ISSN 1995-0802, Lobachevskii Journal of Mathematics, 2018, Vol. 39, No. 9, pp. 1370–1376. © Pleiades Publishing, Ltd., 2018.

Uniqueness of the Critical Point of the Conformal Radius: "Method of Déjà vu"

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(Submitted by A. M. Elizarov)

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Received June 21, 2018

Abstract—New conditions are constructed for the critical point of the conformal radius (hyperbolic derivative) to be unique where the mapping function is holomorphic and locally univalent in the unit disk. We use an approach based on the uniqueness research of the univalence conditions depending on the additional parameters. Such a research has been carried out for the univalence criteria due to Singhs, Szapiel and some other mathematicians.

DOI: 10.1134/S1995080218090408

Keywords and phrases: *Conformal radius, inner radius of domain, hyperbolic derivative, critical points of conformal radius, Gakhov equation.*

1. INTRODUCTION

Jack's lemma in its original statement [1] asserts the existence of the real number k with the properties $k \geq 1$ and

$$\zeta_1 w'(\zeta_1) = k w(\zeta_1), \tag{1}$$

where $w(\zeta)$ is the holomorphic function in the disk $\mathbb{D} = \{\zeta \in \mathbb{C} : |\zeta| < 1\}$ with $w(0) = 0$, and the modulus of this function, $|w(\zeta)|$, attains its maximum value on the circle $|\zeta| = r(< 1)$ at a point $\zeta = \zeta_1$. On the base of this assertion it has been established in the article [2] that the condition

$$|w(\zeta)|^{1-\gamma}|\zeta w'(\zeta)|^{\gamma} < 1, \quad \zeta \in \mathbb{D}, \tag{2}$$

with some $\gamma \geq 0$ for the just mentioned $w(\zeta)$ implies the inequality $|w(\zeta)| < 1$, $\zeta \in \mathbb{D}$.

When

$$w(\zeta) = f'(\zeta) - 1, \tag{3}$$

the condition (2) gives the first of the theorems in the paper [2]. Namely, we have the following

Theorem S. *If a holomorphic function $f(\zeta)$ in $\mathbb{D}$ with the normalization*

$$f(0) = f'(0) - 1 = 0 \tag{4}$$

satisfies the condition

$$|f'(\zeta) - 1|^{1-\gamma}|\zeta f''(\zeta)|^{\gamma} < 1, \quad \zeta \in \mathbb{D}, \tag{5}$$

for some $\gamma \geq 0$, then $f(\zeta)$ is close-to-convex and bounded in $\mathbb{D}$.

We are interested in the uniqueness criteria for the critical point of the hyperbolic derivative

$$h_f(\zeta) = (1 - |\zeta|^2)|f'(\zeta)| \tag{6}$$

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of the holomorphic function f in $\mathbb{D}$ (conformal radius of the domain $f(\mathbb{D})$). In particular, we study the question of whether the inequalities of the form (5) will be such conditions. It is well-known (see, e.g., [3]) that the critical points of the function (6) are exactly the roots of the Gakhov equation

$$f''(\zeta)/f'(\zeta) = 2\overline{\zeta}/(1 - |\zeta|^2). \tag{7}$$

The series of the uniqueness conditions is traditionally opened by the case when the only root of (7) is the zero. The presence of the latter is postulated by the condition

$$f''(0) = 0, \tag{8}$$

which is by (3) equivalent to the equality

$$w'(0) = 0. \tag{9}$$

However, not all of the functions w with (9) satisfy the inequality (2). This fact considerably impoverishes the corresponding classes of functions f. So, for example, the function $w(\zeta) = \zeta^n$ with $n \geq 2$ satisfies the inequality $|w(\zeta)| < 1$ all over the unit disk, but the inequality in (2)—only on its part. In order to correct the situation we have to make the above formulations more precise due to introduction of the power n to them. The results are as follows. Let H be the class of all functions holomorphic in $\mathbb{D}$, let A be its subclass consisting of all functions f with normalization (4), and let H_0 be the subclass of A which elements f are locally univalent functions in $\mathbb{D}$, i.e. $f'(\zeta) \neq 0$ when $\zeta \in \mathbb{D}$. For a subclass $X \subset H$ we denote $\tilde{X} = \{f \in X : f''(0) = 0\}$. Jack's lemma will take the following form.

Lemma 1. *Let $w(\zeta) = \alpha\zeta^n + \ldots \in H$ where $n \in \mathbb{N}$ and $\alpha \neq 0$. If the modulus $|w(\zeta)|$ attains its maximum value on the circle $|\zeta| = r$ at a point ζ_1, then the inequality (1) is fulfilled with the real number $k \geq n$.*

Prototype of the following statement hasn't been explicitly formulated in [2], but it has been contained in the proof of Theorem S.

Lemma 2. *Let $w(\zeta) = \alpha\zeta^n + \ldots$ be a holomorphic function in $\mathbb{D}$ satisfying the condition*

$$|w(\zeta)|^{1-\gamma}|\zeta w'(\zeta)|^{\gamma} < n^{\gamma}, \quad \zeta \in \mathbb{D},$$

where $n \in \mathbb{N}$, $\alpha \neq 0$, and $\gamma \geq 0$ are arbitrary fixed numbers. Then $|w(\zeta)| < 1$, $\zeta \in \mathbb{D}$.

Now we reorientate Theorem S from univalence to uniqueness with adding the condition (8).

Theorem 1. *If $n \geq 2$ and a holomorphic function $f(\zeta) = \zeta + \sum_{k=n+1}^{\infty} a_k\zeta^k$ in $\mathbb{D}$ satisfies the condition*

$$|f'(\zeta) - 1|^{1-\gamma}|\zeta f''(\zeta)|^{\gamma} < n^{\gamma}, \quad \zeta \in \mathbb{D}, \tag{10}$$

for some $\gamma \geq 0$, then $f \in H_0$, and $\zeta = 0$ is the only critical point of the function (6).

According to Lemma 2 the condition (10) implies the inequality

$$|f'(\zeta) - 1| < 1, \quad \zeta \in \mathbb{D}, \tag{11}$$

which is the uniqueness condition (see, e.g., [3]). By virtue of (11) close-to-convexity and boundedness of the function f are also preserved when we passing from (5) to (10).

The other aspect of the generalization of the Theorem S is connected with the extension of the inequality (11) to the subordination

$$f'(\zeta) \prec \frac{1 + \beta\zeta}{1 - \alpha\zeta}, \quad \zeta \in \mathbb{D}, \tag{12}$$

where (α, β) varies over the triangle $\Delta = \{(\alpha, \beta) \in \mathbb{R}^2 : \alpha \leq 1, \beta \leq 1, \alpha + \beta > 0\}$. Inequality (11) is equivalent to the condition (12) when $\alpha = 0$, $\beta = 1$. This aspect develops in Section 2 of the present note. In Section 3 we establish two-parametric extension of the Szapiel univalence condition (see [3]). In Section 4 the uniqueness problem is solved for the classes $\mathcal{T}(\alpha, \beta)$ which prototypes have been studied in [4].

2. LAMÉ CURVES AND QUASIHYPERBOLES

The following theorem has been proved in [3] by the use of the method that goes back to [5]. We will give here another, geometric proof. Let us denote $\Delta^* = \{(\alpha, \beta) \in \Delta : \alpha + \beta \le 1\}$ and $\delta_k = \{(\alpha, \beta) \in \mathbb{R}^2 : |\alpha^{k+1} - (-\beta)^{k+1}| \le 1\}$, $k \in \mathbb{N} \cup \{0\}$.

Theorem 2. *Let $n \ge 2$, and let a holomorphic in $\mathbb{D}$ function $f(\zeta) = \zeta + \sum_{k=n+1}^{\infty} a_k \zeta^k$ satisfies the condition (12). If $(\alpha, \beta) \in \Delta^*$, then the image $f(\mathbb{D})$ has the only zero critical point of the conformal radius except a strip, for which $(\alpha, \beta) = (1, 0)$.*

Proof. Subordination (12) is equivalent to the relation

$$f'(\zeta) = \frac{1 + \beta\varphi(\zeta)}{1 - \alpha\varphi(\zeta)}, \quad \zeta \in \mathbb{D}, \tag{13}$$

where $\varphi(\zeta) = c\zeta^n + \ldots$ satisfies the condition $\varphi(\mathbb{D}) \subset \mathbb{D}$. From (13) we obtain

$$\frac{f''(\zeta)}{f'(\zeta)} = \frac{\beta\varphi'}{1 + \beta\varphi} + \frac{\alpha\varphi'}{1 - \alpha\varphi} = \varphi' \sum_{k=0}^{\infty} \left[\alpha^{k+1} - (-\beta)^{k+1}\right] \varphi^k, \quad \varphi = \varphi(\zeta), \quad \zeta \in \mathbb{D},$$

whence by the Schwarz lemma (see, for example, [6, p. 333])

$$\left|\frac{f''(\zeta)}{f'(\zeta)}\right| \le \frac{2|\zeta|}{1 - |\zeta|^2}(1 - |\varphi|) \sum_{k=0}^{\infty} \left|\alpha^{k+1} - (-\beta)^{k+1}\right| |\varphi|^k, \quad \varphi = \varphi(\zeta), \quad \zeta \in \mathbb{D}. \tag{14}$$

From (14) it is seen that for $(\alpha, \beta) \in \bigcap_{k=0}^{\infty}(\delta_k \cap \Delta)$ we have

$$\left|\frac{f''(\zeta)}{f'(\zeta)}\right| \le \frac{2|\zeta|}{1 - |\zeta|^2}, \quad \zeta \in \mathbb{D}. \tag{15}$$

For even k each boundary $\partial\delta_k$ consists of the Lamé curve ([7, p. 179]) $\alpha^{k+1} + \beta^{k+1} = 1$ and its reflection $\alpha^{k+1} + \beta^{k+1} = -1$ with respect to the asymptote $\alpha + \beta = 0$. There is a chain of inclusions $\Delta^* = \delta_0 \cap \Delta \subset \delta_2 \cap \Delta \subset \delta_4 \cap \Delta \subset \ldots$. For odd k each boundary $\partial\delta_k$ consists of the quasihyperboles $\alpha^{k+1} - \beta^{k+1} = \pm 1$ with asymptotes $\alpha \pm \beta = 0$. At the same time $\Delta^* \subset \delta_{2m+1}$, $m = 0, 1, 2, \ldots$. Thus, $\bigcap_{k=0}^{\infty}(\delta_k \cap \Delta) = \Delta^*$.

Since for any $k \ge 1$ the relation $\partial\delta_k \cap \Delta^* = \{(1, 0), (0, 1)\}$ takes place, the assumption about the equality in (7) (so, in (15)) for $\zeta = \zeta_0 \in \mathbb{D} \setminus \{0\}$ leads to $n = 2$, and also to $(\alpha, \beta) = (1, 0)$ or $(\alpha, \beta) = (0, 1)$, whence by the Schwarz lemma we obtain

$$\varphi(\zeta) = \varepsilon\zeta^2, \quad |\varepsilon| = 1. \tag{16}$$

Solving the corresponding Gakhov equation (7) we conclude that initial assumption about the equality may be fulfilled only in the case $(\alpha, \beta) = (1, 0)$ and only when $f(\mathbb{D})$ is a strip.

Theorem 2 is proved.

Remark 1. When $n = 2$, the result of the Theorem 2 is sharp in the sense that if $(\alpha, \beta) \in \Delta \setminus \Delta^*$, then there exists a function $f \in \tilde{A}$ with the condition (12) such that the hyperbolic derivative (6) has more than one critical point. Such a sharpness is established by the direct solution of the equation (7) for the function f with the representation (13), (16) where $(\alpha, \beta) \in \Delta \setminus \Delta^*$.

Moreover, the just formulated addition about the sharpness remains valid if we remove the condition $n \ge 2$ in the statement of Theorem 2 given above, and if we demand only $a_2 = 0$ in the Taylor expansion of a function $f(\zeta)$ (i.e. the equality (8)). Such a modification of Theorem 2 wins in community, but loses the connection with [2].

It remains only to note that in the conclusion

$$|\varphi'(\zeta)| \le \frac{n|\zeta|^{n-1}}{1 - |\zeta|^{2n}}(1 - |\varphi(\zeta)|^2), \quad \zeta \in \mathbb{D},$$

of Theorem 5 in [6, p. 333], which we use under the name of the Schwarz lemma, the right hand side increases with decreasing n.

Now we are able to establish the following analogy of Theorem 1.

Theorem 3. *If $n \geq 2$ and a holomorphic function $f(\zeta) = \zeta + \sum_{k=n+1}^{\infty} a_k \zeta^k$ in $\mathbb{D}$ satisfies the inequality*

$$|f'(\zeta) - 1|^{1-\gamma} |\zeta f''(\zeta)|^{\gamma} < (n/(\alpha+\beta))^{\gamma} |\alpha f'(\zeta) + \beta|^{1+\gamma}, \quad \zeta \in \mathbb{D},$$

for some $(\alpha, \beta) \in \Delta^*$ and $\gamma \geq 0$, and the derivative $f'(\zeta)$ omit the value $-\beta/\alpha$ in $\mathbb{D}$, then the conformal radius of the domain $f(\mathbb{D})$ has the only zero critical point except a strip, for which $(\alpha, \beta) = (1, 0)$ and $n = 2$.

Proof. Let us define $\varphi(\zeta) = \frac{f'(\zeta)-1}{\alpha f'(\zeta)+\beta}$, where $\alpha f'(\zeta) + \beta \neq 0$, $\zeta \in \mathbb{D}$, by the theorem condition. As

$$\varphi'(\zeta) = \frac{(\alpha+\beta) f''(\zeta)}{(\alpha f'(\zeta) + \beta)^2},$$

we find ourselves in the conditions of Lemma 2. Application of the latter immediately leads to the condition (13), i.e. to the subordination (12), and the proof is completed by the use of Theorem 2.

Let us note that new uniqueness conditions may be similarly constructed from the other univalence conditions in [2].

3. FIRST SZAPIEL'S CRITERION

The first Szapiel condition (there are three of them, see [3]) is a subordination of the form $\ln f'(\zeta) \prec A\zeta$, $\zeta \in \mathbb{D}$; it was shown in [3] that if $A = 1$, then this subordination is the sharp condition for the uniqueness of the zero critical point of the function (6). Here we consider two-parametric extension of the first Szapiel condition, namely, the subordination

$$\ln f'(\zeta) \prec \frac{\beta\zeta}{1 - \alpha\zeta}, \quad \zeta \in \mathbb{D}, \tag{17}$$

where the admissible intervals of varying of the parameters α and β form the semi-strip $\Pi = \{(\alpha, \beta) \in \mathbb{R}^2 : |\alpha| < 1, \beta \geq 0\}$. We have the following

Lemma 3. *Let a function $f \in \tilde{A}$ satisfies the condition (17) for $(\alpha, \beta) \in \Pi$. Then the inequality*

$$\mathcal{F}(\zeta; f) \equiv (1 - |\zeta|^2) \left| \frac{1}{\zeta} \frac{f''(\zeta)}{f'(\zeta)} \right| \leq 2\eta(\alpha, \beta), \quad \zeta \in \mathbb{D},$$

takes place where $\eta(\alpha, \beta) = \beta$ if $|\alpha| \leq 1/2$ and $\eta(\alpha, \beta) = \beta / [4|\alpha|(1 - |\alpha|)]$ if $|\alpha| > 1/2$.

Proof. Let us pass from the subordination (17) to the equivalent relation

$$\ln f'(\zeta) = \frac{\beta\varphi(\zeta)}{1 - \alpha\varphi(\zeta)}, \quad \zeta \in \mathbb{D}, \tag{18}$$

where $|\varphi(\zeta)| \leq |\zeta|^2$. This implies an estimate for the pre-Schwarzian

$$\frac{f''(\zeta)}{f'(\zeta)} = \frac{\beta\varphi'(\zeta)}{(1 - \alpha\varphi(\zeta))^2},$$

namely, by the Schwarz lemma we obtain

$$\left| \frac{f''(\zeta)}{f'(\zeta)} \right| \leq \frac{\beta}{|1 - \alpha\varphi|^2} \frac{2|\zeta|}{1 - |\zeta|^4} (1 - |\varphi|^2) \leq H(\mathrm{Re}\varphi) \frac{2|\zeta|}{1 - |\zeta|^2}, \quad \varphi = \varphi(\zeta), \quad \zeta \in \mathbb{D}, \tag{19}$$

where $H(t) = \beta(1 - |t|)/(1 - \alpha t)^2$. Let us find a maximum $\eta(\alpha, \beta)$ of the function $H(t)$ on the segment $[-1, 1]$. We have $H'_{|t|}(t) = -\beta(1 - \alpha t)^{-3} g(t)$, where $g(t) = 1 - 2\alpha s + \alpha s|t|$ and $s = \mathrm{sgn} t$.

If $|\alpha| \leq 1/2$, then obviously $H'_{|t|}(t) \leq 0$, hence $\eta(\alpha, \beta) = H(0) = \beta$.

Now consider the case $1/2 < |\alpha| < 1$. If $s = -\mathrm{sgn}\alpha$, then $\max_{t \in [0, -\mathrm{sgn}\alpha]} H(t) = H(0) = \beta$, and if $s = \mathrm{sgn}\alpha$, then $\max_{t \in [0, \mathrm{sgn}\alpha]} H(t) = H((2|\alpha| - 1)/\alpha) = \beta / [4|\alpha|(1 - |\alpha|)] > \beta$. Therefore in this case $\eta(\alpha, \beta) = \beta / [4|\alpha|(1 - |\alpha|)]$, and Lemma 3 is proved.

We call the set $\Omega \subset \Pi$ *the uniqueness domain* for the condition (17) if for any $(\alpha, \beta) \in \Omega$ this condition ensures the uniqueness of the (zero) critical point of the hyperbolic derivative of the function $f \in \tilde{A}$ that satisfies (17).

We introduce the sets $\Delta_{-1} = \{(\alpha, \beta) : -1 < \alpha < -1/2, \beta \le 4|\alpha|(1 - |\alpha|)\}$, $\Delta_0 = [-1/2, -1/2] \times [0, 1]$ and $\Delta_1 = \{(\alpha, \beta) : 1/2 < \alpha < 1, \beta < 4|\alpha|(1 - |\alpha|)\}$. The following result is valid.

Theorem 4. *The set $\Omega = \Delta_{-1} \cup \Delta_0 \cup \Delta_1$ is the uniqueness domain for the condition (17).*

Proof. By Lemma 3 the uniqueness takes place for all (α, β) with $\eta(\alpha, \beta) < 1$, i.e. when (α, β) belongs to the interior of the domain Ω or to the interval $(-1, 1) \times \{0\}$.

It remains to consider the equation $\eta(\alpha, \beta) = 1$. When $\beta = 4|\alpha|(1 - |\alpha|)$, $-1 < \alpha < -1/2$, the assumption about the existence of a non-zero root a of the equation (7) for a function f with (17) by virtue of (19) leads to the contradiction: $0 < \varphi(a) = (2|\alpha| - 1)/\alpha < 0$. When $-1/2 \le \alpha \le 1/2$ and $\beta = 1$, the uniqueness of the zero root also follows from the chain (19) since $H(t) < H(0)$ for $t \ne 0$ (see the proof of Lemma 3). Finally, the curve $\beta = 4|\alpha|(1 - |\alpha|)$, $1/2 < \alpha < 1$, is excluded from the domain Δ_1. Theorem 4 is proved.

Remark 2. We should comment on the elimination of the part of the curve $\eta(\alpha, \beta) = 1$ over the domain Δ_1 from the statement of Theorem 4. This theme is connected with the sharpness problem for the result of Theorem 4. As already noted, the assumption on the existence of an additional root of the equation (7) according to (19) leads to a function of the form $\varphi(\zeta) = \varepsilon\zeta^2$, $|\varepsilon| = 1$. Then the condition (18) with this φ defines the function f for which Gakhov's equation has some roots in $\mathbb{D} \setminus \{0\}$. Moreover, the uniqueness cannot be continued across the curve $\eta(\alpha, \beta) = 1$ over $\alpha \in [-1/2, 1)$ out of $\Delta_0 \cup \overline{\Delta_1}$. This is not the case over Δ_{-1}. Complete picture of solvability of Gakhov's equation for the function $f \in \tilde{A}$ in (18) for $\varphi(\zeta) = \zeta^2$ shows that the equation (7) hasn't non-zero roots when $(\alpha, \beta) \in \Delta_{-1}^* = (-1, -1/2) \times [0, 1]$.

Hypothesis. *Maximal (by inclusion) uniqueness domain for the condition (17) will be the domain $\Omega^* = \Delta_{-1}^* \cup \Delta_0 \cup \Delta_1$.*

4. UNIQUENESS IN CLASSES T: CREATION OF THE FOLKLORE

Let $D = \{(a, b) \in \overline{\mathbb{R}}^2 : |a - 1| \le b \le a\}$. Following [4] for any $(a, b) \in D$ we consider the class $T(a, b)$ that consists of all functions $f \in A$ satisfying the condition

$$\sum_{n=2}^{\infty} (n - a + b)|a_n| \le b - |1 - a|. \tag{20}$$

Change of variables (see [5, 3])

$$a = \frac{1 + \alpha\beta}{1 - \alpha^2}, \quad b = \frac{\alpha + \beta}{1 - \alpha^2}, \tag{21}$$

transferring D into the set $\overline{\Delta}$, where $\Delta = \{(a, b) \in \mathbb{R}^2 : \alpha + \beta > 0, \alpha \le 1, \beta \le 1\}$, allows us to rewrite the inequality (20) as

$$\sum_{n=2}^{\infty} \frac{\beta + n\alpha + (n - 1)}{1 + \alpha}|a_n| \le \frac{\alpha + \beta}{1 + |\alpha|}. \tag{20$'$}$$

We will denote $\mathcal{T}(\alpha, \beta)$ the class of functions $f \in A$ satisfying (20′). Thus $T(a, b) = \mathcal{T}(\alpha, \beta)$ if and only if (a, b) and (α, β) are mapped by (21).

Mapping (21) is the orientation-preserving diffeomorphism on $D^0 = \{(a, b) \in \overline{\mathbb{R}}^2 : |a - 1| < b < a\}$. At the exit to the boundary ∂D the correspondence (21) keep its bijectivity only on the part of ∂D—on the ray $b = a$, $a \in (1/2, +\infty)$ that goes into the interval $\beta = 1$, $\alpha \in (-1, 1)$ of the boundary ∂D. Correspondence of the other parts of boundaries ∂D and $\partial\Delta$ is as follows. The point $a = b = \infty$ corresponds to the semi-segment $\alpha = 1$, $\beta \in (-1, 1]$, the ray $b = a - 1$, $a \in (1, +\infty)$—to the point $(\alpha, \beta) = (1, -1)$, the point $(a, b) = (1, 0)$—to the interval $\alpha + \beta = 0$, $\alpha \in (-1, 1)$, and the semi-segment $b = 1 - a$, $a \in [1/2, 1)$—to the point $(\alpha, \beta) = (-1, 1)$.

We exclude three latter parts of the boundary ∂D from our consideration when passing from $\overline{D}$ to Δ: for these parts the condition (20) admits the unique function, namely $f(\zeta) = \zeta$; the corresponding conformal radius will be have the only critical point (maximum-ombilic).

Three Propositions below are possibly well-known, but since Theorem 5 crowning them is new, we consider them to be folklore.

Proposition 1. *If $f \in \mathcal{T}(\alpha, \beta)$ for $(\alpha, \beta) \in \Delta$, then $|a_n| \le 1/n$, $n \in \mathbb{N}$.*

Proof. Let's pass to appropriate $T(a, b)$ by the help of (21). It follows from (20) that the inequality

$$|a_n| \le \frac{b - |1-a|}{n - a + b}, \quad n \ge 2, \tag{22}$$

is valid. The right-hand side of (22) increases with $b \le a$. Substituting in (22) the value $b = a$, we will obtain

$$|a_n| \le \frac{a - |1-a|}{n} = \frac{1}{n} - \frac{|a-1| - (a-1)}{n} \le \frac{1}{n}, \tag{23}$$

as required.

By the use of Haegi's classification [8] we get the following

Corollary. *If $f \in \tilde{\mathcal{T}}(\alpha, \beta)$ for $(\alpha, \beta) \in \Delta$, then $\zeta = 0$ is elliptic or parabolic critical point of the function (6).*

Remark 3. When $a_2 = 0$, the collection of inequalities $|a_n| \le 1/n$, $n \in \mathbb{N}$, isn't sufficient for the uniqueness of the (zero) critical point of the function (6). To confirm this conclusion, it is enough to consider the function $f(\zeta) = \zeta - \zeta^3/3 - \zeta^4/4 - \dots$ for which the equation (7) has four roots in $\mathbb{D}$; logarithmic singularity at $\zeta = 1$ plays the role of five critical point—maximum.

Proposition 1 admits the following complement.

Proposition 2. *Let $f \in \mathcal{T}(\alpha, \beta)$ for $(\alpha, \beta) \in \Delta$. Then*

1) *attainment of the equality in (22) at $n = k(\ge 2)$ implies the vanishing of all a_n, $n \ge 2$, except k-th;*

2) *equality $|a_n| = 1/n$ when $n = k \ge 2$ is possible only in the case $a_n = 0$, $n \ge 2$, $n \ne k$, and $b = a \ge 1$.*

Proof. 1) The desired conclusion follows from the chain

$$b - |1-a| = (k - a + b)|a_k| \le \sum_{n=2}^{\infty} (n - a + b)|a_n| \le b - |1-a|.$$

2) It follows from (22) and (23) that for $|a_n| = 1/n$ the relations $b = a$ and $a \ge 1$ take place. Under given restrictions on a and b the desired equality follows from

$$1 = k|a_k| \le \sum_{n=2}^{\infty} n|a_n| \le 1.$$

Proposition 3. *The following inclusions are valid:* 1) *$\mathcal{T}(\alpha, \beta) = \mathcal{T}(1, 1)$ when $\alpha \ge 0$, $\beta = 1$;*

2) *if $(\alpha, \beta) \in \Delta$, then $\mathcal{T}(\alpha, \beta) \subset \mathcal{T}(\alpha, 1) \subset \mathcal{T}(|\alpha|, 1) = \mathcal{T}(1, 1)$;*

3) *$\mathcal{T}(1, 1) \subset R(0, 1)$ where $R(\alpha, \beta)$ is the class of functions $f \in A$ satisfying (12).*

Proof. The result of 1) is obtained by direct substituting the considered values to the definition of the class $\mathcal{T}(\alpha, \beta)$, as well as the last equality in 2).

Defining relation (20') for $\mathcal{T}(\alpha, \beta)$ is written in the form $\sum_{n=2}^{\infty} F_n(\alpha, \beta)|a_n| \le 1$, where

$$F_n(\alpha, \beta) = \frac{1 + |\alpha|}{1 + \alpha} \cdot \frac{\beta + n\alpha + (n-1)}{\alpha + \beta}$$

is the decreasing function on β. The rest inclusions of 2) are consequences of the inequalities

$$F_n(\alpha, \beta) \ge F_n(\alpha, 1) = \frac{n(1 + |\alpha|)}{1 + \alpha} \ge F_n(|\alpha|, 1) = n = F_n(1, 1).$$

3) Defining relation for the class $\mathcal{T}(1,1)$ has the form $\sum_{n=2}^{\infty} n|a_n| \leq 1$ which immediately gives the inequality (11), but this is the condition (12) for $\alpha = 0$ and $\beta = 1$.

Now we are able to prove the following

Theorem 5. *Maximal uniqueness domain for the condition (20') when* $a_2 = 0$ *is the triangle* Δ.

Proof. Let $(\alpha, \beta) \in \Delta$, and let $f \in \tilde{\mathcal{T}}(\alpha, \beta)$. By virtue of Proposition 3, 3) the function f satisfies the inequality (11) which provides the uniqueness of the zero critical point of the function (6).

Theorem 5 is proved.

ACKNOWLEDGMENTS

This work was funded by the subsidy of the Russian Foundation for Basic Research and the Government of the Republic of Tatarstan, Grant 18-41-160017.

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