in free space

Let us describe the geometry and give the problem statement for a waveguide in the half-space. Denote by Ω_∞ the unbounded domain $\Omega_\infty = \{x \in R^2 : x_1 \in R, x_2 > 0\} \backslash \overline{\Omega}$. The refractive index n_∞ of Ω_∞ is very different from the refractive index of the bottom half-space n_b. Suppose that the refractive indices of Ω and

Ω_∞ are approximately equal. So we can suggest that $u = 0$ for $x_2 = 0$ and use the approximation of weakly guiding waveguide (Fig. 2).

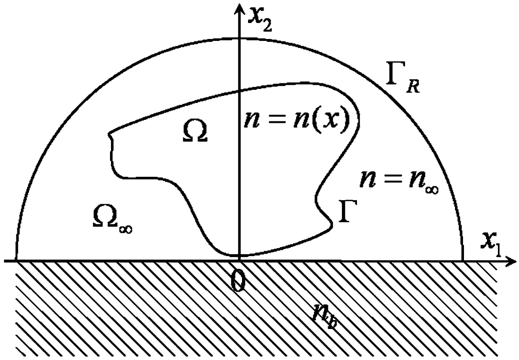

Fig. 2 Cross section of the waveguide which is in the half-space

The problem of the waveguide in the half-space yields as an equivalent Helmholtz equation

$$\left[\Delta + \left(k^2 n^2 - \beta^2\right)\right] u = 0, \quad x \in \Omega \cup \Omega_\infty. \tag{9}$$

Functions u also have to satisfy the following conjugation conditions:

$$u^+ = u^-, \quad \frac{\partial u^+}{\partial \nu} = \frac{\partial u^-}{\partial \nu}, \quad x \in \Gamma. \tag{10}$$

In this case partial condition means that the function u can be represented for all $|x| > R$ as

$$u = \sum_{l=-\infty}^{\infty} a_l H_l^{(1)}(\chi r) \sin(l\varphi). \tag{11}$$

3 Spectrum Properties

In this section the original problems will be reduced to the spectral problems for integral operators. Some results of integral operators spectrum properties will be formulated. If u is an eigenfunction of problem (1)–(3) corresponding to some eigenvalues $\beta \in \Lambda$ and $\omega > 0$, then [2] the following integral presentation is valid:

$$v(x) = \lambda \int_\Omega K_{1,2}(x,y) v(y) dy, \quad x \in \Omega, \tag{12}$$

where

$$K_{1,2}(x,y) = \Phi_{1,2}(\beta;x,y)p(x)p(y), \quad v(x) = u(x)p(x),$$

$$p(x) = \sqrt{((n(x))^2 - n_\infty^2)/(n_+^2 - n_\infty^2)}, \quad \lambda = k^2(n_+^2 - n_\infty^2),$$

$$\Phi_1 = \frac{i}{4}H_0^{(1)}(\chi(\beta)\,|x-y|).$$

An equivalent integral presentation for a waveguide in the half-space is also valid. Note that function

$$\Phi_2 = \frac{i}{4}(H_0^{(1)}(\chi(\beta)\,|x-y|) - H_0^{(1)}(\chi(\beta)\,|x-y^*|))$$

is Green's function of the problem (9)–(11). Here y^* is $(y_1, -y_2)$.

The original problem (1)–(3) is spectrally equivalent [2] to the problem (12). Let the frequency ω have a fixed positive value. Rewrite problem (12) in the form of spectral problem for operator-valued function

$$A(\beta)v = 0, \tag{13}$$

where $A(\beta) = I - \lambda T(\beta) : L_2(\Omega) \to L_2(\Omega)$, T is the operator, defined by the right side of the Eq. (12), and I is the identical operator.

Definition 2. Let $\omega > 0$ be a fixed parameter. A nonzero vector $v \in L_2(\Omega)$ is called an eigenvector of operator-valued function $A(\beta)$ corresponding to an eigenvalue $\beta \in \Lambda$ if the relation (13) is valid. The set of all $\beta \in \Lambda$ for which the operator $A(\beta)$ does not have a bounded inverse operator in $L_2(\Omega)$ is called the spectrum of operator-valued function $A(\beta)$. Denote by $\mathrm{sp}(A) \subset \Lambda$ the spectrum of operator-valued function $A(\beta)$.

Theorem 1. *The following assertions hold:*

1. *For all $\omega > 0$ and $\beta \in \Lambda$ the operator $T(\beta)$ is compact.*
2. *If ω has a fixed positive value, then the spectrum of the operator-valued function $A(\beta)$ can be only a set of isolated points on Λ, moreover on the principal sheet $\Lambda_0^{(1)}$ it can belong only to the set*

$$G = \left\{\beta \in R_0^{(1)} : kn_\infty < |\beta| < kn_+\right\}.$$

3. *Each eigenvalue β of the operator-valued function $A(\beta)$ depends continuously on $\omega > 0$ and can appear and disappear only at the boundary of Λ, i.e., at $\beta = \pm kn_\infty$ and at infinity on Λ.*

This theorem was proved in [3]. The equivalent theorem for the second problem is valid. The proof of this theorem is based on the spectral theory of operator-valued Fredholm holomorphic functions.

The well-known surface modes satisfy to $\beta \in G$. In this case $\chi(\beta) = i\sigma(\beta)$, where $\sigma(\beta) = \sqrt{\beta^2 - k^2 n_\infty^2} > 0$. Let transverse wavenumber σ have a fixed positive value. Rewrite problem (12) in the form of usual liner spectral problem with integral compact operator

$$v = \lambda T(\sigma) v, \quad T: \; L_2(\Omega) \to L_2(\Omega). \tag{14}$$

Definition 3. Let $\sigma > 0$ be a fixed parameter. A nonzero function $v \in L_2(\Omega)$ is called an eigenfunction of the operator T corresponding to a characteristic value λ if the relation (14) is valid. The set of all characteristic values of the operator T is called the spectrum and is denoted by $\mathrm{sp}(T)$.

Theorem 2. *For all positive σ the following statements are valid:*

1. *There exist the denumerable set of positive characteristic values λ_l, $l = 1, 2, \ldots$, with only cumulative point at infinity.*
2. *The set of all eigenfunctions v_l, $l = 1, 2, \ldots$, can be chosen as the orthonormal set.*
3. *The smallest characteristic value λ_1 is positive and simple, corresponding eigenfunction v_1 is positive.*
4. *Each eigenvalue λ_l, $l = 1, 2, \ldots$, depends continuously on $\sigma > 0$,*
5. $\lambda_1 \to 0$ *as* $\sigma \to 0$.

This theorem was proved in [3]. The proving of this theorem is based on the combination of three equivalent statements: original statement, statement in form of spectral problem with integral operator with symmetric weakly polar kernel and on the special variational formulation on the plane and on the half-plane. The corresponding integral operators are self-adjoint and compact; therefore (see, e.g., [5]) there exists a denumerable set of λ_l, $l = 1, 2, \ldots$, with only cumulative point at infinity. We use special variational statement and equivalence of variational and original statements for proving positiveness of these integral operators. Then we obtain that all characteristic values are positive. Moreover, the minimal value λ_1 is simple (it means that multiplicity of λ_1 is equal to one) and $\lambda_1 \to 0$ as $\sigma \to 0$.

However, the last assertion for the problem (9)–(11) has the other form. In particular, $\lambda_1 \to const > 0$ as $\sigma \to 0$. The well-known fundamental mode satisfies to the smallest characteristic value λ_1. We can conclude that the fundamental mode exists for all $\omega > 0$ in case of a waveguide in free space. The fundamental mode will appear from the certain value of ω for a waveguide in the half-space. If some values of the parameters λ and σ are known, then β and ω can be calculated by evidence formulas.

4 Collocation Method

Let us consider the collocation method [11] for numerical approximation of the integral equation (12). We suppose that $\sigma > 0$ is a fixed parameter. We cover Ω with small triangles $\Omega_{j,h}$ such that

$$\max_{1\leq \mathrm{j} \leq \mathrm{N_h}} \mathrm{diam}(\Omega_{\mathrm{j,h}}) \leq h$$

and $\Omega_{i,h} \bigcap \Omega_{j,h} = \emptyset$, if $i \neq j$. Denote by Ω_h the sub-domain $\Omega_h = \bigcup_{j=1}^{N_h} \Omega_{j,h} \subseteq \Omega$. Let $\Xi_h = \{\xi_{j,h}\}_{j=1}^{N_h}$ be a grid for Ω (a finite number of points of Ω) such that $\xi_{j,h}$ is a centroid of $\Omega_{j,h}$, $j = 1,\ldots,N_h$ and

$$\mathrm{dist}(x, \Xi_h) \to 0, \quad h \to 0, \quad \forall x \in \Omega.$$

It is well known that each solution of the equation (12) belongs to a space $E = C(\overline{\Omega})$ [11] with norm

$$||v||_E = \sup_{x\in\Omega} |v(x)|.$$

Introduce the space $E_h = C(\Xi_h)$ of functions on the grid Ξ_h with the norm

$$||v_h||_{E_h} = \max_{1\leq j\leq N_h} |v_h(\xi_{j,h})|, \quad v_h \in E_h.$$

Define $p_h \in L(E, E_h)$ as the operator restricting functions $v \in E$ to the grid Ξ_h: $p_h v \in E_h$ is a grid function with the values

$$(p_h v)(\xi_{j,h}) = v(\xi_{j,h}), \quad j = 1,\ldots,N_h.$$

Then the discrete convergence v_h to v means that

$$\max_{1\leq j\leq N_h} |v_h(\xi_{j,h}) - v(\xi_{j,h})| \to 0, \quad h \to 0.$$

We represent an approximate solution of the integral equation (12) as a piecewise constant function $\tilde{v}_h(x) = \sum_{j=1}^{N_h} v_{j,h} f_{j,h}(x)$, $x \in \Omega_h$, where $f_{j,h}$ are basis functions, $f_{j,h}(x) = 1$, if $x \in \Omega_{j,h}$, $f_{j,h}(x) = 0$, if $x \notin \Omega_{j,h}$. In the integral equation (12) we approximate the domain of integration Ω by Ω_h:

$$v(x) = \lambda \int_{\Omega_h} K(x,y) v(y) dy. \tag{15}$$

Replacing v by $\tilde{v}_h$ and collocating at points $\xi_{i,h}$, we obtain a system of linear algebraic equations to find the values $v_{j,h}$, namely,

$$v_{i,h} = \lambda \sum_{j=1}^{N_h} \int_{\Omega_{j,h}} K(\xi_{i,h}, y) v_{j,h} dy, \quad i = 1, \dots N_h. \tag{16}$$

Let us introduce a discrete analogue of operator T:

$$(T_h \tilde{v}_h)(\xi_{i,h}) = \sum_{j=1}^{N_h} \int_{\Omega_{j,h}} K(\xi_{i,h}, y) \tilde{v}_h(\xi_{i,h}) dy.$$

Therefore, using collocation method for solving linear spectral problem for the integral equation (12), we obtain finite-dimensional linear spectral problem.

Let us formulate the convergence theorem for the linear case.

Theorem 3. *The following assertions hold:*

1. *If* $0 \neq \lambda_0 \in sp(T)$ *then there exists* $\lambda_h \in sp(T_h)$ *such that* $\lambda_h \to \lambda_0$ *as* $h \to 0$.
2. *Conversely, if* $sp(T_h) \ni \lambda_h \to \lambda_0$ *as* $h \to 0$ *then* $\lambda_0 \in sp(T)$.
3. *The convergence rate for a simple characteristic value is estimated as follows: for* $sp(T_h) \ni \lambda_h \to \lambda_0 \in sp(T)$, $\lambda_0 \neq 0$

$$|\lambda_h - \lambda_0| \leq ch^2.$$

The proof of this theorem is based on the discrete convergence theory [11].

Let us describe calculation of integrals in (16). Taking into account that the diagonal elements have singularities, we obtain these formulas:

$$a_{ii} = \frac{p^2(\xi_i)}{2\pi} \left(\frac{\pi R_i^2}{2} - \ln R_i |\Omega_{i,h}| - \ln \frac{\sigma \gamma}{2} |\Omega_{i,h}| \right),$$

where R_i is the minimal distance from the centroid of triangle to triangle's sides. Nondiagonal elements were calculated by following formulas

$$a_{ij} = \frac{|\Omega_{j,h}|}{2\pi} K_0(\sigma |\xi_i - \xi_j|) p(\xi_i) p(\xi_j),$$

where K_0 is McDonald's function.

The latter formulas for a waveguide in the half-space take the forms

$$a_{ii} = \frac{p^2(\xi_i)}{2\pi} \left(\frac{\pi {R_i}^2}{2} - \ln R_i |\Omega_{i,h}| - \ln \frac{\sigma \gamma}{2} |\Omega_{i,h}| - K_0(2\sigma |\xi_2^i|) |\Omega_{i,h}| \right),$$

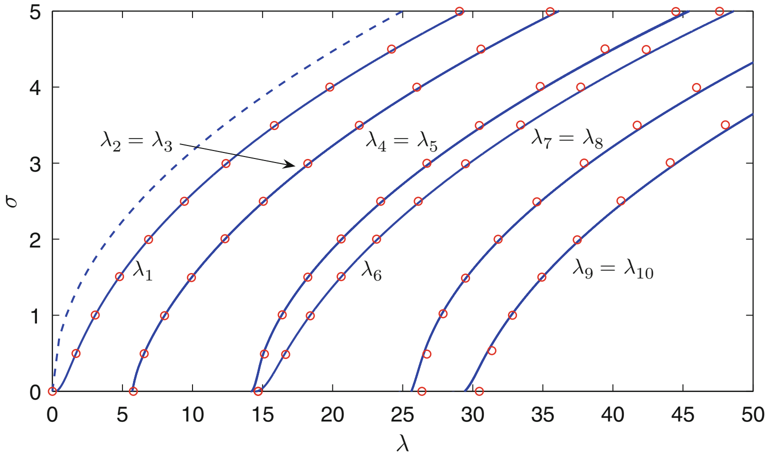

Fig. 3 The first ten dispersion curves for surface modes of circular step-index fiber calculated by the collocation method (*plotted by solid lines*) with comparison to exact solutions (*marked by circles*); $n_+ = \sqrt{2}$, $n_\infty = 1$

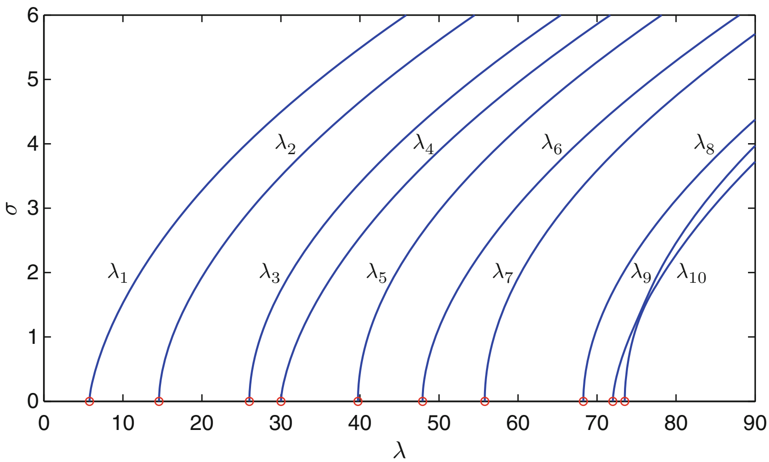

Fig. 4 The first ten dispersion curves for surface modes of semicircle step-index fiber in the half-space calculated by the collocation method; $n_+ = \sqrt{2}$, $n_\infty = 1$

$$a_{ij} = \frac{p(\xi_i)p(\xi_j)|\Omega_j|}{2\pi}\left(K_0(\sigma|\xi_i - \xi_j|) - K_0(\sigma|\xi_i - \xi_j^*|)\right).$$

Now we describe numerical results based on the collocation method. Dispersion curves show dependence for σ of λ. They are presented on the Fig. 3 for the circular waveguide in free space and on the Fig. 4 for the semicircle waveguide in the half-space, respectively. Figures 5 and 6 show the eigenfunction isolines

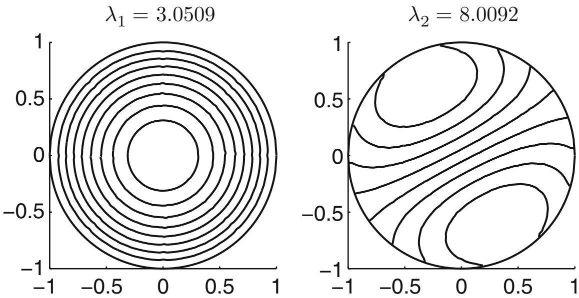

Fig. 5 Eigenfunction isolines for surface waves of circular waveguide in free space; $n_+ = \sqrt{2}$, $n_\infty = 1$

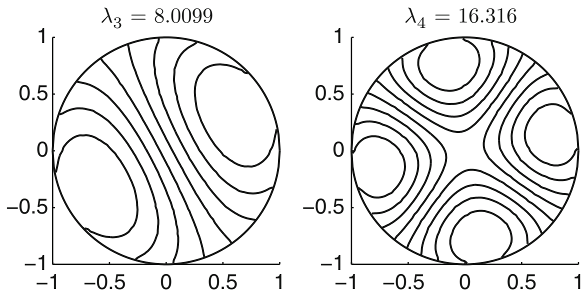

Fig. 6 Eigenfunction isolines for surface waves of circular waveguide in free space; $n_+ = \sqrt{2}$, $n_\infty = 1$

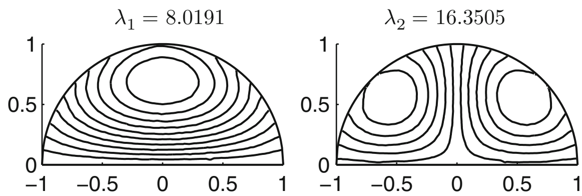

Fig. 7 Eigenfunction isolines for surface waves of semicircle waveguide in the half-space; $n_+ = \sqrt{2}$, $n_\infty = 1$

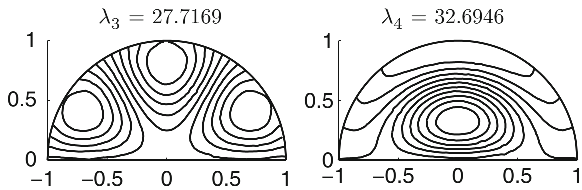

Fig. 8 Eigenfunction isolines for surface waves of semicircle waveguide in the half-space; $n_+ = \sqrt{2}$, $n_\infty = 1$

Table 1 Numerical results for eigenvalue λ_6 ($\sigma = 1$) of circular waveguide in free space; $n_+ = \sqrt{2}$, $n_\infty = 1$

N	64	256	512	1032	2304	4128	6528
h	0.4856	0.2573	0.1551	0.1217	0.0800	0.0618	0.0491
$\tilde{\lambda}_6$	16.0095	17.7529	18.1083	18.2402	18.3337	18.3842	18.4020
e	0.5576	0.5572	0.7315	0.7041	0.8384	0.6868	0.6881
ε	0.1315	0.0369	0.0176	0.0104	0.0054	0.0026	0.0017

Table 2 Numerical results for eigenvalue λ_6 ($\sigma = 1$) of semicircle waveguide in the half-space; $n_+ = \sqrt{2}$, $n_\infty = 1$

N	61	240	506	1059	2024	4236
h	0.3531	0.1693	0.1210	0.0863	0.0605	0.0432
$\tilde{\lambda}_6$	39.3336	48.0972	49.5528	50.2392	50.5952	50.7702
e	1.8172	1.8956	1.7561	1.6377	1.4209	0.9432
ε	0.2266	0.0543	0.0257	0.0122	0.0052	0.0018

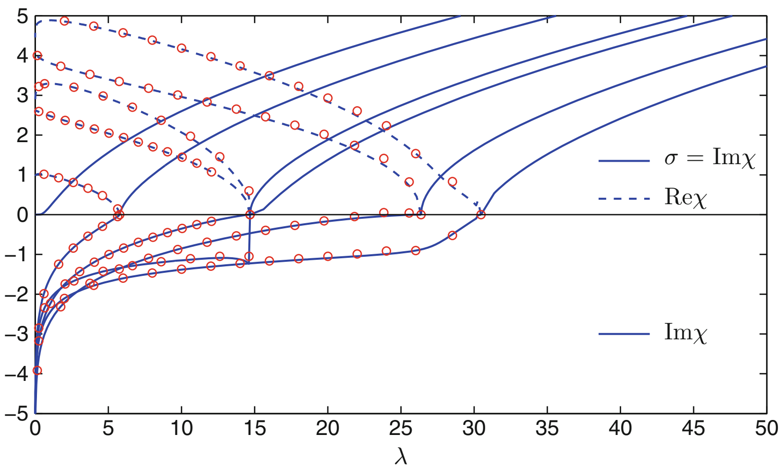

Fig. 9 Dispersion curves for surface and leaky modes of the circular step-index fiber calculated by collocation method (*marked by circles*) with comparison to exact solutions (*plotted by sold lines*); $n_+ = \sqrt{2}$, $n_\infty = 1$

for surface waves of circular waveguide in free space. Figures 7 and 8 show the eigenfunction isolines for surface waves of semicircle waveguide in the half-space. We present a Table 1 for circular waveguide that evaluates dependence for relative error $\varepsilon = |\lambda_6 - \tilde{\lambda}_6|/\lambda_6$ and $e = \varepsilon/(h/R)^2$ of N_h with $\sigma = 1$. Here $\lambda_6 = 18.4324$ is the exact value, $\tilde{\lambda}_6$ is the approximate value, R is the radius of the circular fiber.

The Table 2 describes the behaviour of inner convergence for semicircle waveguide in the half-space. We compare $\tilde{\lambda}_6$ with $\lambda_6 = 50.8596$ which is calculated for $N_h = 8096$. We also applied this method for solving nonlinear problem (13). In this

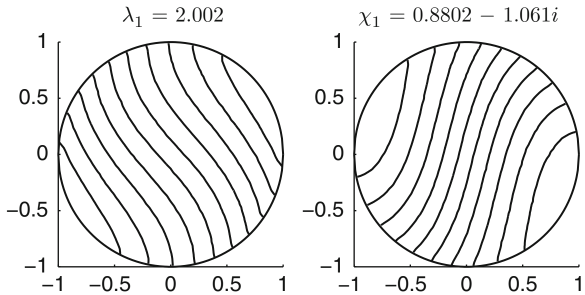

Fig. 10 Isolines for real and imaginary part of the first eigenfunction of circular waveguide; $n_+ = \sqrt{2}$, $n_\infty = 1$

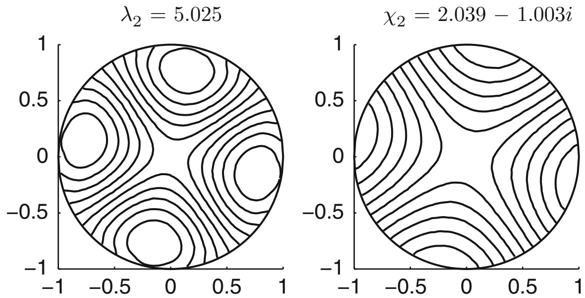

Fig. 11 Isolines for real and imaginary part of the fourth eigenfunction of circular waveguide; $n_+ = \sqrt{2}$, $n_\infty = 1$

Table 3 Numerical results for eigenvalue χ_3 for $\lambda_3 = 10.02$ of circular waveguide; $n_+ = \sqrt{2}$, $n_\infty = 1$

N	512	1032	2304	4128
h	0.1551	0.1217	0.0800	0.0618
$\tilde{\chi}_3$	5.8018-1.0489i	5.8030-1.0655i	5.8047-1.0702i	5.8050-1.0726i
e	0.1859	0.1124	0.1300	0.1099
ε	0.0045	0.0017	0.0008	0.0004

case we fixed value for parameter ω and therefore for λ and find values of β. Let us formulate the convergence theorem for the nonlinear case.

Theorem 4. *Let $A_h(\beta) = I - \lambda T_h(\beta)$. The following assertions hold:*

1. *If $\beta_0 \in sp(A)$ then there exists $\beta_h \in sp(A_h)$ such that $\beta_h \to \beta_0$ as $h \to 0$.*
2. *If $\beta_h \in sp(A_h)$ and $\beta_h \to \beta_0 \in \Lambda$ as $h \to 0$ then $\beta_0 \in sp(A)$.*

The proof of this theorem is based on the discrete convergence theory [11].

The dispersion curves for surface and leaky modes of the circular step-index fiber calculated by collocation method in comparation with exact solutions are presented at Fig. 9. Figures 10 and 11 show isolines of the first and second eigenfunctions for

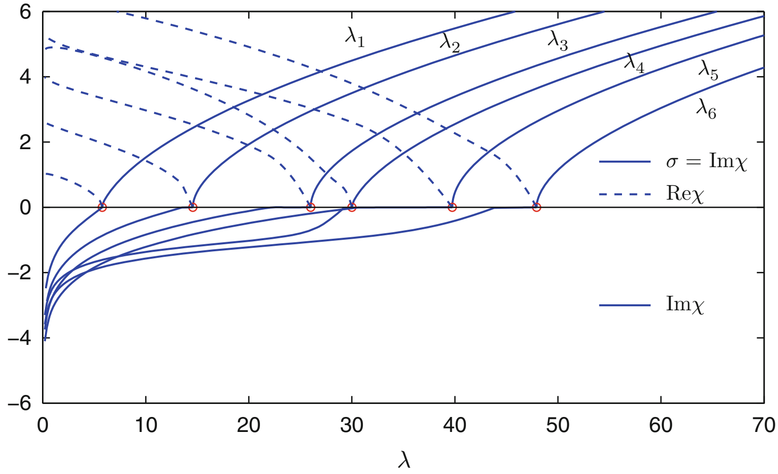

Fig. 12 Dispersion curves for surface and leaky modes of the semicircle step-index waveguide in the half-space; $n_+ = \sqrt{2}$, $n_\infty = 1$

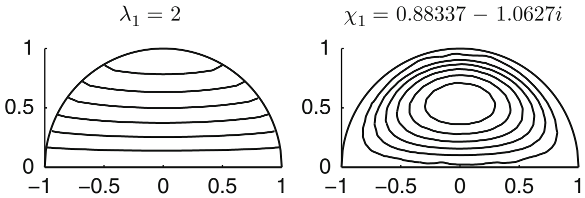

Fig. 13 Isolines for real and imaginary part of the first eigenfunction of semicircle waveguide in the half-space; $n_+ = \sqrt{2}$, $n_\infty = 1$

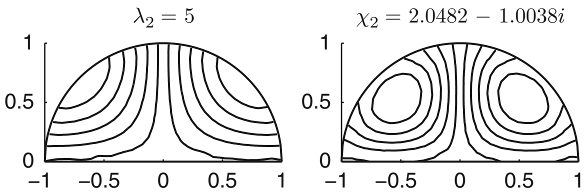

Fig. 14 Isolines for real and imaginary part of the fourth eigenfunction of semicircle waveguide in the half-space; $n_+ = \sqrt{2}$, $n_\infty = 1$

Table 4 Numerical results for eigenvalue χ_4 for $\lambda_4 = 20.02$ of semicircle waveguide; $n_+ = \sqrt{2}$, $n_\infty = 1$

N	240	506	1059	2024
h	0.1693	0.1210	0.0863	0.0605
$\tilde{\chi}_4$	2.7616−0.9311i	2.7897− 1.0195i	2.7978−1.0556i	2.8020−1.0715i
e	1.8019	1.4209	1.1408	0.8241
ε	0.0516	0.0208	0.0085	0.0030

leaky waves for circular waveguide, respectively. The Table 3 shows dependence for relative error $\varepsilon = |\chi_3 - \tilde{\chi}_3|/|\chi_3|$ and $e = \varepsilon/(h/R)^2$ of N_h with $\lambda_3 = 10.02$. Here $\chi_3 = 2.96\text{-}0.8469i$ is the exact value, $\tilde{\chi}_3$ is the approximate value.

We provided the same calculations for the semicircle waveguide in the half-space. The dispersion curves for surface and leaky modes of the semicircle step-index waveguide are presented at Fig. 12. Figures 13 and 14 show isolines of the first and second eigenfunctions for leaky waves for circular waveguide, respectively. The Table 4 shows dependence for relative error ε, value e of N_h. Here $\chi_4 = 2.8042 - 1.0803i$ is the value which is calculated for $N = 4236$ and $\lambda = 20.2$. Our numerical calculations show that the collocation method has the second rate of convergence. This is consistent with the theoretical estimates.

References

1. Abramovitz, M, Stegun, I.:Handbook of Mathematical Functions. Dover, New York (1965)
2. Karchevskii, E.M., Solovi‘ev, S.I.: Investigation of a spectral problem for the Helmholtz operator on the plane. Differ equat. **36**, 631–634 (2000)
3. Kartchevski, E.M., Nosich, A.I., Hanson, G.W.: Mathematical analysis of the generalized natural modes of an inhomogeneous optical fiber. SIAM J. Appl. Math. **65**(6), 2003–2048 (2005)
4. Koshparenok, V.N., Melezhik, P.N., Poedinchuk, A.E., Shestopalov, V.P.: Spectral theory of two-dimensional open resonators with dielectric inserts. USSR Comput. Math. Math. Phys. **25**(2), 151–161 (1985)
5. Kress, R.: Linear Integral Equations. Springer, New York (1999)
6. Marcuse, D.: Theory of Dielectric Optical Waveguides. Academic Press, New York (1974)
7. Shestopalov, Yu.V., Kotik, N.Z.: Interaction and propagation of waves in slotted waveguides. New J. Phys. **4**, 40.1–40.16 (2002)
8. Shestopalov, Yu.V., Okuno, Y., Kotik, N.Z.: Oscillations in Slotted Resonators with Several Slots: Application of Approximate Semi-Inversion, vol. 39, pp. 193–247, Progress In Electromagnetics Research (PIER), Moscow (2003)
9. Shestopalov, Yu.V., Smirnov, Yu.G., and Chernokozhin, E.V.: Logarithmic Integral Equations in Electromagnetics. VSP, Leiden, The Netherlands (2000)
10. Snyder, A.W., Love, J.D.: Optical Waveguide Theory. Chapman and Hall, London (1983)
11. Vainikko, G.: Multidimensional Weakly Singular Integral Equations. Springer, Berlin (1993)