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Discontinuous Mixed Penalty-Free Galerkin Method for Second-Order Quasilinear Elliptic Equations

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Abstract—Discrete schemes for finding an approximate solution of the Dirichlet problem for a second-order quasilinear elliptic equation in conservative form are investigated. The schemes are based on the discontinuous Galerkin method (DG schemes) in a mixed formulation and do not involve internal penalty parameters. Error estimates typical of DG schemes with internal penalty are obtained. A new result in the analysis of the schemes is that they are proved to satisfy the Ladyzhenskaya–Babuska–Brezzi condition (inf-sup) condition.

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1. INTRODUCTION

A method for constructing upwind nonconformal finite element schemes (FEM) of arbitrary order of accuracy for solving linear nonstationary convection–diffusion problems was proposed in [1]. In [2] the method was extended to similar quasilinear equations. The stability of the schemes was analyzed in [1, 2], but no error estimates were obtained. In this paper, we fill the gap and analyze the accuracy of the schemes from [2] as applied to the stationary problem[1]

$$-\nabla \cdot k(x, u, \nabla u) + k_0(x, u, \nabla u) = f, \quad x \in \Omega, \quad u|_{\Gamma_D} = 0 \tag{1}$$

in a polyhedral domain $\Omega \subset R^d$, $d \geq 1$, with boundary Γ_D. The coefficients $k(x, \xi) = (k_1(x, \xi), k_2(x, \xi), \ldots, k_d(x, \xi))$, $k_0(x, \xi)$ are assumed to be continuous functions of $x \in \overline{\Omega}$ for any $\xi \in R^{d+1}$ and, for any $x \in \Omega$, satisfy the estimates

$$|k_i(x, \xi) - k_i(x, \eta)| \leq \beta|\xi - \eta| \quad \forall \xi, \eta \in R^{d+1}, \quad i = 0, \ldots, d, \quad \beta = \text{const} > 0, \tag{2}$$

$$\begin{gathered}(k(x, \sigma_0, \sigma) - k(x, \eta_0, \eta)) \cdot (\sigma - \eta) + (k_0(x, \sigma_0, \sigma) - k_0(x, \eta_0, \eta))(\sigma_0 - \eta_0) \geq \alpha|\sigma - \eta|^2 \\ \forall \sigma_0, \eta_0 \in R, \quad \sigma, \eta \in R^d, \quad \alpha = \text{const} > 0.\end{gathered} \tag{3}$$

(According to theory of monotone operators, these are sufficient conditions for the existence and uniqueness of a weak solution of the problem in the Sobolev space $H_0^1(\Omega)$.)

The scheme under study was obtained as a limit Galerkin–Petrov approximation based on a mixed formulation of Eq. (1) in the form of the system of first-order equations

$$-\nabla \cdot q + q_0 = f, \quad q = k(x, u, \sigma), \quad q_0 = k_0(x, u, \sigma), \quad \sigma = \nabla u. \tag{4}$$

Later, it was found that the scheme belongs to the broader class of discontinuous Galerkin schemes (DG or DGFEM schemes). For a linear elliptic problem, a similar scheme was somewhat earlier proposed and analyzed in [3]. DG methods (see [4]) occupy an intermediate position between the finite-volume method and FEM and combine many good properties of both methods, thus providing the practical foundation

[1] The estimates obtained below are easy to extend to a similar parabolic problem.