Estimations of classes of integral which constructed with the help of the classical warping function

Let G be a multiply connected plane domain. We denote by Γ_0 the outer boundary curve of G, and by $\Gamma_1, \ldots, \Gamma_n$ the internal boundary curves. The boundary-value problem that defines the warping function $u(x, G)$ of G is

$$\begin{cases} \triangle u = -2 & \text{in } G, \\ u = 0 & \text{on } \Gamma_0, \\ u = c_i & \text{on } \Gamma_i, \ i = 1, \ldots, n, \end{cases}$$

where the constants c_i are determined by the conditions

$$\oint_{\Gamma_i} \frac{\partial u}{\partial n} \, \mathrm{d}s = -2a_i, \ i = 1, \ldots, n,$$

$\partial/\partial n$ is the inward normal derivative, and a_i is the area enclosed by Γ_i.

In the next two assertions we give estimates for a class of integrals of the warping function.

Let a function $F(t)$ has the representation

$$F(t) := p \int\limits_0^t s^{p-1} f(s) \mathrm{d}s,$$

where $p > 0$, and $f(s)$ is another function, whose properties play an important role, as we see below.

Theorem 1. Let G be a multiply connected domain and let $p > 0$ such that $\mathbf{T}_p(G) < +\infty$. Then
1) If $f(s)$ is a non decreasing function, then

$$\int_G F(u(x, G)) \, \mathrm{dA} \leq \int_{R_p} F(u(x, R_p)) \, \mathrm{dA}.$$

2) if $f(s)$ is a non increasing function, then an inverse inequality holds

$$\int_G F(u(x, G)) \, \mathrm{dA} \geq \int_{R_p} F(u(x, R_p)) \, \mathrm{dA}.$$

Here R_p is a concentric ring with the same joint area of the holes as on G, and the ring R_p satisfy the equality $\mathbf{T}_p(R_p) = \mathbf{T}_p(G)$. Both equalities hold if and only if G is a ring bounded by two concentric circles.

Using the functionals $\mathbf{T}_p(G)$ and $\mathbf{u}(G)$ we can get explicit bounds for integrals of the warping function.

Theorem 2. Under the assumptions of Theorem 1 the following estimates hold:

$$\int_G F(u(x,G))\,\mathrm{dA} \leq \frac{\mathbf{T}_p(G)}{\mathbf{u}(G)^p} F(\mathbf{u}(G)) - \frac{2\pi \mathbf{u}(G) F(\mathbf{u}(G))}{p+1} + 2\pi \int_0^{\mathbf{u}(G)} F(t)\,\mathrm{d}t,$$

where $f(s)$ is a non decreasing function, and

$$\int_G F(u(x,G))\,\mathrm{dA} \geq \frac{\mathbf{T}_p(G)}{\mathbf{u}(G)^p} F(\mathbf{u}(G)) - \frac{2\pi \mathbf{u}(G) F(\mathbf{u}(G))}{p+1} + 2\pi \int_0^{\mathbf{u}(G)} F(t)\,\mathrm{d}t,$$

here $f(s)$ is a non increasing function.

Equalities hold iff G is a concentric ring.