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Saturated-unsaturated seepage from Kornev's subsurface element : comparison of analytic and numerical solutions

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Аннотация

Subsurface emitters (SEs) are modeled as line sources with descending Darcian seepage impeded by either a natural impervious horizont or by designed and constructed barrier, which makes a wedge beneath SE. An analytical model assumes a tension-saturated steady-state 2D flow (Laplace's governing PDE) near an emitter, with a capping phreatic line, along which the stream function linearly depends on the horizontal coordinate that allows to use the Polubarinova-Kochina technique, *videlicet* a conformal mapping of a circular trigon in the hodograph domain on a reference half-plane. In the finite element model (HYDRUD2D, the Richards-Richardson PDE), a transient initial value problem (giving an asymptotic steady-state limit is solved in a fixed domain (an isosceles curvilinear tetragon or trapeziumIsobars, isohumes, streamlines,isotachs and the Christiansen uniformity coefficient are computed.

Keywords: conformal mapping of circular polygons, Riemann-Hilbert's problem, HYDRUS2D modeling, seepage flow topology, field experiments.

For saturated flows (Strack, 1989), the Darcian velocity, $\overrightarrow{V}(x, y)$ obeys the relation $\overrightarrow{V}(x, y) = -k\nabla h$. The hydraulic (piezometric) head $h(x, y) = p(x, y) + y$ in homogeneous incompressible soils, k is a constant saturated hydraulic conductivity and involves the pressure head, $p(x, y)$, is positive everywhere. For incompressible pore water in Welsh's seepage domain, G_z (see Fig.1a), a free boundary, BMC, caps G_z. In the Vedernikov-Bouwer model for steady-state seepage $h(x, y)$ is a harmonic function. A complex potential $w = \varphi + i\psi$ is introduced, where $\varphi = -kh$ is the velocity potential, and ψ is a stream function.

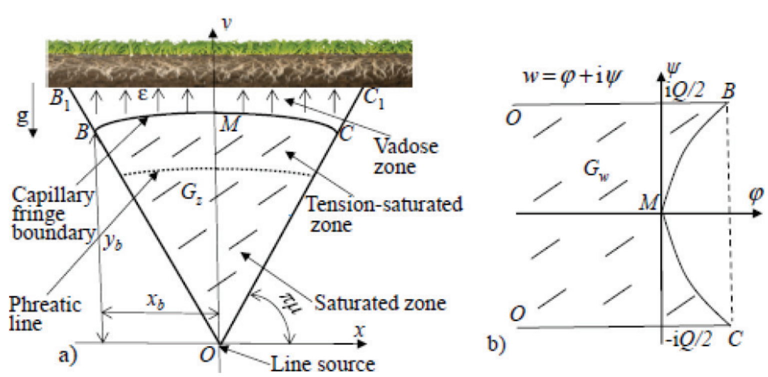

Fig 1: Vertical cross-section of the physical flow domain (a), complex potential domain (b).

A complex Darcian velocity (an antiholomorphic function) is $V = u + iv$, where $u(x, y)$ and $v(x, y)$ are the horizontal and vertical components of $\overrightarrow{V}(x, y, t)$. The complex potential domain G_w is shown in Fig.1b (point M is fiducial). The shape of BMC in G_w is not known. The hodograph domain, corresponding to G_z and G_w, is a circular triangle $G_V = \{V : \pi\mu <$

$\arg V <: \pi(1-\mu), |2V + \mathrm{i}(k-\varepsilon)| > (k+\varepsilon)\}$. The boundary-value problem (BVP) in G_z is formulated as:

$$OB\colon \psi = Q/2,\ y = \tan\pi\mu x; OC\colon \psi = -Q/2,\ y = -\tan\pi\mu x; BMC\colon \varphi + ky = -p_c, \psi = \varepsilon x. \quad (1)$$

where ε and k are constants such that $0 < \varepsilon < \infty$, $0 < k < \infty$, ε is the intensity of evapotranspiration from BMC, and p_c is a the height of capillary rise in a vertical soil column.

To solve BVP (1) the upper half of a reference (auxiliary) ζ-plane G_ζ is conformally mapped onto the circular triangle $G_\Omega = \{\Omega : \pi\mu < \arg\Omega < \pi(1-\mu),), |2k\varepsilon\Omega + \mathrm{i}(k-\varepsilon)| < (k+\varepsilon)\}$ in the plane $\Omega = \partial z/\partial w$. Here G_Ω is a circular triangle onto which the function $1/V$ maps a triangle symmetrical with G_V relative to the real axis. An analytical function mapping the upper half of the reference plane onto the triangle G_Ω is

$$\Omega(\zeta) = \mathrm{e}^{\mathrm{i}\pi\mu} R\zeta^{1-2\mu} f(\zeta; 1-\mu)/f(\zeta;\mu), \quad (2)$$

where $f(\zeta;\mu), = \mathrm{F}((\mu-\nu)/2 - 1/4, (\mu-\nu)/2 + 1/4; 1/2+\mu; \zeta^2)$ (F is the hypergeometric function), and parameters ν, R are determined as $\pi\nu = \arccos\sqrt{1 - \left(\frac{k-\varepsilon}{k+\varepsilon}\right)^2 (\cos\pi\mu)^2}$, $R = \dfrac{k-\varepsilon}{2^{2-2\mu}k\varepsilon}\left[\sin\pi\mu + \sqrt{\left(\frac{k+\varepsilon}{k-\varepsilon}\right)^2 - (\cos\pi\mu)^2}\right]\dfrac{\Gamma(3/2-\mu+\nu)\Gamma(1/2+\mu)}{\Gamma(1/2-\mu)\Gamma(1/2+\mu+\nu)}$.

We introduce the following functions:

$$W(\zeta) = \mathrm{d}w/\mathrm{d}\zeta, \qquad Z(\zeta = \mathrm{d}z/\mathrm{d}\zeta, \quad (3)$$

such that $Z(\zeta) = \Omega(\zeta)W(\zeta)$. Next, we show that the BWP (1) is reduced to the following simpler one:

$$\mathrm{Im}W(\xi) = 0,\ -1 < \xi < 1; \quad \mathrm{Im}[(k\Omega(\xi) + \mathrm{i})W(\xi)] = 0,\ \xi < -1, \xi > 1. \quad (4)$$

The Riemann BVP (4) has an unique (up to real multiplier d) solution. This solution $W(\zeta)$ and the corresponding function $Z(\zeta)$ could be written down in the following form:

$$W(\zeta) = d\zeta^{-1}(1-\zeta^2)^{-3/4+\mu/2-\nu/2} f(\zeta,\mu),\ Z(\zeta) = d\mathrm{e}^{\mathrm{i}\pi\mu} R\zeta^{-2\mu}(1-\zeta^2)^{-3/4+\mu/2-\nu/2} f(\zeta, 1-\mu). \quad (5)$$

In accordance with (3), (5)

$$w(\zeta) = d\int_{-\infty}^{\zeta}(1-\tau^2)^{-3/4+\mu/2-\nu/2} f(\tau;\mu)\frac{\mathrm{d}\tau}{\tau},\ z(\zeta) = d\mathrm{e}^{\mathrm{i}\pi\mu} R\int_0^{\zeta}\tau^{-2\mu}(1-\tau^2)^{-3/4+\mu/2-\nu/2} f(\tau; 1-\mu)\mathrm{d}\tau.$$

A real constant d is found from the condition $\mathrm{Im}w(-1) = Q/2$, which gives $d = Q/\pi$. Eventually, the free boundary and flow net are plotted.

In the unsaturated-saturated flow model for transient seepage, p and the volumetric moisture content θ are interrelated via the Van Genuchten relationship, $k(p)$ is another characteristic function of the soil, such that a nonlinear parabolic Richards-Richardson equation holds in a fixed flow domain. Initial boundary value problems are solved by the finite element method with the help of HYDRUS2D package (Radcliffe and Simunek, 2018). Three seepage problems are modeled. First, for comparisons with the analytical solution, a curvilinear tetragon is considered as a flow tube, with a circular arc serving as a "feeding" positive-pressure isobar and horizontal segment of the soil surface as an evaporating isobar such that a 2-D ascending flow crosses an aposteriori determined phreatic line and makes a vadose zone above it. Second, we model infiltration in a lysimeter of Moscow State University station (Umarova et al., 2021). The flow domain is a trapezium with a tilted bottom and a blanket drain on its part. A "perched" phreatic line emerges above such drain with a vadose zone making a mini-bubble (Fig.2).

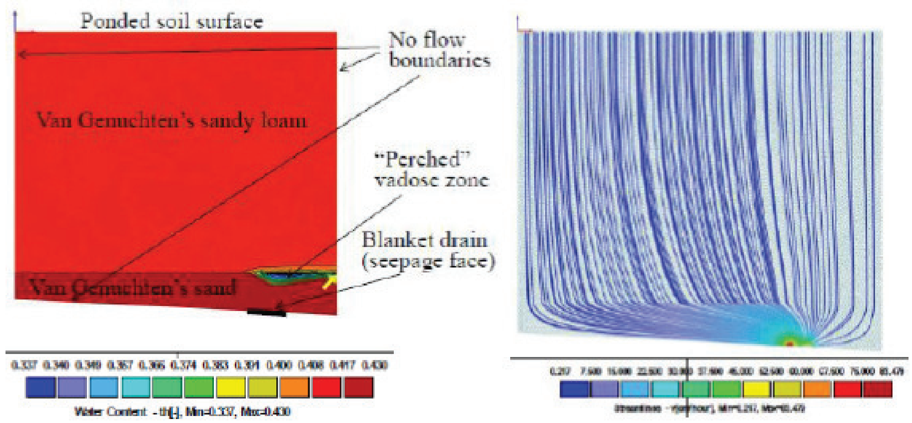

Fig 2: Results of HYDRUS2D computations (steady-state limit). Volumetric moisture content and streamlines (left and right panels, correspondingly).

Third, we model infiltration in a two-component composite, which consists of a bulk sandy soil and a cylindrical lens of peat (Smagin, 2005) or fine-textured loam such that an essentially axisymmetric seepage is transformed from a purely unsaturated to saturated-unsaturated one, involving nontrivial phreatic surfacessimilar to one in Fig.2. For all three cases we reconstruct the vector fields of Darcian velocity, and scalar field of isobars, isotachs and isohumes.

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Насыщенная и ненасыщенная фильтрация из оросителя-дрены Корнева: сравнение аналитических и численных решений

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Аннотация

Подземные оросители моделируются линейными источниками, с фильтрацией, которая блоикруется либо естественным непроницаемым горизонтом либо искуственным барьером, который образуют клин под оросителем. Аналитическая модель предполагает насыщенный стационарный двумерный поток (описываемый уравнением

Лапласа) в облестаи со свободной границей, вдоль которой функция потока линейно зависит от горизонтальной координаты, что позволяет использовать технику Полубариновой-Кочиной, использующей конформное отображение кругового треугольника в области годографа на вспомогательную полуплоскость. Методом конечных элементов (пакет HYDRUD2D, уравнение Ричардса-Ричардсона) нестационарная начальная задача (дающая асимптотический предел стационарного состояния) решается в фиксированной области (равнобедренный криволинейный четырехугольник или трапеция). Вычисляются изобары, изохьюмы, линии тока, изотахи и коэффициент однородности Кристиансена.

Ключевые слова: конформное отображение круговых многоугольников, задача Римана-Гильберта, моделирование на HYDRUS2D, топология фильтрационного течения, полевые эксперименты