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INVESTIGATION OF A NUMERICAL METHOD FOR SOLVING THE SPECTRAL PROBLEM OF THE THEORY OF DIELECTRIC WAVEGUIDES

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Investigation of spectral problems of the theory of dielectric waveguides and development of numerical methods attract large attention (see, e. g., [1]–[4]). In [5]–[7] numerical methods were suggested for searching eigenwaves of cylindric dielectric waveguides on the basis of the representation of their amplitudes as a superposition of potentials of simple and double layers. During the last years in solving a series of spectral problems of electrodynamics authors successfully apply representation of fields in the form of potentials of a simple layer (see, e. g., [8]–[10]). This enables to shorten essentially computer time.

The present article is devoted to the investigation of a numerical method for searching constants of propagation of surface eigenwaves of cylindric dielectric waveguides with a smooth contour of the cross section, which is based on representation of the desired functions in the form of potentials of simple layer. Under assumption of closeness of the refractive indices of the waveguide and the environment, the problem is reduced to a nonlinear spectral problem for a system of singular integral equations. On the basis of the known regularization procedure (see, e.g., [11], p. 14), we construct the system of the Galyorkin method. Zeros of the determinant of the matrix of this system are taken in the capacity of approximate solution of this problem. For investigation of the convergence of the method we use the results of [12]. A similar approach was applied to substantiation of the method for calculation of microband lines in [8], [9].

1. The problem of determination of the constants of propagation of eigen surface waves of a cylindric dielectric waveguide under assumption of closeness of the refractive indices of the waveguide and environment can be reduced (see [1], item 2) to the search of values of the parameter β, under which nontrivial solutions $u(x, y)$ of the boundary problem

$$\Delta u + \chi_1^2 u = 0, \quad (x,y) \in S, \tag{1}$$

$$\Delta u + \chi_2^2 u = 0, \quad (x,y) \notin \overline{S}, \tag{2}$$

$$u^+ = u^-, \quad \frac{\partial u^+}{\partial \nu} = \frac{\partial u^-}{\partial \nu}, \quad (x,y) \in C, \tag{3}$$

$$u \quad \text{exponentially decreases as} \quad r = \sqrt{x^2+y^2} \to \infty \tag{4}$$

exist. Here S is a bounded domain with the boundary C, $\chi_i^2 = k_0^2 n_i^2 - \beta^2$, $k_0^2 = \omega^2 \varepsilon_0 \mu_0$, ε_0 is the electric constant, μ_0 is the magnetic constant, ω is the frequency of electromagnetic oscillations, n_1, n_2 are values of the refractive indices of waveguide and environment, $\partial u/\partial \nu$ is the correct normal derivative, f^+ (f^-) is the limit value of the function f inside (outside) the contour C.

We shall seek solution of problem (1)–(4) in the class of functions continuous in $\overline{S}$ and $R^2 \setminus S$ and twice continuously differentiable in $R^2 \setminus C$. In what follows we shall assume that the contour C is a twice continuously differentiable curve, $k_0 > 0$, $n_1 > n_2$.

It is well known that in the simplest case where the contour C is a circumference, nontrivial solutions of the problem under consideration can exist only if $k_0 n_2 \le \beta \le k_0 n_1$. Moreover, the corresponding values of the constants of propagation β can be easily computed in this situation as the roots of the transcendent equation (see, e. g., [13]). In the present article we investigate the case of a contour C of arbitrary form.

As usual, we shall assume that $\operatorname{Re}\beta > 0$ (see, e. g., [14], p. 265). Consider multivalued functions $\chi_j(\beta) = \sqrt{k_0^2 n_j^2 - \beta^2}$ of the complex variable β. On the complex plane we make cuts which connect the bifurcation points $k_0 n_j$, $-k_0 n_j$ of the functions χ_j via the infinite point. Let us agree to choose in what follows univalent branches of the functions χ_j such that $\operatorname{Im}\chi_j > 0$ for $\beta > k_0 n_j$.

Analogous to Theorem 3.40 in [15] one can prove

Theorem 1. *For any $k_0 > 0$, $n_1 > n_2$, nontrivial solutions of problem* (1)–(4) *can exist only for real β which are on the segment $k_0 n_2 \le \beta \le k_0 n_1$.*

We shall seek solutions of equations (1) and (2) in the form of potential of simple layer (see [15]) with the continuous densities φ_1 and φ_2, respectively:

$$u(M) = \int_C \Phi_1(\beta; M, M_0)\varphi_1(M_0)\,dl_{M_0}, \quad M \in S, \tag{5}$$

$$u(M) = \int_C \Phi_2(\beta; M, M_0)\varphi_2(M_0)\,dl_{M_0}, \quad M \notin \overline{S}. \tag{6}$$

Here [1]

$$\Phi_j(\beta; M, M_0) = \frac{i}{4} H_0^{(1)}(\chi_j r_{MM_0}), \quad j = 1, 2, \quad M = (x, y), \quad M_0 = (x_0, y_0).$$

The function $u(x, y)$, given by (6), satisfies condition (4) for any $\beta \in G = (k_0 n_2,\ k_0 n_1)$. Using the boundary conditions (3) and limit properties of potentials of simple layer (see [15]), we obtain the problem: Find $\beta \in G$, for which the nontrivial continuous solutions φ_1, φ_2 of the system of integral equations

$$\int_C (\Phi_1(\beta; M, M_0)\varphi_1(M_0) - \Phi_2(\beta; M, M_0)\varphi_2(M_0))\,dl_{M_0} = 0, \tag{7}$$

$$\frac{1}{2}(\varphi_1(M) + \varphi_2(M)) + \int_C \left(\frac{\partial \Phi_1}{\partial \nu_M}(\beta; M, M_0)\varphi_1(M_0) - \frac{\partial \Phi_2}{\partial \nu_M}(\beta; M, M_0)\varphi_2(M_0)\right) dl_{M_0} = 0, \quad M \in C \tag{8}$$

exist.

Similar to Theorem 2 from [17] one can prove

Lemma 1. *If, for a certain $\beta \in G$, problem* (1)–(4) *can have only the trivial solution, then, for the same β, system* (7), (8) *can have only the trivial solution.*

We should note that in [17] investigation was carried out in the class of densities which are continuous by Hölder. In this situation, normal derivative on the contour is understood in the usual sense. Investigation in the class of continuous densities is to be carried out in a similar way if by the normal derivative on the contour C one means the correct normal derivative.

From Lemma 1 we immediately have

Theorem 2. *If, for a certain $\beta \in G$, system* (7), (8) *has a nontrivial solution, then a nontrivial solution of problem* (1)–(4) *corresponds to it.*

[1] For special functions we use the notation from [16].

2. Let the contour C be given parametrically $r = r(t)$, $t \in [0, 2\pi]$. Passing to the integration variable t, selecting explicitly the logarithmic singularity of the kernels $\Phi_1(M, M_0)$, $\Phi_2(M, M_0)$, we transform system (7), (8) to the form

$$Sx^{(1)} + R^{(1,1)}(\beta)x^{(1)} + R^{(1,2)}(\beta)x^{(2)} = 0, \quad t \in [0, 2\pi], \tag{9}$$

$$x^{(2)} + R^{(2,1)}(\beta)x^{(1)} + R^{(2,2)}(\beta)x^{(2)} = 0, \quad t \in [0, 2\pi]. \tag{10}$$

Here

$$Sx^{(1)} = -\frac{1}{2\pi}\int_0^{2\pi} \ln\left|\sin\frac{t-t_0}{2}\right| x^{(1)}(t_0)dt_0, \quad t \in [0, 2\pi],$$

$$R^{(i,j)}(\beta)x^{(j)} = \frac{1}{2\pi}\int_0^{2\pi} h^{(i,j)}(\beta; t, t_0)x^{(j)}(t_0)dt_0, \quad t \in [0, 2\pi],$$

$$x^{(1)}(t_0) = (\varphi_1(M_0) - \varphi_2(M_0))|r'(t_0)|, \quad x^{(2)}(t_0) = \varphi_1(M_0) + \varphi_2(M_0),$$

$$h^{(1,1)}(\beta; t, t_0) = 2\pi(G^{(1,1)}(\beta; t, t_0) + G^{(1,2)}(\beta; t, t_0)),$$

$$h^{(1,2)}(\beta; t, t_0) = 2\pi(G^{(1,1)}(\beta; t, t_0) - G^{(1,2)}(\beta; t, t_0))|r'(t_0)|,$$

$$h^{(2,1)}(\beta; t, t_0) = 4\pi(G^{(2,1)}(\beta; t, t_0) + G^{(2,2)}(\beta; t, t_0)),$$

$$h^{(2,2)}(\beta; t, t_0) = 4\pi(G^{(2,1)}(\beta; t, t_0) - G^{(2,2)}(\beta; t, t_0))|r'(t_0)|,$$

$$G^{(1,j)}(\beta; t, t_0) = \Phi_j(\beta; M, M_0) + \frac{1}{2\pi}\ln\left|\sin\frac{t-t_0}{2}\right|,$$

$$G^{(2,j)}(\beta; t, t_0) = \frac{\partial}{\partial \nu_M}\Phi_j(\beta; M, M_0), \quad M = M(t), \quad M_0 = M_0(t_0).$$

Using the known properties of the Hankel function (see, e.g., in [16]), one can easily verify that the following proposition is valid.

Lemma 2. *The functions $h^{(i,j)}(\beta; t, t_0)$, $i, j = 1, 2$, are analytic in the interval $G = (k_0 n_2, k_0 n_1)$ with respect to the real parameter β for each point $(t, t_0) \in [0, 2\pi] \times [0, 2\pi]$.*

Each of the functions $h^{(i,j)}(\beta; t, t_0)$, $i, j = 1, 2,$, considered first for real values of $\beta \in G$, for each point $(t, t_0) \in [0, 2\pi] \times [0, 2\pi]$ admits the unique analytic continuation with respect to the complex parameter β into the complex plane with cuts connecting the infinite point with the points $k_0 n_1$, $k_0 n_2$. Thus, for any interval $G' \subset G$ in the complex plane with cuts, a neighborhood Λ exists such that $k_0 n_2, k_0 n_1 \notin \Lambda$. Thus, the following lemma takes place.

Lemma 3. *The functions $h^{(i,j)}(\beta; t, t_0)$, $i, j = 1, 2$, are analytic in the domain Λ for each point $(t, t_0) \in [0, 2\pi] \times [0, 2\pi]$.*

By using the known properties of the Hankel functions, one can easily verify the following proposition.

Lemma 4. *The functions $h^{(i,j)}(\beta; t, t_0)$, $i, j = 1, 2$, with any $\beta \in \Lambda$ are continuous and continuously differentiable functions of $(t, t_0) \in [0, 2\pi] \times [0, 2\pi]$.*

In the construction and investigation of the numerical method, it is convenient to treat system (9)–(10) as an operator equation in a certain Hilbert space. It is known (see, e.g., [11], p. 10) that the operator $S : L_2 \to W_2^1$ is continuously invertible, the inverse operator $S^{-1} : W_2^1 \to L_2$ is determined via the formula

$$S^{-1}(y; t) = \frac{c_0(y)}{\ln 2} + 2\sum_{k=-\infty}^{\infty} |k| c_k(y) e^{ikt}, \quad y \in W_2^1, \tag{11}$$

where

$$c_k(y) = \frac{1}{2\pi}\int_0^{2\pi} y(t_0)e^{-ikt_0}dt_0$$

are the Fourier coefficients of the function y, and

$$\|S^{-1}\| = 2. \qquad (12)$$

Let us note that, for any $\beta \in \Lambda$, by virtue of Lemma 4 the operators $R^{(2,1)}(\beta)$, $R^{(2,2)}(\beta) : L_2 \to L_2$, $R^{(1,1)}(\beta)$, $R^{(1,2)}(\beta) : L_2 \to W_2^1$ are completely continuous. Thus, system (9), (10) is equivalent to the operator equation

$$A(\beta)y \equiv (I + B(\beta))y = 0, \qquad (13)$$

where

$$y = (y^{(1)},\ y^{(2)}),\quad y^{(1)} = Sx^{(1)} \in W_2^1,\quad y^{(2)} = x^{(2)} \in L_2,$$

the operator B which acts in the Hilbert space $H = W_2^1 \times L_2$, is defined via the equality

$$By = (R^{(1,1)}S^{-1}y^{(1)} + R^{(1,2)}y^{(2)},\ R^{(2,1)}S^{-1}y^{(1)} + R^{(2,2)}y^{(2)}), \qquad (14)$$

I is the unit operator.

We denote by $\rho(A) = \{\beta : \beta \in \Lambda,\ \exists A(\beta)^{-1} : H \to H\}$ a set of regular points of the operator $A(\beta)$, $\sigma(A) = \Lambda \setminus \rho(A)$ is a set of singular points of the operator $A(\beta)$. By virtue of the complete continuity of the operator $B(\beta)$, for any $\beta \in \Lambda$, the operator $A(\beta)$ is Fredholm, and, consequently, every singular point is a characteristic number of this operator. By using the known properties of integral operators with weakly singular kernels (see, e. g., [11]), one can easily show that to any eigenvector $y \in H$ of the operator A there corresponds a nontrivial solution φ_1, φ_2 of problem (7), (8) from the space of continuous functions.

The approximate solution $y_n = (y_n^{(1)},\ y_n^{(2)})$ of equation (13) will be sought in the form

$$y_n^{(j)}(t) = \sum_{k=-n}^{n} \alpha_k^{(j)} e^{ikt},\quad n \in N,\quad j = 1, 2.$$

We shall determine coefficients $\alpha_k^{(j)}$ by means of the Galyorkin method:

$$\int_0^{2\pi} (Ay_n)^{(k)}(t)e^{-ijt}dt = 0,\quad j = -n, \dots, n,\quad k = 1, 2. \qquad (15)$$

By virtue of (11) we have

$$S^{-1}(y_n^{(1)}; t) = \frac{\alpha_0^{(1)}}{\ln 2} + 2\sum_{k=-n}^{n} |k|\alpha_k^{(1)}e^{ikt}.$$

Therefore equalities (15) are equivalent to the system of linear algebraic equations

$$\alpha_j^{(1)} + \sum_{k=-n}^{n} h_{jk}^{(1,1)}(\beta)d_j\alpha_k^{(1)} + \sum_{k=-n}^{n} h_{jk}^{(1,2)}(\beta)\alpha_k^{(2)} = 0,\quad j = -n, \dots, n, \qquad (16)$$

$$\alpha_j^{(2)} + \sum_{k=-n}^{n} h_{jk}^{(2,1)}(\beta)d_j\alpha_k^{(1)} + \sum_{k=-n}^{n} h_{jk}^{(2,2)}(\beta)\alpha_k^{(2)} = 0,\quad j = -n, \dots, n. \qquad (17)$$

Here $d_j = \{1/\ln 2 \text{ for } j = 0,\ 2|j| \text{ for } j \neq 0\}$,

$$h_{jk}^{(l,m)}(\beta) = \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} h^{(l,m)}(\beta; t, t_0)e^{-ijt}e^{ikt_0}dt\, dt_0.$$

Let L_n be a set of all trigonometric polynomials of order not exceeding n. We denote by H_n a subspace of H composed by the elements $(y_n^{(1)},\ y_n^{(2)})$, $(y^{(1)},\ y^{(2)}) \in H_n^T$. We introduce into consideration the projection operator $p_n : H \to H_n$: $p_n y = (\Phi_n y^{(1)}, \Phi_n y^{(2)})$, $y = (y^{(1)}, y^{(2)}) \in H$, where

$$\Phi_n(\varphi; t) = \sum_{k=-n}^{n} c_k(\varphi) e^{ikt}$$

is the Fourier operator. Clearly,

$$\|p_n\| = 1. \tag{18}$$

The system of linear algebraic equations (16), (17) is equivalent to the linear operator equation

$$A_n(\beta) y_n \equiv p_n A(\beta) y_n \equiv (I + p_n B(\beta)) y_n \equiv (I + B_n(\beta)) y_n = 0.$$

Here $A_n : H_n \to H_n$, I is the unit operator in the space H_n.

We denote by $\rho(A_n) = \{\beta : \beta \in \Lambda, \exists A_n(\beta)^{-1} : H_n \to H_n\}$ a set of regular points of the operator $A_n(\beta)$, $\sigma(A_n) = \Lambda \setminus \rho(A_n)$ is a set of singular points of the operator $A_n(\beta)$, which coincides (by virtue of the operator being finite-dimensional) with the set of its characteristic numbers. Approximate values β_n of the propagation constants β will be sought as characteristic numbers of the operator $A_n(\beta)$, i. e., as zeros of the determinant of the matrix of system (16), (17).

3. As concerns the convergence of the method described in the previous item, the following theorem takes place.

Theorem 3. *The sets $\sigma(A)$ and $\sigma(A_n)$ consist of isolated points. Let $\beta_0 \in \sigma(A)$, then a sequence $\{\beta_n\}_{n \in N}$, $\beta_n \in \sigma(A_n)$, exists such that $\beta_n \to \beta_0$, $n \in N$. Let $\beta_n \in \sigma(A_n)$, $\beta_n \to \beta_0 \in \Lambda$, $n \in N' \subseteq N$. Then $\beta_0 \in \sigma(A)$.*

Here and in what follows we denote by N', N'', and N''' infinite subsets of the set of natural numbers N. By the convergence $z_n \to z$, $n \in N'$, we mean the convergence as $n \to \infty$, when the index n runs over the set N'.

The validity of Theorem 3 follows from Theorem 1 in [12] and the following lemmas, where, in fact, we verify the condition of this theorem.

Lemma 5. *The operator $p_n : H \to H_n$ have the properties:*

$$\|p_n y\|_H \to \|y\|_H, \quad n \in N \ \ \forall y \in H;$$
$$\|p_n(\alpha y + \alpha' y') - (\alpha p_n y + \alpha' p_n y')\|_H \to 0, \quad n \in N \ \ \forall y, y' \in H, \quad \alpha, \alpha' \in C.$$

The first of the properties holds in view of the evident limit relations $\|\Phi_n x\| \to \|x\|$, $n \in N$, $x \in L_2$, W_2^1. The second one is a consequence of the linearity of the operator p_n.

Lemma 6. *The operator functions $A(\beta)$, $A_n(\beta)$ are holomorphic in the domain Λ.*

Proof. The operator functions $R^{(2,1)}(\beta)$, $R^{(2,2)}(\beta)$, $R^{(1,1)}(\beta)$, and $R^{(1,2)}(\beta)$ are holomorphic in the domain Λ by virtue of Lemma 3 (see [9], p. 71). Hence and from (12)–(14) we conclude that $A(\beta)$ is a holomorphic operator function in the domain Λ. Consequently, the same property is possessed also by $A_n(\beta) = p_n A(\beta)$. □

Lemma 7. *The following estimate takes place $\|A(\beta)\| \le c(\beta)$, $\beta \in \Lambda$, where*

$$c(\beta) = 1 + 2(c_{11}^2(\beta) + d_{11}^2(\beta))^{\frac{1}{2}} + (c_{12}^2(\beta) + d_{12}^2(\beta))^{\frac{1}{2}} + 2c_{21}(\beta) + c_{22}(\beta)$$

is a function continuous in the domain Λ. Here

$$c_{ij}^2(\beta) = \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} |h^{(i,j)}(\beta; t, t_0)|^2 dt\, dt_0, \quad i, j = 1, 2,$$
$$d_{1j}^2(\beta) = \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} \left| \frac{d}{dt} h^{(1,j)}(\beta; t, t_0) \right|^2 dt\, dt_0, \quad j = 1, 2.$$

Proof. The validity of Lemma follows from the inequality

$$\|A(\beta)\| \le 1 + \|R^{(1,1)}(\beta)\|\,\|S^{-1}\| + \|R^{(1,2)}(\beta)\| + \|R^{(2,1)}(\beta)\|\,\|S^{-1}\| + \|R^{(2,2)}(\beta)\|,$$

equality (12), and obvious (by virtue of Lemma 4) estimates

$$\|R^{(2,j)}(\beta)\| \le c_{2,j}(\beta), \qquad R^{(2,j)}(\beta) : L_2 \to L_2,$$
$$\|R^{(1,j)}(\beta)\|^2 \le c_{1,j}^2(\beta) + d_{1,j}^2(\beta), \quad R^{(1,j)}(\beta) : L_2 \to W_2^1, \ j = 1, 2. \quad \square$$

From the definition of the operator $A_n(\beta)$ and equality (18) one can derive

Lemma 8. *The following estimate takes place:* $\|A_n(\beta)\| \le \|A(\beta)\|$, $n \in N$, $\beta \in \Lambda$.

Lemma 9. *For any* $\beta \in \Lambda$, *the sequence of operators* $\{A_n(\beta)\}_{n\in N}$ *properly converges to the operator* $A(\beta)$.

Proof. In correspondence with the definition from [12] for the proof of Lemma we have to show that with any $\beta \in \Lambda$ the following conditions are fulfilled:

1. the fact that the sequence $\{y_n\}_{n\in N}$, $y_n \in H_n$ P-converges to $y \in H$ implies that the sequence $\{A_n y_n\}_{n\in N}$ P-converges to Ay;
2. the uniform boundedness of $\{y_n\}_{n\in N}$, $\|y_n\| \le$ const, $n \in N$, and P-compactness of the sequence $\{A_n y_n\}_{n\in N}$ implies that the sequence $\{y_n\}_{n\in N}$ is P-compact.

The P-convergence of $\{y_n\}_{n\in N}$ to $y \in H$ means that $\|y_n - p_n y\| \to 0$, $n \in N$, and thus the validity of the first condition follows from the estimate $\|A_n y_n - p_n A y\| \le \|A_n\|\,\|y_n - p_n y\| + \|p_n\|\,\|A\|\,\|p_n y - y\|$, $n \in N$, Lemmas 7, 8, equality (18), and an evident limit relation $\|p_n y - y\| \to 0$, $n \in N$.

Let us verify the second of the conditions. The P-compactness of the sequence $\{A_n y_n\}_{n\in N}$ means that, for any $N' \subseteq N$, $N'' \subseteq N'$ exists such that the sequence $\{A_n y_n = y_n + B_n y_n\}_{n\in N''}$ P-converges to $z \in H$. If $\|y_n\| \le$ const, $n \in N''$, then a weakly convergent subsequence $\{y_n\}_{n\in N'''}$, $N''' \subset N''$ exists. The completely continuous operator B sends it to a strong convergent one: $\|By_n - u\| \to 0$, $n \in N'''$, $u \in H$. Hence by virtue of the inequality $\|B_n y_n - p_n u\| \le \|p_n\|\,\|By_n - u\|$ and equality (18) we conclude that the sequence $\{B_n y_n\}_{n\in N'''}$ P-converges to $u \in H$. Thus, $\{y_n\}_{n\in N'''}$ P-converges to $y = z - u \in H$, and the second condition is fulfilled. $\square$

Lemma 10. *The norms* $\|A_n(\beta)\|$ *are bounded uniformly with respect to* n *and* β *on each compact* $\Lambda_0 \subset \Lambda$.

The validity of this Lemma evidently follows from Lemmas 8 and 7.

Lemma 11. *For any* $k_0 > 0$, $n_1 > n_2$ *in the interval* $G = (k_0 n_2,\ k_0 n_1)$, β *exist with which problem* (1)–(4) *has only the trivial solution.*

Proof. If $\beta \in G$, then $\chi_1 > 0$, $\chi_2 = i\rho_2$, $\rho_2 > 0$. Let u be a solution of problem (1)–(4). Applying in the domains S, $S_R = \{(x,y) \notin \overline{S} : r < R\}$ the Green formula, by taking into account conditions (1)–(3), we arrive at the equality

$$\rho_2^2 \int_{S_R} |u|^2 ds + \int_{S_R} |\nabla u|^2 ds + \int_S |\nabla u|^2 ds = \int_{C_R} u \frac{\partial \overline{u}}{\partial r} dl + \chi_1^2 \int_S |u|^2 ds. \tag{19}$$

From (19), (4) it follows that R_0 exists such that for any $R \ge R_0$ the inequality takes place

$$\rho_2^2 \int_{S_R} |u|^2 ds + \int_{S_R} |\nabla u|^2 ds + \frac{1}{2} \int_S |\nabla u|^2 ds \le \chi_1^2 \int_S |u|^2 ds. \tag{20}$$

Let us apply the embedding theorems (see, e. g., [18]):

$$\int_S |u|^2 ds \le c_{S,1} \Big(\int_S |\nabla u|^2 ds + \int_C |u|^2 dl \Big), \tag{21}$$

$$\int_C |u|^2 dl \le c_{S,2} \Big(\int_{S_R} |\nabla u|^2 ds + \int_{S_R} |u|^2 ds \Big). \tag{22}$$

Here $c_{S,1}$, $c_{S,2}$ are constants which can depend only on the domain S. By substituting (21), (22) into (20), we get the inequality

$$(\rho_2^2 - \chi_1^2 c_{1,S} c_{2,S}) \int_{S_R} |u|^2 ds + \left(\frac{1}{2} - \chi_1^2 c_{1,S}\right) \int_S |\nabla u|^2 ds + (1 - \chi_1^2 c_{1,S} c_{2,S}) \int_{S_R} |\nabla u|^2 ds \le 0. \quad (23)$$

Let us require that in (23) all the factors at integrals be positive. This requirement will be fulfilled if

$$\beta > \max_{i=1,2,3} \beta_i. \quad (24)$$

Here

$$\beta_1^2 = \frac{k_0^2 n_2^2}{1 + c_S} + \frac{k_0^2 n_1^2 c_S}{1 + c_S}, \quad \beta_2^2 = k_0^2 n_1^2 - \frac{1}{2} \frac{1}{c_{1,S}},$$

$$\beta_3^2 = k_0^2 n_1^2 - \frac{1}{c_S}, \quad c_S = c_{1,S} c_{2,S}.$$

Clearly, for arbitrary $k_0 > 0$, $n_1 > n_2$, the set β from the interval G, for which (24) is fulfilled, is nonempty. To every such β only the trivial solution of problem (1)–(4) can correspond. Indeed, from (23) it follows

$$\int_{S_R} |u|^2 ds = 0.$$

Thus, $u(M) = 0$, $M \in S_R$. Consequently, $u(M) = 0$, $M \in R^2$. □

Lemma 12. *The set $\rho(A)$ is nonempty, i. e., $\sigma(A) \ne \Lambda$.*

Result of this Lemma follows directly from Fredholm property of the operator A and Lemmas 11 and 1.

Thus, all the conditions of theorem 1 from [12] in the case under our consideration are fulfilled and the assertions of Theorem 3 have been proved.

The method described above was applied to solving problems on eigenwaves of dielectric waveguides, in both simplified statement considered here and in the complete electrodynamic statement (see, e. g., [1]) which leads to nonlinear spectral problems, for systems of singular equations with the Cauchy kernel. Numerical experiments, partially described in [19], [20], show high efficiency of the method. For example, to determine basic waves' propagation constants of waveguides of various cross sectional shapes, it turned out to be sufficient to take at most three terms of the trigonometric series.

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