ISSN 0037–4466, Siberian Mathematical Journal, 2020, Vol. 61, No. 2, pp. 248–254.

INEQUALITIES FOR DETERMINANTS AND CHARACTERIZATION OF THE TRACE

A. M. Bikchentaev UDC 512.643:517.982

Abstract: Let tr be the canonical trace on the full matrix algebra $\mathscr{M}_n$ with unit I. We prove that if some analog of classical inequalities holds for the determinant and trace (or the permanent and trace) of matrices for a positive functional φ on $\mathscr{M}_n$ with $\varphi(I) = n$, then $\varphi = \mathrm{tr}$. Also, we generalize Fischer's inequality for determinants and establish a new inequality for the trace of the matrix exponential.

DOI: 10.1134/S0037446620020068

Keywords: linear functional, matrix, trace, determinant, permanent, matrix exponential, Fischer inequality

Introduction

Let tr be the canonical trace on the full matrix algebra $\mathscr{M}_n = \mathbb{M}_n(\mathbb{C})$ and let $\det(A)$ stand for the determinant of $A \in \mathscr{M}_n$. Let $\mathscr{M}_n^{\mathrm{pr}}$, $\mathscr{M}_n^{\mathrm{id}}$, $\mathscr{M}_n^{\mathrm{sa}}$, and $\mathscr{M}_n^{+}$ be the lattice of projections ($P = P^2 = P^*$), the set of idempotents ($P = P^2$), the Hermitian part, and the cone of nonnegative definite matrices in $\mathscr{M}_n$ respectively. Let I be the unit of $\mathscr{M}_n$. We obtain the following generalization of Fischer's inequality for determinants. Suppose that $\{P_k\}_{k=1}^m \subset \mathscr{M}_n^{\mathrm{id}}$ with $P_iP_k = 0$ for $i \neq k$, $i, k = 1, 2, \dots, m$, and $\sum_{k=1}^m P_k = I$. Then $\det(\mathscr{P}(A)) \geq \det(A)$ for all $A \in \mathscr{M}_n^{+}$, where $\mathscr{P}(A) = \sum_{k=1}^m P_kAP_k^*$ (Theorem 1). For $\{P_k\}_{k=1}^m \subset \mathscr{M}_n^{\mathrm{pr}}$, we demonstrate that $\mathrm{tr}(\exp(\mathscr{P}(A))) \leq \mathrm{tr}(\mathscr{P}(\exp(A)))$ for all $A \in \mathscr{M}_n^{+}$ (Theorem 2).

It is well known that validity of each of the Young, Hölder, Cauchy–Bunyakovskii–Schwartz, Golden–Thompson, Peierls–Bogoliubov, and Araki–Lieb–Thirring inequalities implies the equality $\varphi = \mathrm{tr}$ for an arbitrary positive functional φ on $\mathscr{M}_n$ with $\varphi(I) = n$ (see [1–4]). Suppose that $\varphi = \mathrm{tr}$, while $\mathrm{per}(A)$ is the permanent, and $\lambda_t(A)$ ($t = 1, \dots, n$) are the eigenvalues of $A \in \mathscr{M}_n$. Then the following relations hold:

• Schur's inequality [5, Chapter III, § 1.4]

$$\sum_{t=1}^{n} |\lambda_t(A)|^2 \leq \sum_{i,j=1}^{n} |a_{ij}|^2 (= \varphi(AA^*)) \quad \text{for all } A \in \mathscr{M}_n;$$

the equality is attained if and only if A is normal;

• the equality [5, Chapter I, § 4.16, formula (1)]

$$\det(\exp(A)) = \exp(\varphi(A)) \quad \text{for all } A \in \mathscr{M}_n; \tag{1}$$

• the inequality [6, Problem 3, p. 163] $\det(A)^{\frac{1}{n}} \leq \frac{1}{n}\varphi(A)$ for all $A \in \mathscr{M}_n^{+}$;

• the inequality [5, Chapter II, § 4.4.12] $\mathrm{per}(A) \leq \frac{1}{n}\varphi(A^n)$ for all nonnegative matrices $A \in \mathscr{M}_n^{\mathrm{sa}}$.

We will demonstrate that validity of each of these four relations implies that $\varphi = \mathrm{tr}$ (Theorems 3 and 4) for an arbitrary positive functional φ on $\mathscr{M}_n$ with $\varphi(I) = n$.

The research was funded by the subsidy allocated to Kazan Federal University for the State Assignment in the Sphere of Scientific Activities, Project 1.13556.2019/13.1.

Original article submitted September 25, 2019; revised September 30, 2019; accepted October 18, 2019.

1. Definitions and Notation

A C^**-algebra* is a complex Banach $*$-algebra $\mathscr{A}$ such that $\|A^*A\| = \|A\|^2$ for all $A \in \mathscr{A}$. Denote by $\mathscr{A}^{\mathrm{pr}}$, $\mathscr{A}^{\mathrm{id}}$, and $\mathscr{A}^+$ the subsets of projections, idempotents, and positive elements of a C^*-algebra $\mathscr{A}$. Let $\mathscr{H}$ be a Hilbert space over $\mathbb{C}$ and let $\mathscr{B}(\mathscr{H})$ be the $*$-algebra of all bounded linear operators on $\mathscr{H}$. Each C^*-algebra can be realized as a C^*-subalgebra in $\mathscr{B}(\mathscr{H})$ for some Hilbert space $\mathscr{H}$ (Gelfand–Naimark, see [7, Theorem 3.4.1]).

Recall that $A^* = [\overline{a_{ji}}]_{i,j=1}^n$ for $A = [a_{ij}]_{i,j=1}^n \in \mathscr{M}_n$. A linear functional φ on $\mathscr{M}_n$ is called *Hermitian* if $\varphi(A^*) = \overline{\varphi(A)}$ for all $A \in \mathscr{M}_n$ and *positive* if φ is Hermitian and $\varphi(\mathscr{M}_n^+) \subset \mathbb{R}^+$. A positive functional φ on $\mathscr{M}_n$ is called *faithful* if $\varphi(A) = 0$ $(A \in \mathscr{M}_n^+) \Rightarrow A = 0$.

Let $\{P_k\}_{k=1}^m \subset \mathscr{M}_n^{\mathrm{id}}$ with $P_iP_k = 0$ for $i \neq k$, $i,k = 1,2,\dots,m$, and $\sum_{k=1}^m P_k = I$. Define the mapping $\mathscr{P} : \mathscr{M}_n \to \mathscr{M}_n$ by the formula

$$\mathscr{P}(A) = \sum_{k=1}^m P_k A P_k^* \quad \text{for all } A \in \mathscr{M}_n.$$

If $\{P_k\}_{k=1}^m \subset \mathscr{M}_n^{\mathrm{pr}}$, then $\mathscr{P}$ is a block projection operator whose properties are studied in [8–10]. The formula $S = 2P - I$ $(P \in \mathscr{M}_n^{\mathrm{id}})$ establishes a bijection between $\mathscr{M}_n^{\mathrm{id}}$ and the set $\mathscr{M}_n^{\mathrm{sym}}$ of all symmetries $(S^2 = I)$ from $\mathscr{M}_n$.

2. New Inequalities for Determinants and the Trace

Lemma 1. *Suppose that* $\{P_k\}_{k=1}^m \subset \mathscr{M}_n^{\mathrm{id}}$ *with* $P_iP_k = 0$ *for* $i \neq k$, $i,k = 1,2,\dots,m$, *and* $\sum_{k=1}^m P_k = I$. *Then* $\operatorname{tr}(A) = \operatorname{tr}\left(\sum_{k=1}^m P_k A P_k\right)$ *for all* $A \in \mathscr{M}_n$. *In particular,* $\operatorname{tr}(\mathscr{P}(A)) = \operatorname{tr}(A)$, $A \in \mathscr{M}_n$, *for* $\{P_k\}_{k=1}^m \subset \mathscr{M}_n^{\mathrm{pr}}$.

Proof. If $A \in \mathscr{M}_n$, then

$$\operatorname{tr}(A) = \operatorname{tr}\left(\sum_{k=1}^m P_k A\right) = \sum_{k=1}^m \operatorname{tr}(P_k A) = \sum_{k=1}^m \operatorname{tr}(P_k A P_k) = \operatorname{tr}\left(\sum_{k=1}^m P_k A P_k\right). \quad \square$$

Lemma 2 [11, Theorem 1.3]. *Let* $\mathscr{A}$ *be a* C^**-algebra and* $P \in \mathscr{A}^{\mathrm{id}}$. *There is a unique decomposition* $P = \widetilde{P} + Z$, *where* $\widetilde{P} \in \mathscr{A}^{\mathrm{pr}}$ *and the nilpotent* Z *belongs to* $\mathscr{A}$ *with* $Z^2 = 0$; *moreover,* $Z\widetilde{P} = 0$ *and* $\widetilde{P}Z = Z$.

Proposition 1. *Let* $\mathscr{A}$ *be a unital* C^**-algebra,* $A \in \mathscr{A}^+$ *is invertible,* $P \in \mathscr{A}^{\mathrm{id}}$, *and* $P = \widetilde{P} + Z$ *is the decomposition described in Lemma 2. Then* PAP^* *is invertible in the reduced algebra* $\widetilde{P}\mathscr{A}\widetilde{P}$.

Proof. There exists $\varepsilon > 0$ such that $A \geq \varepsilon I$. Consider the multiplicative representation $P = \widetilde{P}T$ with an invertible $T \in \mathscr{A}^+$ [12, Lemma 3]. Let $\delta > 0$ be such that $T \geq \delta I$. Then $T^2 \geq \delta^2 I$ and

$$PAP^* \geq \varepsilon PP^* = \varepsilon \widetilde{P} T^2 \widetilde{P} \geq \varepsilon \delta^2 \widetilde{P}.$$

It remains to take into account the fact that $\widetilde{P}P = P$, $\widetilde{P}PAP^*\widetilde{P} = PAP^*$, and $\widetilde{P}$ is the unit of the reduced algebra $\widetilde{P}\mathscr{A}\widetilde{P}$. $\square$

Theorem 1. $\det(\mathscr{P}(A)) \geq \det(A)$ *for all* $\{P_k\}_{k=1}^m \subset \mathscr{M}_n^{\mathrm{id}}$ *with* $P_iP_k = 0$ *for* $i \neq k$, $i,k = 1,2,\dots,m$, *and* $\sum_{k=1}^m P_k = I$ *for all* $A \in \mathscr{M}_n^+$.

Proof. By the Determinant Product Theorem, $\det(S) \in \{-1, +1\}$ for each $S \in \mathscr{M}_n^{\mathrm{sym}}$. Since $\mathscr{P}(\mathscr{M}_n^+) \subset \mathscr{M}_n^+$ and $\det(X) \geq 0$ for all $X \in \mathscr{M}_n^+$, it suffices to verify the claim only for invertible matrices. The results of [13, 14] imply that the function

$$A \mapsto \log\det(A) \tag{2}$$

is concave on the set of invertible matrices $A \in \mathscr{A}^+$ (see also [15, Chapter 10, § 2, Theorem 9′]). By Lemma 2 from [10],

$$\mathscr{P}(A) = \frac{1}{2^{m-1}} \sum_{j=1}^{2^{m-1}} S_j A S_j^* \tag{3}$$

for 2^{m-1} collections $\{t_{jk}\}_{k=1}^m$ with $t_{jk} \in \{-1, +1\}$, where $S_j = \sum_{k=1}^m t_{jk} P_k \in \mathscr{M}_n^{\mathrm{sym}}$ for all $j = 1, 2, 3, \dots, 2^{m-1}$. Therefore, $\det(S_j) = \det(S_j^*) \in \{-1, +1\}$ for all $j = 1, 2, 3, \dots, 2^{m-1}$. The invertibility of $\mathscr{P}(A)$ for an invertible $A \in \mathscr{M}_n^+$ follows from the representation of (3) where each summand $S_j A S_j^*$ lies in $\mathscr{M}_n^+$ and is invertible by the Invertible Product Theorem. Concavity of (2), the Determinant Product Theorem, and (3) imply that

$$\begin{aligned} \log\det(\mathscr{P}(A)) &\geq \sum_{j=1}^{2^{m-1}} \frac{1}{2^{m-1}} \log\det(S_j A S_j^*) \\ &= \sum_{j=1}^{2^{m-1}} \frac{1}{2^{m-1}} \log\det(A) = \log\det(A). \end{aligned}$$

Therefore,

$$\det(\mathscr{P}(A)) \geq \det(A) \tag{4}$$

due to strict monotonicity of the logarithmic function on the half-axis $(0, +\infty)$. □

Remark 1. Relation (4) for a particular case when $\{P_k\}_{k=1}^m \subset \mathscr{M}_n^{\mathrm{pr}}$ is known as Fischer's inequality [16, Problem II.5.6]. Hence, by Lemma 1 and (1), we obtain

$$\det(\mathscr{P}(\exp(A))) \geq \det(\exp(A)) = \exp(\mathrm{tr}(A)) = \exp(\mathrm{tr}(\mathscr{P}(A)))$$

for all $A \in \mathscr{M}_n^+$.

Corollary 1. $\det(\mathscr{P}(A)) \geq \exp(\mathrm{tr}(\log A))$ *for each positive definite matrix* $A \in \mathscr{M}_n^+$.

Proof. We have

$$\det(\mathscr{P}(A)) = \det(\mathscr{P}(\exp(\log A))) \geq \det(\exp(\log A)) = \exp(\mathrm{tr}(\log A))$$

for a positive definite matrix $A \in \mathscr{M}_n^+$. □

Proposition 2. *Let* $n \in \mathbb{N}$ *be odd,* $A \in \mathscr{M}_n$*, and* $S, T \in \mathscr{M}_n^{\mathrm{sym}}$ *with* $\det(S) = \det(T)$*. Then* $\det(A - SAT) = 0$.

Proof. The claim follows from the relations

$$S(A - SAT)T = -(A - SAT), \quad \det(S) = \det(T) \in \{-1, +1\}$$

and the Determinant Product Theorem. □

Here the oddness of $n \in \mathbb{N}$ is essential. Consider the matrices

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \text{ and } S = \begin{pmatrix} 1 & x \\ 0 & -1 \end{pmatrix}, \quad \text{where } x \in \mathbb{R},$$

in $\mathscr{M}_2$. Then $S \in \mathscr{M}_2^{\mathrm{sym}}$ and $\det(A - SAS) = x^2 + 2x - 4 \neq 0$ for $2x \neq -1 \pm \sqrt{5}$. The trace $\mathrm{tr}(A - SAS^*) = x^2 + 2x$ can take arbitrary values from the interval $[-1, +\infty)$. We have $\mathrm{tr}(PAP^*) - \mathrm{tr}(\widetilde{P}A\widetilde{P}) = x + x^2/4$ for the idempotent $P = (I + S)/2$, while the projection $\widetilde{P}$ is defined in Lemma 2. Since $\mathrm{tr}(\mathscr{P}(A)) - \mathrm{tr}(A) = x + x^2/2$ for the pair $P_1 = P$, $P_2 = I - P$, the requirement $\{P_k\}_{k=1}^m \subset \mathscr{M}_n^{\mathrm{pr}}$ is essential in Lemma 1.

Lemma 3. *Suppose that $A \in \mathscr{M}_n^+$ and $B \in \mathscr{M}_n$ with the operator norm $\|B\| \le 1$, $1 \le p < \infty$. Then*

$$\lambda_t((BAB^*)^p) \le \lambda_t(BA^pB^*) \quad \text{for all } t = 1, 2, \ldots, n. \tag{5}$$

Proof. Since the real function $s \mapsto s^q$ ($s \in \mathbb{R}^+$) is operator convex for $1 \le q \le 2$, we have

$$(BXB^*)^q \le BX^qB^*$$

for all $X \in \mathscr{M}_n^+$ and $B \in \mathscr{M}_n$ with $\|B\| \le 1$ by [17, Theorem 2.1]. By monotonicity of eigenvalues (i.e., $\lambda_t(X) \le \lambda_t(Y)$ for all $t = 1, 2, \ldots, n$ for $0 \le X \le Y$) this matrix inequality leads to the claim of the lemma for $1 \le q \le 2$. Let $t \in \{1, 2, \ldots, n\}$ and let $2 < p < \infty$ be fixed. Choose $j \in \mathbb{N}$ such that $2^{j-1} < p \le 2^j$ and put $q = \sqrt[j]{p}$. Then $j \ge 2$ and $1 < 2^{\frac{j-1}{j}} < q \le 2$. We have

$$\begin{aligned} \lambda_t(BA^pB^*) &= \lambda_t(B(A^{p/q})^qB^*) \ge \lambda_t((BA^{p/q}B^*)^q) = \lambda_t((BA^{p/q}B^*))^q \\ &= \lambda_t(B(A^{p/q^2})^qB^*)^q \ge \cdots \ge \lambda_t(BA^{p/q^j}B^*)^{q^j} = \lambda_t(BAB^*)^p = \lambda_t((BAB^*)^p) \end{aligned}$$

by monotonicity of the power functions $s \mapsto s^b$ ($s \in \mathbb{R}^+$) and the equality $\lambda_t(X^b) = \lambda_t(X)^b$ for all $X \in \mathscr{M}_n^+$ and the reals $b > 0$. $\square$

Theorem 2. *Let $\{P_k\}_{k=1}^m \subset \mathscr{M}_n^{\mathrm{pr}}$ with $P_iP_k = 0$ for $i \ne k$, $i, k = 1, 2, \ldots, m$, and $\sum_{k=1}^m P_k = I$. Then $\mathrm{tr}(\exp(\mathscr{P}(A))) \le \mathrm{tr}(\mathscr{P}(\exp(A)))$ for all $A \in \mathscr{M}_n^+$.*

Proof. It is easy to see that

$$\begin{aligned} \exp(\mathscr{P}(A)) &= I + \sum_{k=1}^m P_kAP_k + \sum_{k=1}^m \frac{(P_kAP_k)^2}{2!} + \cdots + \sum_{k=1}^m \frac{(P_kAP_k)^j}{j!} + \cdots \\ &= -(m-1)I + \left(I + P_1AP_1 + \frac{(P_1AP_1)^2}{2!} + \cdots + \frac{(P_1AP_1)^j}{j!} + \cdots\right) \\ &\quad + \cdots + \left(I + P_mAP_m + \frac{(P_mAP_m)^2}{2!} + \cdots + \frac{(P_mAP_m)^j}{j!} + \cdots\right), \\ \mathscr{P}(\exp(A)) &= P_1\left(I + A + \frac{A^2}{2!} + \cdots + \frac{A^j}{j!} + \cdots\right)P_1 \\ &\quad + \cdots + P_m\left(I + A + \frac{A^2}{2!} + \cdots + \frac{A^j}{j!} + \cdots\right)P_m \\ &= -(m-1)I + \left(I + P_1AP_1 + \frac{P_1A^2P_1}{2!} + \cdots + \frac{P_1A^JP_1}{j!} + \cdots\right) \\ &\quad + \cdots + \left(I + P_mAP_m + \frac{P_mA^2P_m}{2!} + \cdots + \frac{P_mA^jP_m}{j!} + \cdots\right); \end{aligned}$$

the matrix series converges in norm (i.e., elementwise). Since the matrix trace coincides with the spectral trace and is a continuous linear functional, Theorem 2 follows from Lemma 3. $\square$

3. The Inequalities for Determinants Characterize the Trace

Theorem 3. *The following are equivalent for a positive functional φ on the algebra $\mathscr{M}_n$ with $\varphi(I) = n$:*

(i) $\varphi = \mathrm{tr}$;

(ii) $\det(\mathscr{P}(\exp(A))) \ge \exp(\varphi(A))$ *for all $\mathscr{P}$ and $A \in \mathscr{M}_n^+$;*

(iii) $\det(A)^{\frac{1}{n}} \le \frac{1}{n}\varphi(A)$ *for all $A \in \mathscr{M}_n^+$;*

(iv) $\operatorname{per}(A) \le \frac{1}{n}\varphi(A^n)$ *for all nonnegative matrices* $A \in \mathscr{M}_n^{\mathrm{sa}}$*;*
(v) $\det(I+\varepsilon A) = 1 + \varepsilon\varphi(A) + o(\varepsilon)$ *as* $\varepsilon \to 0+$ *for all* $A \in \mathscr{M}_n^+$*.*
Moreover, if φ *is faithful, then* (i)–(v) *are equivalent to the conditions:*
(vi) $\det(\exp(A)) \le \exp(\varphi(A))$ *for all* $A \in \mathscr{M}_n^+$*;*
(vii) $\varphi(A^p)^{\frac{1}{p}} \le \varphi(A^q)^{\frac{1}{q}}$ *for all* $A \in \mathscr{M}_n^+$ *and* $0 < q < p$*.*

PROOF. The implication (i) $\Rightarrow$ (ii) follows from Theorem 1 and (1); see the implication (i) $\Rightarrow$ (v) in [18, Chapter 6, § 9, Exercise 1].

Without loss of generality, assume that $\varphi(X) = \operatorname{tr}(S_\varphi X)$ for all $X \in \mathscr{M}_n$, where

$$S_\varphi = \operatorname{diag}(s_1, \dots, s_n) \in \mathscr{M}_n^+$$

and $s_1 + \cdots + s_n = n$. We need to show that

$$s_1 = \cdots = s_n = 1. \tag{6}$$

(ii) $\Rightarrow$ (i): If (6) is not valid, then there exists $k \in \{1, \dots, n\}$ such that $s_k > 1$. By the Spectral Theorem in finite dimensions, $\exp(A) = \exp(1)\cdot A + \exp(0)\cdot(I-A)$ for the projection

$$A = \operatorname{diag}(\underbrace{0, \dots, 0}_{k-1 \text{ times}}, 1, 0, \dots, 0) \in \mathscr{M}_n^{\mathrm{pr}}, \tag{7}$$

while, by (ii), $\exp(1) \ge \exp(s_k)$ for the mapping $\mathscr{P}$ associated with all projections of the form (7) with $k = 1, 2, \dots, n$. Consequently, $s_k \le 1$; a contradiction.

(iii) $\Rightarrow$ (i): If (6) is not valid, then there exists $k \in \{1, \dots, n\}$ such that $s_k > 1$. Given a real $\varepsilon > 0$, introduce the matrix $A_\varepsilon = (1+\varepsilon)I - \varepsilon A$, where A is from (7). Inserting $A_\varepsilon (\in \mathscr{M}_n^+)$ in (iii), we obtain

$$\begin{aligned}(1+\varepsilon)^{\frac{n-1}{n}} &\le \frac{1}{n}((1+\varepsilon)s_1 + \cdots + (1+\varepsilon)s_{k-1} + s_k \\ &\quad + (1+\varepsilon)s_{k+1} + \cdots + (1+\varepsilon)s_n) \\ &= \frac{1}{n}((1+\varepsilon)n - \varepsilon s_k) = 1 + \varepsilon - \frac{s_k}{n}\varepsilon.\end{aligned}$$

Recall the Taylor formula with Peano's remainder:

$$(1+\varepsilon)^{\frac{n-1}{n}} = 1 + \frac{n-1}{n}\varepsilon + o(\varepsilon) \quad \text{as } \varepsilon \to 0+.$$

Now, (iii) takes the form

$$1 + \frac{n-1}{n}\varepsilon + o(\varepsilon) \le 1 + \varepsilon - \frac{s_k}{n}\varepsilon \quad \text{as } \varepsilon \to 0+.$$

Consequently, $s_k \le 1$; a contradiction.

(iv) $\Rightarrow$ (i): If (6) is not valid, then there exists $k \in \{1, \dots, n\}$ such that $s_k > 1$. Given $1 > \varepsilon > 0$, introduce the matrix $A_\varepsilon = I - \varepsilon A$, where A is from (7). Inserting A_ε in (iv), we obtain

$$1 - \varepsilon \le \frac{1}{n}(s_1 + \cdots + s_{k-1} + (1-\varepsilon)^n s_k + s_{k+1} + \cdots + s_n).$$

Write the Taylor formula with Peano's remainder:

$$(1-\varepsilon)^n = 1 - n\varepsilon + o(\varepsilon) \quad \text{as } \varepsilon \to 0+.$$

Now, (iv) takes the form $1 - \varepsilon \le 1 - s_k\varepsilon + o(\varepsilon)$ as $\varepsilon \to 0+$. Consequently, $s_k \le 1$; a contradiction.

(v) ⇒ (i): If (6) is not valid, then there exists $k \in \{1, \dots, n\}$ such that $s_k > 1$. By (v), we obtain

$$1 + \varepsilon = 1 + s_k \varepsilon + o(\varepsilon) \quad \text{as } \varepsilon \to 0+$$

for the projection A from (7). Consequently, $s_k = 1$; a contradiction.

(vi) ⇒ (i): If (6) is not valid, then there exists $k \in \{1, \dots, n\}$ such that $0 < s_k < 1$. By (vi), $\exp(1) \le \exp(s_k)$ for the projection A from (7). Consequently, $s_k \ge 1$; a contradiction.

(i) ⇒ (vii): Without loss of generality, assume that $A = \operatorname{diag}(a_1, \dots, a_n)$ with $a_j \ge 0$ for all $j = 1, \dots, n$. Then $A^r = \operatorname{diag}\big(a_1^r, \dots, a_n^r\big)$ for all $r > 0$. By Jensen's inequality (see [19, Theorem 19]),

$$\varphi(A^p)^{\frac{1}{p}} = \big(a_1^p + \dots + a_n^p\big)^{\frac{1}{p}} \le \big(a_1^q + \dots + a_n^q\big)^{\frac{1}{q}} = \varphi(A^q)^{\frac{1}{q}}$$

for all $0 < q < p$.

(vii) ⇒ (i): If (6) is not valid, then there exists $k \in \{1, \dots, n\}$ such that $0 < s_k < 1$. By (vii), $s_k^q \le s_k^p$ for the projection A from (7). Consequently, $s_k \ge 1$; a contradiction. Recall that if $1 < p < \infty$ and φ is a positive functional on $\mathscr{M}_n$ with $\varphi(A^p) \le \varphi(B^p)$ for $0 \le A \le B$, then $\varphi = \lambda \operatorname{tr}$ with some $\lambda \in \mathbb{R}^+$ [20, Theorem]. □

Corollary 2. *For a positive functional φ on $\mathscr{M}_n$ with $\varphi(I) = n$ the following are equivalent:*
(i) $\varphi = \operatorname{tr}$;
(ii) $\det(\exp(A)) \ge \exp(\varphi(A))$ *for all* $A \in \mathscr{M}_n^+$.

Remark 2. In connection with the inequality from Theorem 3(iii), recall that

$$\det(A)^{\frac{1}{n}} = \min_{B \in \mathscr{M}_n^+,\ \det(B)=1} \frac{\operatorname{tr}(AB)}{n}$$

for all positive definite real matrices $A \in \mathscr{M}_n^+$ [21, Chapter II, § 21, Theorem 14].

Theorem 4. *For a positive functional φ on $\mathscr{M}_n$ with $\varphi(I) = n$ the following are equivalent:*
(i) $\varphi = \operatorname{tr}$;
(ii) $\sum_{t=1}^n \lambda_t(A)^2 \le \varphi(A^2)$ *for all* $A \in \mathscr{M}_n^+$;
(iii) $\left|\lambda_t(A) - \frac{\varphi(A^*A)}{n}\right| \le \left(\frac{n-1}{n}\left(\varphi(A^*A) - \frac{|\varphi(A)|^2}{n}\right)\right)^{1/2}$ *for all* $A \in \mathscr{M}_n$ *and* $t = 1, \dots, n$;
(iv) $\sum_{i=1}^n a_{ii}^2 \le \varphi(A^2)$ *for all* $A = [a_{ij}] \in \mathscr{M}_n^+$;
(v) $\varphi(A^2) \le \operatorname{tr}(A)^2$ *for all* $A \in \mathscr{M}_n^+$;
(vi) $\sqrt{\operatorname{tr}(A)} \le \varphi(\sqrt{A})$ *for all* $A \in \mathscr{M}_n^+$;
(vii) $\varphi(\sqrt{A}) \le \sum_{i=1}^n \sqrt{a_{ii}}$ *for all* $A = [a_{ij}] \in \mathscr{M}_n^+$.

Proof. The implication (i) ⇒ (ii) is the aforementioned Schur's inequality. See the implication (i) ⇒ (iii) in [16, Problem I.6.16, p. 172] and the implications (i) ⇒ (iv)–(vii) in [6, Problem 16, p. 24].

Show the converse implications. Without loss of generality, assume that $\varphi(X) = \operatorname{tr}(S_\varphi X)$ for all $X \in \mathscr{M}_n$, where $S_\varphi = \operatorname{diag}(s_1, \dots, s_n) \in \mathscr{M}_n^+$ and $s_1 + \dots + s_n = n$. We need to verify relations (6). If (6) is not valid, then there exist $m, j \in \{1, \dots, n\}$ such that $s_m < 1$ and $s_j > 1$.

(ii) ⇒ (i): By (ii), $1 = \sum_{t=1}^n \lambda_t(A)^2 > \varphi(A^2) = s_j$ for a projection A (with $j = k$) from (7); a contradiction.

(v) ⇒ (i) and (vii) ⇒ (i): For the matrix A indicated above, inequality (v) (or (vii)) gives $s_j \le 1$; a contradiction.

(iii) ⇒ (i): Inequality (iii) for $t = 1$ implies $s_m \ge 1$ for a projection A (with $m = k$) from (7); a contradiction.

(iv) ⇒ (i) and (vi) ⇒ (i): Inequality (iv) (or (vi)) gives $s_m \ge 1$ for the projection A (with $m = k$) from (7); a contradiction. □

About other characterizations of the trace, see [22–25] and references therein.

References

1. Bikchentaev A. M. and Tikhonov O. E., "Characterization of the trace by Young's inequality," J. Inequal. Pure Appl. Math., vol. 6, no. 2, Article 49 (2005).
2. Cho K. and Sano T., "Young's inequality and trace," Linear Algebra Appl., vol. 431, no. 8, 1218–1222 (2009).
3. Bikchentaev A. M., "Commutation of projections and trace characterization on von Neumann algebras. II," Math. Notes, vol. 89, no. 4, 461–471 (2011).
4. Bikchentaev A. M., "The Peierls–Bogoliubov inequality in C^*-algebras and characterization of tracial functionals," Lobachevskii J. Math., vol. 32, no. 3, 175–179 (2011).
5. Marcus M. and Mink H., *A Survey of Matrix Theory and Matrix Inequalities*, Allyn and Bacon, Boston (1964).
6. Zhang F., *Matrix Theory. Basic Results and Techniques*, Springer-Verlag, New York (1999 (Universitext).
7. Murphy G., *C^*-Algebras and Operator Theory*, Academic Press, Boston (1990).
8. Gokhberg I. Ts. and Krein M. G., *An Introduction to the Theory of Linear Nonselfadjoint Operators in Hilbert Space*, Amer. Math. Soc., Providence (1969).
9. Chilin V. I., Krygin A. V., and Sukochev Ph. A., "Extreme points of convex fully symmetric sets of measurable operators," Integral Equations Operator Theory, vol. 15, no. 2, 186–226 (1992).
10. Bikchentaev A. M., "Block projection operators in normed solid spaces of measurable operators," Russian Math. (Iz. VUZ), vol. 56, no. 2, 75–79 (2012).
11. Koliha J. J., "Range projections of idempotents in C^*-algebras," Demonstratio Math., vol. 24, no. 1, 91–103 (2001).
12. Bikchentaev A. M., "On representation of elements of a von Neumann algebra in the form of finite sums of products of projections," Sib. Math. J., vol. 46, no. 1, 24–34 (2005).
13. Bellman R., "Notes on Matrix Theory. II," Amer. Math. Monthly, vol. 60, no. 3, 173–175 (1953).
14. Mirsky L., "An inequality for positive definite matrices," Amer. Math. Monthly, vol. 62, no. 6, 428–430 (1955).
15. Lax P. D., *Linear Algebra and Its Applications.* Enlarged second edition, Wiley- Intersci. [John Wiley & Sons], Hoboken, NJ (2007) (Pure Appl. Math. (Hoboken)).
16. Bhatia R., *Matrix Analysis*, Springer-Verlag, New York (1997) (Grad. Texts Math.; Vol. 169).
17. Hansen F. and Pedersen K. G., "Jensen's inequality for operators and Löwner's theorem," Math. Ann., vol. 258, no. 3, 229–241 (1982).
18. Bellman R., *Introduction to Matrix Analysis.* 2nd ed., SIAM, Philadelphia, PA, USA (1997) (Class. Appl. Math.).
19. Hardy G. H., Littlewood J. E., and Pólya G., *Inequalities*, Cambridge University Press, Cambridge (UK) etc. (1988).
20. Bikchentaev A. M. and Tikhonov O. E., "Characterization of the trace by monotonicity inequalities," Linear Algebra Appl., vol. 422, no. 1, 274–278 (2007).
21. Beckenbach E. F. and Bellman R., *Inequalities* [Russian translation], Mir, Moscow (1965).
22. Bikchentaev A. M., "Commutativity of projections and characterization of traces on von Neumann algebras," Sib. Math. J., vol. 51, no. 6, 971–977 (2010).
23. Bikchentaev A. M., "Commutation of projections and characterization of traces on von Neumann algebras. III," Int. J. Theor. Phys., vol. 54, no. 12, 4482–4493 (2015).
24. Bikchentaev A. M., "Trace and differences of idempotents in C^*-algebras," Math. Notes, vol. 105, no. 5, 641–648 (2019).
25. Bikchentaev A. M., "Metrics on projections of the von Neumann algebra associated with tracial functionals," Sib. Math. J., vol. 60, no. 6, 952–956 (2019).

A. M. Bikchentaev
Lobachevskii Institute of Mathematics and Mechanics
of Kazan (Volga Region) Federal University, Kazan, Russia
E-mail address: `Airat.Bikchentaev@kpfu.ru`