Technical Note

Size and Shape of Steady Seawater Intrusion and Sharp-Interface Wedge: The Polubarinova-Kochina Analytical Solution to the Dam Problem Revisited

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Abstract: Rescaling of the geometrical sizes and the value of hydraulic conductivity in the classical problem of steady two-dimensional (2D) potential seepage through a rectangular earth dam with an empty tailwater is shown to result in a mathematically equivalent problem of seawater intrusion with a sharp interface into a confined horizontal aquifer, which discharges fresh groundwater to the sea through a vertical segment of the beach. The shape of the interface, the vertical and horizontal sizes of the static intrusion wedge, and its cross-sectional area are written in an explicit form, using the Polubarinova-Kochina formulas, rectified. The densities of the two liquids and the aquifers' hydraulic conductivity and thickness, as well as the incident hydraulic gradient serve as input parameters. With reduction of the incident groundwater gradient far upstream from the intrusion zone (due to, e.g., freshwater abstraction by wells), the sizes of the wedge rapidly increase. The analytical solution has been validated with recent sand tank experiments. **DOI: 10.1061/(ASCE)HE.1943-5584.0001385.** *© 2016 American Society of Civil Engineers.*

Author keywords: Seawater intrusion; Steady potential phreatic flow; Sharp-interface model; Earth dam problem; Exact solution.

Introduction

Fresh groundwater discharge as submarine springs or outseeps is important for the global hydrological balance and catchment-scale assessments of travel times of groundwater particles, hydrogeochemistry of coastal sea water and discharging aquifers, ecology of coral reefs and fish in Oman affected by groundwater-imported nutrients, paleohydrogeology-anthropology, hydrology of global climate changes, and planning of wellfield operations in coastal zones, among others (e.g., Burnett et al. 2006; Faure et al. 2002; Ferguson and Gleeson 2012; Hoefel and Evans 2001; Sherif et al. 2014; Taniguchi et al. 2002; Uchiyama et al. 2000; Zektser and Loaiciga 1993). Seawater intrusion (SWI) in pristine (anthropogenically intact) aquifers is conceptualized as a wedge (tongue) of a relatively dense seawater encroaching along the aquifer bottom [e.g., Fig. 1 of Burnett et al. (2006), Fig. 1 of Strack and Ausk (2015), and Fig. 1(a)] against the direction of groundwater discharge. SWI, especially with upstream freshwater pumping (the wedge is then bumped in shape and blurred) has a detrimental effect on water supply from coastal aquifers in Oman and other Gulf countries, especially on agricultural and municipal wells. Modeling of SWI is carried out by sharp interface and variable density codes, both analytically and numerically (e.g., Al-Bitar and Ababou 2005; Bakker 2014; Bear and Dagan 1964; Bereslavski 2007; Detournay and Strack 1988; Cheng and Ouazar 1999; De Josselin De Jong and Van Duijn 1984; Glover 1959; Hocking and Forbes 2004; Kacimov and Sherif 2006; Kacimov et al. 2009; Kashef 1983; Kourakos and Mantoglou 2015; Lu et al. 2015; Llopis-Albert and Pulido-Velazquez 2015; Mazi et al. 2014; Paster and Dagan 2008; Sherif et al. 2012; Strack 1989; Strack and Ausk 2015; Werner et al. 2012). Field studies of SWI are based on surface and downhole geophysics. Laboratory experiments aimed at measuring the sizes of the wedge [$|B_iD_{1i}|$ and $|A_iD_{1i}|$ in Fig. 1(a)] were carried out in sand-filled boxes (e.g., Bertorette 2014; Chang and Clement 2012; Goswami and Clement 2007).

The main question in mathematical models of SWI is what size of wedge is a quasi-triangle $A_iB_iD_{1i}A_i$ (the subscript i indicates intrusion) in a vertical cross section of Fig. 1(a), i.e., what are the length and height of the wedge? This paper answers this question using an exact analytical solution for a sharp-interface steady-state, Darcian fresh groundwater discharge over a static saline wedge.

Two-Element Freshwater Discharge into Coastal Aquifer and Seepage through a Rectangular Dam

Similar to Kashef (1983), a confined, isotropic, homogeneous aquifer [Fig. 1(a)] of thickness H_{1i}, with an impermeable caprock and bedrock, $E_{1i}D_i$ and $E_{2i}D_{1i}$ (two horizontal rays) as their boundaries is considered. Hydraulic conductivity of the aquifer is k_i. Fresh groundwater of density r_f moves from the left (Jabal Al-Akdar mountains in Oman) to the right (shore of the Gulf) and discharges into the sea as a submarine outlet through an outflow face D_iA_i of the beach. The density of seawater is r_s, $r_s > r_f = 1{,}000\ \mathrm{kg/m^3}$. Unlike Kashef (1983) hydrostratigraphy, the aquifer in Fig. 1(a) is not hydraulically commingled with the superjacent or subjacent aquifers. A static SWI wedge is bounded from the right by a vertical segment A_iD_{1i}, and from above by a sharp interface B_iA_i.

The origin of a Cartesian (x_iy_i) coordinate system is selected as point D_i. The tip B_i of the wedge is at $x_i = -l_i$ ($l_i > 0$ is unknown). The hydraulic head $h_i(x, y)$ (a harmonic function within the flow domain) of the moving freshwater is counted from point D_i, i.e., $h_i(0,0) = 0$. The tip A_i is at the depth $y_i = -H_{0i}$ ($H_i > 0$ is

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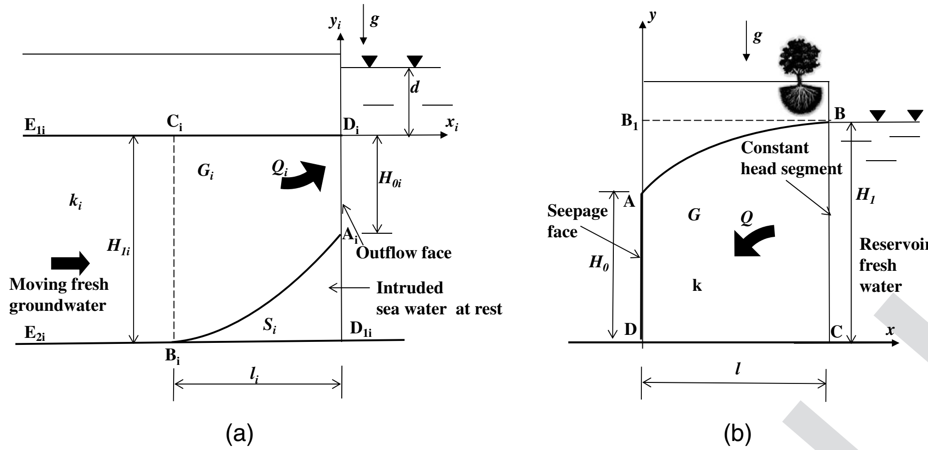

Fig. 1. (a) Vertical cross section of seawater intrusion zone; (b) vertical cross section of seepage in an earth dam

also unknown). The discharge rate of two-dimensional (2D) flow in the plane $x_i D_i y_i$ (per unit length in the direction perpendicular to this plane), $Q_i = \text{const} > 0$ (m^2/s), is given.

On a catchment scale the value of Q_i is often known from the hydrological balance or piezometric data upstream of the SWI zone. In the laboratory experiments of Goswami and Clement (2007) (GC), this value was directly measured.

The goal of this paper is to analyze how the shape of $A_i B_i$ in Fig. 1(a) depends on Q_i, in particular, how l_i, H_{0i}, and the cross-sectional area, S_i, of the wedge in Fig. 1(a) (area between $A_i B_i$ and $B_i D_{1i}$) vary with Q_i.

Kacimov and Obnosov (2001) (KO) gave a full analytical solution to the 2D flow problem in Fig. 1(a), even for a tilted beach face $D_i D_{1i}$. This solution has not, however, won the hearts and minds of groundwater engineers because of its apparent mathematical complexity. A simplified version of this solution is presented here, which combines the rigor of the full 2D potential model with the classical analytical solution of Polubarinova-Kochina (1962, 1977) (PK) to the so-called earth-dam problem shown in Fig. 1(b). KO mentioned the commonality of the problems in Figs. 1(a and b) but—to the best of the authors' knowledge—nobody exploited this analogy in practical groundwater hydrology. Thus, for the sake of methodological lucidity, this paper repeats here what is well-known to geotechnical engineers and applied mathematicians (Crank 1984; Craster 1994; Hornung and Krueger 1985) as the PK dam problem and its solution. This paper illustrates how this solution can be rescaled to the problem of SWI in Fig. 1(a).

A rectangular dam, whose vertical cross section is shown in Fig. 1(b), has the width l and is made of a homogenous isotropic soil of hydraulic conductivity k. The dam stands on an impermeable horizontal foundation CD. The upper pool is filled with a fresh water up to a level H_1, which is counted from the Dx-axis of a Cartesian coordinate system xy. The vertical face BC is a constant-head boundary. The tailwater is empty and seepage is from the right to the left. The phreatic surface BC is a sharp interface that separates a fully saturated flow domain beneath from an absolutely dry soil above BC. The dry soil triangle BAB$_1$ in Fig. 1(b) is an analogue of the wedge in Fig. 1(a). Similar to ignoring dispersion and diffusion in the problem of Fig. 1(a), the PK model of seepage in Fig. 1(b) ignores capillarity of the soil. The outlet vertical segment AD in Fig. 1(b) is a seepage face (i.e., an isobar of atmospheric pressure). The shape of AB in Fig. 1(b), in particular the locus of point A (H_0), is a part of the mathematical solution, as well as the flow rate Q. The hydraulic head, $h(x, y)$, in Fig. 1(b) is counted from point D.

Analytical Solution

As in Kashef (1983), it was assumed that the dashed vertical line $B_i C_i$ in Fig. 1(a) is a line of constant head h_i. This is, of course, only an approximation in terms of the full solution of KO. With this assumption, flow in Fig. 1(a) decouples into two analytic elements: a purely confined trivial one-dimensional (1D) flow in the half-strip $E_{1i} C_i B_i E_{2i}$ and a free-boundary 2D flow on the right of $C_i B_i$. The Dupuit-Forchheimer (DF) approximation (e.g., Bakker 2014; Koussis et al. 2015) adopted by Kashef (1983) was not assumed, and the segment $D_i A_i$ in Fig. 1(a) was not assumed to be a constant hydraulic head boundary. This segment is an outflow boundary as in Strack and Ausk (2015). Unlike Kashef (1983), for the right fragment $B_i C_i D_i A_i B_i$ (a quasi-trapezium G_i) in Fig. 1(a), a potential sharp-interface model of PK was utilized. As usual, a complex potential $w_i = \phi_i + i\ \psi_i$ is introduced, where i is an imaginary unit and $\phi_i = -k_i\ h_i$, which is the velocity potential according to the Darcy law $\mathbf{V}_i = -k_i \nabla h_i$, where $\mathbf{V}_i$ is the Darcian velocity vector and ψ_i is a stream function, complexly conjugated with ϕ_i. Parameter ψ_i will be counted from the streamline $C_i D_i$.

Then the boundary value problem (BVP) for flow in Fig. 1(a) is

$$
\begin{aligned}
&B_i C_i{:}\varphi_i = -k_i \delta H_{1i}, \qquad \text{at}\, x_i = -l_i, \qquad -H_{1i} < y_i < 0\\
&C_i D_i{:}\psi_i = 0, \qquad \text{at}\, y_i = 0, \qquad -l_i < x_i < 0\\
&D_i A_i{:}\ \varphi i - k_i \delta y_i = 0, \qquad \text{at}\, x_i = 0, \qquad -H_{0i} < y_i < 0\\
&A_i B_i{:}\psi_i = -Q_i, \qquad \varphi i - k_i \delta y_i = 0 \qquad \text{at} -l_i < x_i < 0,\\
&\quad -H_{0i} > y_i > -H_{1i} \qquad (1)
\end{aligned}
$$

where $\delta = (r_s - r_f)/r_f$. The freshwater head at point B_i [the first line in Eq. (1)] follows from the Pascal law (see KO) and definition of pressure $p_i = r_f g(-\phi_i/k_i - y_i + r_s/r_f d)$, where d is the depth of seawater above point D_i in Fig. 1(a). The boundary condition along $D_i A_i$ is neither constant head nor constant flux, although in regional-scale numerical models like MODFLOW and SEAWAT these simplified conditions are also used (e.g., Motz and Sedighi 2009).

A standard trick (e.g., Kashef 1983) is to rescale the geometric and flow variables of Fig. 1(a) as $(x, y, \phi, \psi, k) = (-\delta x_i, -\delta y_i,\ \delta\phi_i, \delta\psi_i, \delta k_i)$, where $w = \varphi + i\psi$ is the complex potential of flow in Fig. 1(b). In other words, G_i in Fig. 1(a) is mapped symmetrically with respect to point D, and stretched to get G. A porous medium of conductivity k_i is also replaced by a medium of conductivity k.

Then BVP in Eq. (1) is reduced to

$$\begin{aligned} &\mathrm{BC}: \varphi = -kH_1, \quad x = l, \quad H_1 > y > 0 \\ &\mathrm{CD}: \psi = 0, \quad y = 0, \quad l > x > 0 \\ &\mathrm{DA}: \varphi + ky = 0, \quad x = 0, \quad H_1 > y > 0 \\ &\mathrm{AB}: \psi = -Q, \quad \varphi + ky = 0 \end{aligned} \tag{2}$$

The rescaled BVP in Eq. (2) exactly corresponds to the dam flow problem in Fig. 1(b). A full solution to the BVP in Eq. (2) is given in PK and Crank (1984). Therefore, the back-scaling immediately solves the BVP in Eq. (1). Rephrasing, if G is flipped in Fig. 1(b) (including the phreatic surface) over the center of symmetry D and stretched 33 times (the density of seawater in the Gulf corresponds to $\delta \approx 0.03$), the result is G_i in Fig. 1(a) (including the sharp interface).

The flow rates Q and Q_i are calculated by the Charny formula (PK)

$$Q = k\frac{H_1^2}{2l}, \qquad Q_i = \frac{Q}{\delta} = k_i \delta \frac{H_{1i}^2}{2l_i} \tag{3}$$

which has been recently extended to layered aquifers by Strack and Ausk (2015).

Eq. (3) for the discharges in both SWI [Fig. 1(a)] and dam [Fig. 1(b)] problems are exact. Q and Q_i given by Eq. (3) coincide with those derived from the DF model (see PK for details) but the sharp interface and phreatic surface clearly do not coincide in the exact 2D and approximate 1D (DF) models.

Eq. (3) is used to calculate the areas of G_i in Fig. 1(a) and G in Fig. 1(b). Then for the wedge area

$$S_i = H_{1i}l_i + \int_{-l_i}^{0} y_{i\mathrm{BA}}(x_i)dx_i = \frac{H_1 l}{\delta^2} - \frac{1}{\delta^2}\int_0^l y_{\mathrm{AB}}(x)dx \tag{4}$$

The last integral in Eq. (4) [the saturated area of G in Fig. 1(b)] requires some effort to evaluate. Unfortunately, in both PK (1962, 1977) there are numerous typos and an ambiguous statement on the shape of AB in Fig. 1(b). Namely, after Eq. (10.41) in PK (1977, Chapter 7) [the same mistake is in PK (1962)], the authors incorrectly wrote that the parametric equation of the free surface involves an arbitrary constant. PK suggests equating this constant to 1. In reality, this constant is not arbitrary but has to be determined from PK [(1962, 1977, Eqs. (10.34) and (10.35)]. These two equations are rewritten in a dimensionless form as one equation

$$l^* = \frac{l_i}{H_{1i}} = \frac{l}{H_1} = \frac{\int_0^{\pi/2} \frac{K[\beta \sin^2\chi]}{\sqrt{1-\beta\sin^2\chi}} d\chi}{\int_0^{\pi/2} \frac{K[\beta+(1-\beta)\sin^2\chi]}{\sqrt{\beta+(1-\beta)\sin^2\chi}} d\chi} \tag{5}$$

where K = complete elliptic integral of the first kind; and $0 \le \beta \le 1$ = parameter [the affix of a conformal mapping whose preimage is point D in Fig. 1(b)], to be determined. The notations of PK (1977) are kept, although some of them, like for the aquifer thickness, H_{1i}, in Fig. 1(a), may look bizarre to groundwater hydrologists. Eqs. (10–34) and (10–35) in PK (1962, 1977) are written for a general case of a nonempty tailwater [Fig. 1(b)]. Correspondingly, they contain another parameter α, which is zero for this case and hence vanished in Eq. (5).

At the time of PK's work, determination of the two parameters (α, β) by solving a system of nonlinear equations with integrals whose integrands were special (elliptic) functions was prohibitively complicated. So, PK (1962) presented some asymptotic expansions of integrals and in PK (1977) even these expansions were dropped. Neither PK (1977, 1962) contain a systematic analysis of the shape of AB in Fig. 1(b). Hornung and Krueger (1985) extended the PK (1962, 1977) analysis and presented numerical results for several l/H_1 values in Fig. 1(b). Their motivation was "Though Polubarinova-Kochina published her formulas in 1962, her solution was seldomly used as a reference to test numerical methods. This may be due to the fact that the evaluation of these formulas is not straightforward." The solution of the dam problem was published in the 1930s; half a century later, geotechnical engineers did not use the PK solution and spurred the Hornung and Krueger (1985) analysis; 30 years after their paper the situation is the same: with all the juggernauts of FEFLOW, HUDRUS2D, and MODFLOW, the PK (1962, 1977) results are not in the arsenal of numerical modelers and practitioners.

Nowadays, solving Eq. (5) or a system of equations for (α, β), i.e., for the most general case of the dam problem with an arbitrary tailwater level in Fig. 1(b), is a routine of Wolfram's (1991) *Mathematica* (or other computer algebra packages like *MATLAB*). The FindRoot, EllipticK, and NIntegrate built-in functions of *Mathematica* were used and Eq. (5) was solved as $\beta = \beta(l/H_1)$. Then Eq. (10.37) was used for determining the size of the seepage face in Fig. 1(b)

$$H_0^* = \frac{H_{0i}}{H_{1i}} = \frac{H_0}{H_1} = \frac{\int_0^{\pi/2} \frac{K[\cos^2\chi]\sin\chi}{\sqrt{1-(1-\beta)\sin^2\chi}} d\chi}{\int_0^{\pi/2} \frac{K[\beta+(1-\beta)\sin^2\chi]}{\sqrt{\beta+(1-\beta)\sin^2\chi}} d\chi} \tag{6}$$

The corrected PK parametric equations of BA follow from Eq. (10.41):

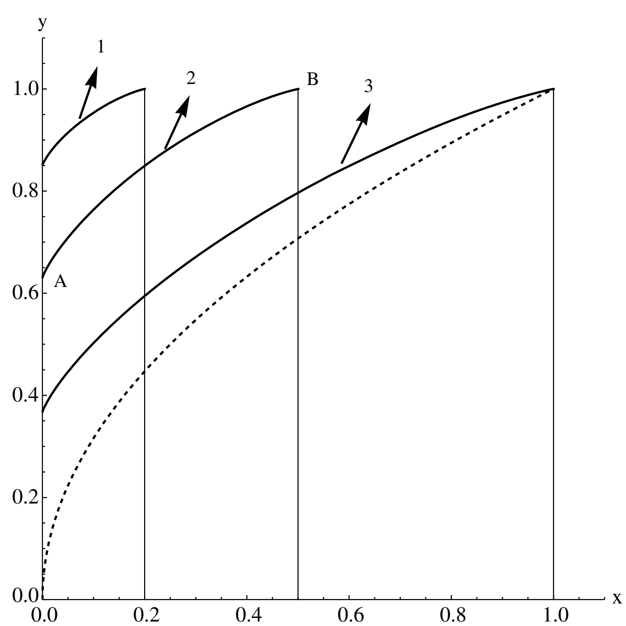

Fig. 2. Shapes of phreatic surface $y(x)$ in Fig. 1(b) for $l^* = 0.2$, 0.5, and 1.0 (Curves 1–3, respectively, solid lines) and the DF parabola (dashed line) for $l^* = 1.0$

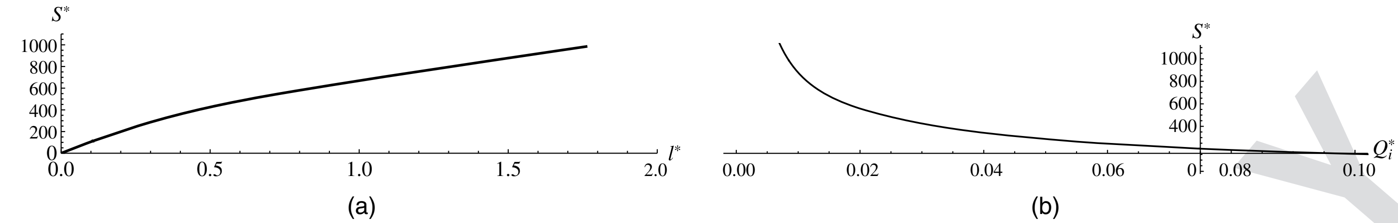

Fig. 3. (a) Dimensionless cross-sectional area S^* of the wedge as a function of the dimensionless wedge base l^* for $\delta = 0.03$; (b) wedge area as a function of the incident gradient

$$x^*(\theta) = -\frac{x_i}{H_{1i}} = \frac{x}{H_1} = \frac{l}{H_1} - \frac{\int_0^\theta \frac{K[\sin^2\chi]\sin\chi}{\sqrt{1-\beta\sin^2\chi}} d\chi}{\int_0^{\pi/2} \frac{K[\beta+(1-\beta)\sin^2\chi]}{\sqrt{\beta+(1-\beta)\sin^2\chi}} d\chi},$$

$$y^*(\theta) = -\frac{y_i}{H_{1i}} = \frac{y}{H_1} = \frac{H_0}{H_1} + \frac{\int_0^\theta \frac{K[\cos^2\chi]\sin\chi}{\sqrt{1-\beta\sin^2\chi}} d\chi}{\int_0^{\pi/2} \frac{K[\beta+(1-\beta)\sin^2\chi]}{\sqrt{\beta+(1-\beta)\sin^2\chi}} d\chi},$$

$$0 \le \theta \le \pi/2 \tag{7}$$

Superscripts in x and y are dropped for the sake of brevity. Then Eq. (4) is rewritten in a dimensionless form

$$S^* = \frac{S_i}{H_{1i}^2} = \frac{1}{\delta^2}\left[\frac{l}{H_1} - \int_0^{\pi/2} y(\chi)\frac{dx(\chi)}{d\chi} d\chi\right] \tag{8}$$

where $dx(\chi)/d\chi$ is evaluated from the first equation in Eq. (7).

Fig. 2 shows $y(x)$ for $l^* = l/H_1 = 0.2$, 0.5, and 1.0 (Curves 1–3, respectively), i.e., in a benign context of the dam problem of Fig. 1(b). Table 2 of Hornung and Krueger (1985) was also checked and *Mathematica* gave exactly the same H_0/H_1 values. For comparison, at $l^* = 1.0$ a DF parabolic phreatic surface $y = \sqrt{x}$ is also plotted in Fig. 2 as a dashed line. For the selected values of l^* in Fig. 2, the DF approximation is not appropriate.

Fig. 3(b) uses the same Eqs. (8) and (3) to depict the area of the SWI zone in the context of SWI management. Fig. 3(b) shows a graph of $S^*(Q_i^*)$, where $Q_i^* = Q_i/(k_i H_{1i})$ is the uniform hydraulic gradient upstream of the SWI zone [compare with a relevant Fig. 3 (a) of Ferguson and Gleeson (2012)]. At $Q_i^* \to 0$, the whole aquifer in Fig. 1(a) is occupied by seawater, i.e., the curve in Fig. 3(a) goes up to the left. The recent alarmism about the rise of the global seawater level (H_{1i}) pedals mostly the ensued damage to on-shore structures, although Fig. 3(b) illustrates the invisible tongue extension deep inland, with a potential deleterious impact on agricultural land that is irrigated from coastal aquifers.

Fig. 4 shows as = $(H_{1i} - H_{oi})/l_i$ as a function of Q_i^*. The ratio as [the vertical size of the SWI wedge in Fig. 1(a) to its horizontal size] quantifies the degree of the hydrodynamic push of the wedge by flowing groundwater. In the case of no SWI $Q_i^* \to \infty$, $\beta \to 0$, the area of the wedge and both its sizes approach zero but the aspect ratio $as \to 8\mathrm{Ca}/\pi^2 \approx 0.74$, where Ca is the Catalan constant (see the horizontal asymptote in Fig. 4), as it should be according to PK (1977) in the dam problem (see PK's Case 2).

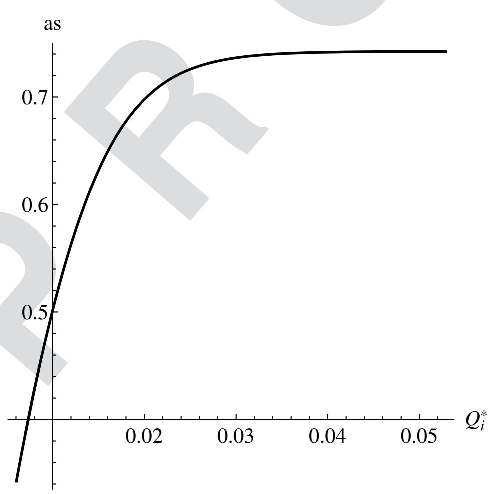

Fig. 4. Aspect ratio of the vertical to horizontal sizes of the SWI wedge in Fig. 1(a)

Comparison with Sand Tank Experiments

GC conducted experiments (see their Fig. 2) in a sand-filled tank with the following values: $k_i = 1{,}050$ m/day; $Q_i = 1.42/2.7$ cm^2/s; $H_{1i} = 26$ cm; $l_i = 15$ cm; $r_f = 1$ g/cm^3; and $r_s = 1.026$ g/cm^3. Although GC's experimental flow was unconfined as in Kashef (1983), i.e., instead of the caprock $E_{1i}C_iD_i$ in Fig. 1(a), GC had a phreatic surface, the slope of this surface was relatively small. In numerical modeling GC used a confined flow-transport model. The GC numerical and experimental results matched well. Therefore, the replacement of GC's free surface by a horizontal no-flow caprock as in Fig. 1(a) is reasonable for the selected experimental setup.

The theoretical value for GC's experiment, according to Eq. (3), is $Q_i = 6.15$ m^2/day. GC's measured discharge is $Q_i = 4.54$ m^2/day. GC's experimental value was also used for l_i in the left-hand side of Eq. (5) and the root of this equation was found to be $\beta = 0.58$. Then this β was put into the right-hand side of Eq. (6) and $H_{0i} = 10.93$ cm was found, while the GC size of the discharge window was $H_{0i} = 13$ cm.

Bertorelle (2014) conducted similar experiments and SUTRA-based numerical modeling for a sandbox with $k_i = 1.8 \times 10^{-3}$ m/s, $Q_i = 2.5/0.3 \times 10^{-3}$ m^2/h, $H_{1i} = 41$ cm, $r_f = 1$ g/cm^3, and $r_s = 1.025$ g/cm^3. Now her experimental data are converted into dimensionless format. The theoretical formula Eq. (3) gives $Q_i^* = 0.004$, while the Italians measured the discharge of $Q_i^* = 0.0027$. The FindRoot routine was again used to solve Eq. (5) and β was found to be 0.9999334. This was used in Eq. (6), which gave a theoretical $H_0^* = 0.17$, which agrees well with Fig. 9.28 of Bertorelle's experiment and numerical modeling. Therefore, both GC's and Bertorelle's (2014) results match well the theory presented in this paper.

Conclusion

Steady SWI with a sharp interface in a confined aquifer is mathematically equivalent to the classical PK problem of a phreatic surface seepage through an earth dam. This mathematical commonality is well known since the comparisons of the Glover (1959) SWI problem, having a parabolic sharp interface, with the Pavlovsky [26] problem (PK) of flow toward a Zhukovsky drain, which has a parabolic phreatic surface.

With modern computer algebra tools and rectification of PK's typos and errors, the analytical solution to the dam problem is metamorphosed into solution to a SWI problem, modulo stretching-rescaling. The sharp-interface model matches well the experiments of GC and Bertorelle (2014), as well as their numerical modelling by variable density codes SEAWAT and SUTRA.

Fig. 3(b) corroborates the results obtained in Kacimov et al. (2009) in terms of the DF model for an unconfined coastal aquifer, viz, SWI increases rapidly with the decrease of the incident gradient [uniform in the left element of Fig. 1(a)] when the gradient is relatively small. A similar conclusion was drawn by Ferguson and Gleeson [2012, Fig. 3(a)]. For example, from Fig. 3(b), with the decrease of the incident gradient Q_i^* from 0.033 to 0.0083 the dimensionless area of the nasty SWI wedge in Fig. 1(a) increases from 400 to 1,000. Unfortunately, in the Gulf countries a continuing overabstraction of fresh groundwater from deeper and deeper aquifers, which submarinely discharges into the sea, results in a drastic SWI. The wedge encroachment inland is pretty limited when the incident fresh groundwater gradient is above a certain threshold level; below it a catastrophic expansion of the SWI zone takes place.

While phreatic coastal aquifers can be replenished by relatively cheap managed aquifer recharge schemes, like infiltration basins, the fate of deep confined aquifers is bleak because these aquifers require more expensive well injection for recuperation of SWI.

The sad fact of a highly nonlinear nastiness of the wedge size, evidenced in the increase of the curve in Fig. 3(b) at small incident fresh groundwater gradients caused either by droughts overpumping (decrease of recharge Q_i) or increase of seawater level H_i, has to win the hearts and minds of water resource managers in the Gulf and other SWI-prone regions. Hopefully groundwater engineers will sympathize with the authors' predilection for analytical solutions, in particular, the old PK one for the dam problem, which was exploited in this paper.

Acknowledgments

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References

Al-Bitar, A., and Ababou, R. (2005). "Random field approach to seawater intrusion in heterogeneous coastal aquifers: Unconditional simulations and statistical analysis." *Geostatistics for environmental applications*, P. Renard, H. Demougeot-Renard, and R. Froidevaux, eds., Springer, Heidelberg, Germany, 233–248.

Bakker, M. (2014). "Exact versus Dupuit interface flow in anisotropic coastal aquifers." *Water Resour. Res.*, 50(10), 7973–7983.

Bear, J., and Dagan, G. (1964). "Some exact solutions of interface problems by means of the hodograph method." *J. Geophys. Res.*, 69(8), 1563–1572. [29]

Bereslavski, E. N. (2007). "Calculation of the salt-water intrusion in coastal regions of the sea bed." *Doklady Phys.*, 52(3), 146–150.

Bertorelle, E. (2014). "Laboratory experiments on the saltwater intrusion process." M.S. thesis, Univ. of Padua, Padua, Italy. [30]

Burnett, W. C., et al. (2006). "Quantifying submarine groundwater discharge in the coastal zone via multiple methods." *Sci. Total Environ.*, 367(2–3), 498–543.

Chang, S. W., and Clement, T. P. (2012). "Experimental and numerical investigation of saltwater intrusion dynamics in flux-controlled groundwater systems." *Water Resour. Res.*, 48, W09527. [31] [32]

Cheng, A. H.-D., and Ouazar, D. (1999). "Analytical solutions." *Seawater intrusion in coastal aquifers: Concepts, methods and practices*, J. Bear, A. H.-D. Cheng, S. Sorek, D. Ouazar, and I. Herrera, eds., Kluwer. [33]

Crank, J. (1984). *Free and moving boundary problems*, Clarendon Press, Oxford. [34]

Craster, R. V. (1994). "Two related free boundary problems." *IMA J. Appl. Math.*, 52(3), 253–270. [35]

De Josselin De Jong, G., and Van Duijn, C. J. (1984). "Transverse dispersion from an originally sharp fresh-salt interface caused by shear flow." *J. Hydrol.*, 84(1–2), 55–79. [36]

Detournay, C., and Strack, O. D. L. (1988). "A new approximate technique for the hodograph method in groundwater flow and its application to coastal aquifers." *Water Resour. Res.*, 24(9), 1971–1981. [37]

Faure, H., Walter, R. C., and Grant, D. R. (2002). "The coastal oasis: Ice age springs on emerged continental shelves." *Global Planet. Change*, 33(1–2), 47–56. [38]

Ferguson, G., and Gleeson, T. (2012). "Vulnerability of coastal aquifers to groundwater use and climate change." *Nat. Clim. Changes*, . [39]

Glover,, R. E. (1959). "The pattern of freshwater flow in a coastal aquifer." *J. Geophys. Res.*, 64(4), 457–459. [40]

Goswami, R. R., and Clement, T. P. (2007). "Laboratory-scale investigation of saltwater intrusion dynamics." *Water Resour. Res.*, 43, W04418. [41] [42]

Hocking, G. C., and Forbes, L. K. (2004). "The lens of freshwater in a tropical island—2d withdrawal." *Comput. Fluids*, 33(1), 19–30. [43]

Hoefel, F. G., and Evans, R. L. (2001). "Impact of low salinity porewater on seafloor electromagnetic data: A means of detecting submarine groundwater discharge?" *Estuarine Coastal Shelf Sci.*, 52(2), 179–189. [44]

Hornung, U., and Krueger, T. (1985). "Evaluation of the Polubarinova-Kochina formula for the dam problem." *Water Resour. Res.*, 21(3), 395–398. [45]

Kacimov, A. R., and Obnosov, Y. V. (2001). "Analytical solution for a sharp interface problem in sea water intrusion into a coastal aquifer." *Proc. R. Soc. London A*, 457(2016), 3023–3038.

Kacimov, A. R., and Sherif, M. M. (2006). "Sea water intrusion into a confined aquifer with controlled pumping: Analytical solution." *Water Resour. Res.*, 42(6), W06501. [46]

Kacimov, A. R., Sherif, M. M., Perret, J. S., and Al-Mushikhi, A. (2009). "Control of sea-water intrusion by salt-water pumping: Coast of Oman." *Hydrogeol. J.*, 17(3), 541–558. [47] [48]

Kashef, A. I. (1983). "Harmonizing Ghyben-Herzberg interface with rigorous solutions." *Ground Water*, 21(2), 153–159.

Kourakos, G., and Mantoglou, A. (2015). "An efficient simulation-optimization coupling for management of coastal aquifers." *Hydrogeol. J.*, 23(6), 1167–1179. [49]

Koussis, A. D., Mazi, K., Riou, F., and Destouni, G. (2015). "A correction for Dupuit-Forchheimer interface flow models of seawater intrusion in unconfined coastal aquifers." *J. Hydrol.*, 525, 277–285. [50]

Llopis-Albert, C., and Pulido-Velazquez, D. (2015). "Using MODFLOW code to approach transient hydraulic head with a sharp-interface solution." *Hydrol. Processes*, 29(8), 2052–2064. [51]

Lu, C., Werner, A. D., Simmons, C. T., and Luo, J. (2015). "A correction on coastal heads for groundwater flow models." *Groundwater*, 53(1), 164–170. [52]

Mazi, K., Koussis, A. D., and Destouni, G. (2014). "Intensively exploited Mediterranean aquifers: Resilience and proximity to critical points of seawater intrusion." *Hydrol. Earth Syst. Sci.*, 18(5), 1663–1677. [53]

Motz, L., and Sedighi, A. (2009). "Representing the coastal boundary condition in regional groundwater flow models." *J. Hydrol. Eng.*, 10.1061/(ASCE)HE.1943-5584.0000049, 821–831.

Paster, A., and Dagan, G. (2008). "Mixing at the interface between fresh and salt waters in 3D steady flow with application to a pumping well in a coastal aquifer." *Adv. Water Resour.*, 31(12), 1565–1577. [54]

Polubarinova-Kochina, P. Ya. (1962). *Theory of ground water movement*, 2nd Ed., Princeton University Press, NJ.

Sherif, M., Mohamed, M., Kacimov, A., and Shetty, A. (2012). "Modeling groundwater flow and seawater intrusion in the coastal aquifer of Wadi Ham, UAE." *Water Resour. Manage.*, 26(3), 751–774.

Sherif, M., Sefelnasr, A., Ebraheem, A., and Javadi, A. (2014). "Quantitative and qualitative assessment of seawater intrusion in Wadi Ham under different pumping scenarios." *J. Hydrol. Eng.*, 10.1061/(ASCE)HE.1943-5584.0000907, 855–866.

Strack, O. D. L. (1989). *Groundwater mechanics*, Prentice Hall, Englewood Cliffs, NJ.

Strack, O. D. L., and Ausk, B. K. (2015). "A formulation for vertically integrated groundwater flow in a stratified coastal aquifer." *Water Resour. Res.*, 51(8), 6756–6775.

Taniguchi, M., Burnett, W. C., Cable, J. E., and Turner, J. V. (2002). "Investigation of submarine groundwater discharge." *Hydrol. Processes*, 16(11), 2115–2129.

Uchiyama, Y., Nadaoka, K., Rolke, P., Adachi, K., and Yagi, H. (2000). "Submarine groundwater discharge into the sea and associated nutrient transport in a sandy beach." *Water Resour. Res.*, 36(6), 1467–1479.

Werner, A. D., et al. (2012). "Seawater intrusion processes, investigation and management: Recent advances and future challenges." *Adv. Water Resour.*, 51, 3–26.

Wolfram, S. (1991). *Mathematica: A system for doing mathematics by computer*, Addison-Wesley, Redwood City.

Zektser, I. S., and Loaiciga, H. A. (1993). "Groundwater fluxes in the global hydrologiccycle: Past, present and future." *J Hydrol.*, 144(1–4), 405–427.

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