ISSN 0081-5438, Proceedings of the Steklov Institute of Mathematics, 2024, Vol. 324, pp. 44–52. © Pleiades Publishing, Ltd., 2024.
Russian Text © The Author(s), 2024, published in Trudy Matematicheskogo Instituta imeni V.A. Steklova, 2024, Vol. 324, pp. 51–59.

Continuity of Operator Functions in the Topology of Local Convergence in Measure

A. M. Bikchentaev[a] **and O. E. Tikhonov**[a]

Received June 20, 2023; revised October 5, 2023; accepted October 6, 2023

Dedicated to Academician A. S. Holevo on the occasion of his 80th birthday

Abstract—Let a von Neumann algebra $\mathcal{M}$ of operators act on a Hilbert space $\mathcal{H}$, and let τ be a faithful normal semifinite trace on $\mathcal{M}$. Let $t_{\tau \mathrm{l}}$ be the topology of τ-local convergence in measure on the *-algebra $S(\mathcal{M},\tau)$ of all τ-measurable operators. We prove the $t_{\tau \mathrm{l}}$-continuity of the involution on the set of all normal operators in $S(\mathcal{M},\tau)$, investigate the $t_{\tau \mathrm{l}}$-continuity of operator functions on $S(\mathcal{M},\tau)$, and show that the map $A \mapsto |A|$ is $t_{\tau \mathrm{l}}$-continuous on the set of all partial isometries in $\mathcal{M}$.

Keywords—Hilbert space, linear operator, von Neumann algebra, normal trace, measurable operator, local convergence in measure, continuity of operator functions.

MSC: 46L10, 46L51, 46L52 **DOI:** 10.1134/S008154382401005X

1. INTRODUCTION

Let a von Neumann algebra $\mathcal{M}$ of operators act on a Hilbert space $\mathcal{H}$, and let τ be a faithful normal semifinite trace on $\mathcal{M}$. This paper continues the research initiated in [1, 3–8, 10, 11, 15, 16] into the properties of the topologies $t_{\tau \mathrm{l}}$ and $t_{\mathrm{w}\tau \mathrm{l}}$ of τ-local and weakly τ-local convergence in measure, respectively, on the *-algebra $S(\mathcal{M},\tau)$ of all τ-measurable operators. We prove the $t_{\tau \mathrm{l}}$-continuity of the involution on the subset of all normal operators in $S(\mathcal{M},\tau)$ (Theorem 4.8), study the $t_{\tau \mathrm{l}}$-continuity of operator functions on $S(\mathcal{M},\tau)$ (Theorem 4.18) using some ideas and methods from [9, 13], and show that the map $A \mapsto |A|$ is $t_{\tau \mathrm{l}}$-continuous on the subset of all partial isometries in the algebra $\mathcal{M}$ (Corollary 4.3).

Note that the continuity of operator functions in the topology t_τ of convergence in measure on $S(\mathcal{M},\tau)$ was studied by the second author in [19], and on algebras of locally measurable operators, by M. A. Muratov and V. I. Chilin in [14]. Some of our results are new even for the *-algebra $\mathcal{M}=\mathcal{B}(\mathcal{H})$ of all bounded linear operators on $\mathcal{H}$ equipped with the canonical trace $\tau=\mathrm{tr}$.

2. NOTATION AND DEFINITIONS

Let $\mathcal{M}$ be a von Neumann algebra of operators in a Hilbert space $\mathcal{H}$, $\mathcal{M}^{\mathrm{pr}}$ the lattice of orthoprojections ($P=P^2=P^*$) in $\mathcal{M}$, I the identity in $\mathcal{M}$, $P^{\perp}=I-P$ for $P\in\mathcal{M}^{\mathrm{pr}}$, and $\mathcal{M}^+$ the cone of positive elements in $\mathcal{M}$.

A map $\varphi\colon \mathcal{M}^+\to[0,+\infty]$ is called a *trace* if $\varphi(X+Y)=\varphi(X)+\varphi(Y)$ and $\varphi(\lambda X)=\lambda\varphi(X)$ for all $X,Y\in\mathcal{M}^+$ and $\lambda\ge 0$ (with $0\cdot(+\infty)\equiv 0$) and in addition $\varphi(Z^*Z)=\varphi(ZZ^*)$ for all $Z\in\mathcal{M}$.

[a] N. I. Lobachevsky Institute of Mathematics and Mechanics, Kazan Federal University, Kremlevskaya ul. 35, Kazan, 420008 Russia.

✉ Airat.Bikchentaev@kpfu.ru (A. M. Bikchentaev), Oleg.Tikhonov@kpfu.ru (O. E. Tikhonov).

A trace φ is said to be

- *faithful* if $\varphi(X) > 0$ for all $X \in \mathcal{M}^+$, $X \neq 0$;
- *semifinite* if $\varphi(X) = \sup\{\varphi(Y)\colon Y \in \mathcal{M}^+,\ Y \le X,\ \varphi(Y) < +\infty\}$ for all $X \in \mathcal{M}^+$;
- *normal* if whenever $X_i \nearrow X$ ($X_i, X \in \mathcal{M}^+$), one has $\varphi(X) = \sup \varphi(X_i)$

(see [17, Ch. V, § 2]).

An operator on $\mathcal{H}$ (not necessarily bounded or densely defined) is said to be *affiliated with the von Neumann algebra* $\mathcal{M}$ if it commutes with every unitary operator in the commutant $\mathcal{M}'$ of $\mathcal{M}$. In what follows, τ is a faithful normal semifinite trace on $\mathcal{M}$ and $\mathcal{M}^{\mathrm{pr}}_\tau = \{P \in \mathcal{M}^{\mathrm{pr}}\colon \tau(P) < \infty\}$.

A closed densely defined operator X affiliated with $\mathcal{M}$, with domain $\mathcal{D}(X)$ in $\mathcal{H}$, is said to be *τ-measurable* if for every $\varepsilon > 0$ there exists a $P \in \mathcal{M}^{\mathrm{pr}}$ such that $P\mathcal{H} \subset \mathcal{D}(X)$ and $\tau(P^\perp) < \varepsilon$. The set $S(\mathcal{M}, \tau)$ of all τ-measurable operators is a *-algebra with respect to taking the adjoint operator, multiplication by scalars, and the operations of strong addition and multiplication obtained by closing the ordinary operations [18, Ch. IX]. Let $S(\mathcal{M}, \tau)^{\mathrm{nor}}$ be the set of all normal ($A^*A = AA^*$) operators in $S(\mathcal{M}, \tau)$. For a family $\mathcal{L} \subset S(\mathcal{M}, \tau)$ we denote by $\mathcal{L}^+$ and $\mathcal{L}^{\mathrm{h}}$ its positive and Hermitian parts, respectively. The partial order in $S(\mathcal{M}, \tau)^{\mathrm{h}}$ generated by the proper cone $S(\mathcal{M}, \tau)^+$ will be denoted by $\le$. If $X \in S(\mathcal{M}, \tau)$ and $X = U|X|$ is the polar decomposition of X, then $U \in \mathcal{M}$ and $|X| = \sqrt{X^*X} \in S(\mathcal{M}, \tau)^+$. For operators $A \in S(\mathcal{M}, \tau)$ we will also use the notation

$$\operatorname{Re} A = \frac{1}{2}(A + A^*) \qquad \text{and} \qquad \operatorname{Im} A = \frac{1}{2\mathrm{i}}(A - A^*).$$

The *-algebra $S(\mathcal{M}, \tau)$ is equipped with the topology t_τ of convergence in measure [18, Ch. IX, § 2], for which a fundamental system of neighborhoods of zero is formed by the sets

$$\mathcal{U}(\varepsilon, \delta) = \bigl\{X \in S(\mathcal{M}, \tau)\colon\ \exists Q \in \mathcal{M}^{\mathrm{pr}}\ (\|XQ\| \le \varepsilon\ \text{ and }\ \tau(Q^\perp) \le \delta)\bigr\}, \qquad \varepsilon > 0, \quad \delta > 0.$$

The algebra $\langle S(\mathcal{M}, \tau), t_\tau\rangle$ is known to be a complete metrizable topological *-algebra, and the algebra $\mathcal{M}$ is dense in $\langle S(\mathcal{M}, \tau), t_\tau\rangle$. To denote the convergence of a net $\{X_j\}_{j\in J} \subset S(\mathcal{M}, \tau)$ to an operator $X \in S(\mathcal{M}, \tau)$ in the topology t_τ, we write $X_j \xrightarrow{\tau} X$; in this case $\{X_j\}_{j\in J}$ is said to converge to X in measure τ.

Let $\mu(X; t)$ denote the *singular value function* of an operator $X \in S(\mathcal{M}, \tau)$, i.e., the nonincreasing right continuous function $\mu(X; \cdot)\colon (0, \infty) \to [0, \infty)$ defined as

$$\mu(X; t) = \inf\bigl\{\|XP\|\colon\ P \in \mathcal{M}^{\mathrm{pr}},\ \tau(P^\perp) \le t\bigr\}, \qquad t > 0.$$

The set $S_0(\mathcal{M}, \tau) = \{X \in S(\mathcal{M}, \tau)\colon\ \lim_{t\to\infty} \mu(X; t) = 0\}$ of τ-compact operators is an ideal in the *-algebra $S(\mathcal{M}, \tau)$. The topology t_τ is also generated by the F-norm $\rho_\tau(X) = \inf_{t>0} \max\{t, \mu(X; t)\}$ for $X \in S(\mathcal{M}, \tau)$.

Lemma 2.1 [12]**.** *Let $X, Y, X_j \in S(\mathcal{M}, \tau)$, $j \in J$. Then the following assertions hold*:

(i) $\mu(X; t) = \mu(|X|; t) = \mu(X^*; t)$ *for all* $t > 0$;

(ii) $\mu(X^*X; t) = \mu(XX^*; t)$ *for all* $t > 0$;

(iii) *if* $|X| \le |Y|$, *then* $\mu(X; t) \le \mu(Y; t)$ *for all* $t > 0$;

(iv) *if* $X \in \mathcal{M}$, *then* $\lim_{t\to+0} \mu(X; t) = \sup_{t>0} \mu(X; t) = \|X\|$;

(v) $\mu(XY; t + s) \le \mu(X; t)\mu(Y; s)$ *for all* $t, s > 0$;

(vi) $\mu(X + Y; t + s) \le \mu(X; t) + \mu(Y; s)$ *for all* $t, s > 0$;

(vii) $\mu(|X|^\alpha; t) = \mu(X; t)^\alpha$ *for all* $\alpha > 0$ *and* $t > 0$;

(viii) $X_j \xrightarrow{\tau} X$ *if and only if* $\mu(X_j - X; t) \to 0$ *for every* $t > 0$;

(ix) *if* $A, Z \in \mathcal{M}$, *then* $\mu(AYZ; t) \le \|A\|\mu(Y; t)\|Z\|$ *for all* $t > 0$.

3. TOPOLOGIES OF LOCAL CONVERGENCE IN MEASURE ON $S(\mathcal{M},\tau)$

The topology t_τ of convergence in measure can be localized as follows. For $\varepsilon,\delta>0$ and $P\in\mathcal{M}_\tau^{\mathrm{pr}}$ we define the sets

$$\mathcal{V}(\varepsilon,\delta,P)=\bigl\{X\in S(\mathcal{M},\tau)\colon \exists Q\in\mathcal{M}^{\mathrm{pr}}\ \bigl(Q\le P,\ \|XQ\|\le\varepsilon,\ \tau(P-Q)\le\delta\bigr)\bigr\},$$
$$\mathcal{W}(\varepsilon,\delta,P)=\bigl\{X\in S(\mathcal{M},\tau)\colon \exists Q\in\mathcal{M}^{\mathrm{pr}}\ \bigl(Q\le P,\ \|QXQ\|\le\varepsilon,\ \tau(P-Q)\le\delta\bigr)\bigr\}.$$

The space $S(\mathcal{M},\tau)$ becomes a topological vector space with respect to the topology $t_{\tau\mathrm{l}}$ of *τ-local convergence in measure*, with a basis of neighborhoods of zero given by the family $\Theta=\{\mathcal{V}(\varepsilon,\delta,P)\}_{\varepsilon,\delta>0,\ P\in\mathcal{M}_\tau^{\mathrm{pr}}}$, as well as with respect to the topology $t_{\mathrm{w}\tau\mathrm{l}}$ of *weak τ-local convergence in measure*, with a basis of neighborhoods of zero given by the family $\Theta=\{\mathcal{W}(\varepsilon,\delta,P)\}_{\varepsilon,\delta>0,\ P\in\mathcal{M}_\tau^{\mathrm{pr}}}$. We will write $X_i\xrightarrow{\tau\mathrm{l}}X$ and $X_i\xrightarrow{\mathrm{w}\tau\mathrm{l}}X$ to denote the $t_{\tau\mathrm{l}}$- and $t_{\mathrm{w}\tau\mathrm{l}}$-convergence, respectively. Using the standard technique of reducing von Neumann algebras, one can show (see also [11, 16]) that $X_i\xrightarrow{\tau\mathrm{l}}X$ if and only if $X_iP\xrightarrow{\tau}XP$ for all $P\in\mathcal{M}_\tau^{\mathrm{pr}}$ (cf. [8, p. 114]), and that $X_i\xrightarrow{\mathrm{w}\tau\mathrm{l}}X$ if and only if $PX_iP\xrightarrow{\tau}PXP$ for all $P\in\mathcal{M}_\tau^{\mathrm{pr}}$ (cf. [8, p. 114] and [10, p. 746]). It is clear that $t_{\mathrm{w}\tau\mathrm{l}}\le t_{\tau\mathrm{l}}\le t_\tau$ and the $t_{\mathrm{w}\tau\mathrm{l}}$-convergence coincides with the convergence in measure with respect to $\langle S(P\mathcal{M}P)=PS(\mathcal{M},\tau)P,t_{\tau_P}\rangle$ for all $P\in\mathcal{M}_\tau^{\mathrm{pr}}$, where $\tau_P(X)=\tau(PXP)$. The topologies $t_{\tau\mathrm{l}}$ and $t_{\mathrm{w}\tau\mathrm{l}}$ can also be defined in terms of nonincreasing rearrangements. The family $\widetilde{\Theta}=\{\widetilde{\mathcal{V}}(\varepsilon,\delta,P)\}_{\varepsilon,\delta>0,\ P\in\mathcal{M}_\tau^{\mathrm{pr}}}$ with $\widetilde{\mathcal{V}}(\varepsilon,\delta,P)=\{X\in S(\mathcal{M},\tau)\colon \mu(XP;\delta)<\varepsilon\}$ also defines a basis of neighborhoods of zero for $t_{\tau\mathrm{l}}$. If $\tau(I)<\infty$, then $t_\tau=t_{\tau\mathrm{l}}=t_{\mathrm{w}\tau\mathrm{l}}$; note that t_τ is the minimal metrizable topology consistent with the ring structure in $S(\mathcal{M},\tau)$ (see [2]).

If $\mathcal{M}$ is the *-algebra $\mathcal{B}(\mathcal{H})$ of all bounded linear operators on $\mathcal{H}$ and $\tau=\mathrm{tr}$ is the canonical trace, then $S(\mathcal{M},\tau)$ and $S_0(\mathcal{M},\tau)$ coincide with $\mathcal{B}(\mathcal{H})$ and with the ideal $\mathfrak{S}_\infty(\mathcal{H})$ of compact operators on $\mathcal{H}$, respectively. The topology t_τ coincides with the norm topology generated by the norm $\|\cdot\|$, and $t_{\tau\mathrm{l}}$ and $t_{\mathrm{w}\tau\mathrm{l}}$ coincide with the topologies of strong and weak operator convergence, respectively. We have $\mu(X;t)=\sum_{n=1}^{\infty}s_n(X)\chi_{[n-1,n)}(t)$, $t>0$, where $\{s_n(X)\}_{n=1}^{\infty}$ is the sequence of s-numbers of a completely continuous operator X and χ_A is the indicator of a set $A\subset\mathbb{R}$.

If $\mathcal{M}$ is abelian (i.e., commutative), then $\mathcal{M}\simeq L^\infty(\Omega,\Sigma,\nu)$ and $\tau(f)=\int_\Omega f\,\mathrm{d}\nu$, where (Ω,Σ,ν) is a localizable measure space, and the algebra $S(\mathcal{M},\tau)$ coincides with the algebra of all measurable complex functions f on (Ω,Σ,ν) that are bounded everywhere except for sets of finite measure. In this case, the topology t_τ is the ordinary topology of convergence in measure, and $t_{\tau\mathrm{l}}=t_{\mathrm{w}\tau\mathrm{l}}$ coincide with the well-known topology of convergence in measure on sets of finite measure.

4. ON THE CONTINUITY OF OPERATOR FUNCTIONS

Lemma 4.1 [1, Theorem 1, part 1]**.** *Let a net $\{A_\alpha\}\subset S(\mathcal{M},\tau)$ converge in the topology $t_{\tau\mathrm{l}}$ to an operator $A\in S(\mathcal{M},\tau)$. Then $A_\alpha B\xrightarrow{\tau\mathrm{l}}AB$ for any $B\in S(\mathcal{M},\tau)$.*

Using the definitions of $t_{\tau\mathrm{l}}$- and $t_{\mathrm{w}\tau\mathrm{l}}$-convergence and the t_τ-continuity of the involution and product in the algebra $S(\mathcal{M},\tau)$, we easily obtain the following.

Proposition 4.2. *If $A,A_\alpha\in S(\mathcal{M},\tau)$ and $A_\alpha\xrightarrow{\tau\mathrm{l}}A$, then $|A_\alpha|^2\xrightarrow{\mathrm{w}\tau\mathrm{l}}|A|^2$.*

Note that by Lemma 3.1(i) in [3] the map $A\mapsto|A|$ $(A\in S(\mathcal{M},\tau))$ is $t_{\tau\mathrm{l}}$-continuous at the point $A=0$.

Corollary 4.3. *If operators $A,A_\alpha\in\mathcal{M}$ are partial isometries and $A_\alpha\xrightarrow{\tau\mathrm{l}}A$, then $|A_\alpha|\xrightarrow{\tau\mathrm{l}}|A|$.*

Proof. Since $|A_\alpha|,|A|\in\mathcal{M}^{\mathrm{pr}}$, we have $|A_\alpha|=|A_\alpha|^2\xrightarrow{\mathrm{w}\tau\mathrm{l}}|A|^2=|A|$, i.e., $|A_\alpha|\xrightarrow{\mathrm{w}\tau\mathrm{l}}|A|$. Now by Lemma 3.7(i) in [3] we obtain $|A_\alpha|\xrightarrow{\tau\mathrm{l}}|A|$. $\square$

Corollary 4.4. *If $A_\alpha\in\mathcal{M}_r:=\{X\in\mathcal{M}\colon \|X\|\le r\}$, $r>0$, and $A_\alpha\xrightarrow{\mathrm{w}\tau\mathrm{l}}A\in S(\mathcal{M},\tau)$, then $A\in\mathcal{M}_r$.*

Proof. Suppose that $A \notin \mathcal{M}_r$, i.e., $r < \|A\| \le +\infty$. Since $\|A^*A\| = \|A\|^2$, we have $r^2 < \|A^*A\| \le +\infty$. There exists a spectral orthoprojection Q of the operator A^*A and a number $a > r^2$ such that

$$A^*A \ge aQ. \tag{4.1}$$

Since the trace τ is semifinite, there exists a nonzero orthoprojection $P \in \mathcal{M}_\tau^{\mathrm{pr}}$ such that $P \le Q$. Then from (4.1) we obtain $PA^*AP \ge aP$. Since $\|A_\alpha^* A_\alpha\| \le r^2$, we have $PA_\alpha^*A_\alpha P \le r^2 P$ and

$$PA^*AP - PA_\alpha^*A_\alpha P \ge (a - r^2)P.$$

Therefore, by Lemma 2.1(ii),

$$\mu\big(PA^*AP - PA_\alpha^*A_\alpha P; t\big) \ge (a - r^2)\mu(P;t) = (a - r^2)\chi_{(0,\tau(P)]}(t).$$

Consequently, from Lemma 2.1(viii) we have $PA_\alpha^*A_\alpha P \stackrel{\tau}{\not\to} PA^*AP$. Thus, we have arrived at a contradiction. □

Theorem 4.5. *Let $A, A_n \in S(\mathcal{M},\tau)$, $A_n \xrightarrow{\mathrm{w}\tau\mathrm{l}} A$ as $n \to \infty$, and the sequence $\{A_n\}$ be t_τ-bounded. Then $BA_n \xrightarrow{\tau\mathrm{l}} BA$ as $n \to \infty$ for every operator $B \in S_0(\mathcal{M},\tau)$ and $BA_nC \xrightarrow{\tau} BAC$ as $n \to \infty$ for every pair of operators $B, C \in S_0(\mathcal{M},\tau)$.*

Proof. Let $X, X_n \in S(\mathcal{M},\tau)$. Then

$$X_n \xrightarrow{\mathrm{w}\tau\mathrm{l}} X \quad \text{as} \ \ n \to \infty \qquad \Leftrightarrow \qquad PX_nQ \xrightarrow{\tau} PXQ \quad \text{as} \ \ n \to \infty \quad \forall P, Q \in \mathcal{M}_\tau^{\mathrm{pr}}$$

(see [1, p. 20]). Since $X_n \xrightarrow{\mathrm{w}\tau\mathrm{l}} X$ as $n \to \infty$ if and only if $X_n^* \xrightarrow{\mathrm{w}\tau\mathrm{l}} X^*$ as $n \to \infty$, we have $PA_n^* \xrightarrow{\tau\mathrm{l}} PA^*$ as $n \to \infty$ for every $P \in \mathcal{M}_\tau^{\mathrm{pr}}$. Now, by [1, Theorem 2] we get

$$PA_n^*B^* \xrightarrow{\tau} PA^*B^* \qquad \text{as} \ \ n \to \infty \qquad \forall P \in \mathcal{M}_\tau^{\mathrm{pr}}, \quad \forall B \in S_0(\mathcal{M},\tau) \tag{4.2}$$

(recall that $B^* \in S_0(\mathcal{M},\tau)$). Passing to the adjoint operators in (4.2) and taking into account the t_τ-continuity of the involution in $S(\mathcal{M},\tau)$, we have $BA_nP \xrightarrow{\tau} BAP$ as $n \to \infty$. Since the orthoprojection $P \in \mathcal{M}_\tau^{\mathrm{pr}}$ is arbitrary, we obtain $BA_n \xrightarrow{\tau\mathrm{l}} BA$ as $n \to \infty$. Applying [1, Theorem 2] once again, we conclude that $BA_nC \xrightarrow{\tau} BAC$ as $n \to \infty$ for every pair of operators $B, C \in S_0(\mathcal{M},\tau)$. □

Corollary 4.6. *Let $A, A_n \in \mathcal{B}(\mathcal{H})$, $A_n \to A$ as $n \to \infty$ in the weak operator topology, and the sequence $\{A_n\}$ be $\|\cdot\|$-bounded. Then $BA_n \to BA$ as $n \to \infty$ in the strong operator topology for every operator $B \in \mathfrak{S}_\infty(\mathcal{H})$, and $\|B(A_n - A)C\| \to 0$ as $n \to \infty$ for every pair of operators $B, C \in \mathfrak{S}_\infty(\mathcal{H})$.*

Example 4.7. The condition that the sequence $\{A_n\}$ is t_τ-bounded is essential in Theorem 4.5. In the abelian von Neumann algebra $\mathcal{M} \simeq L^\infty(\mathbb{R}^+, \mathrm{d}\nu)$ with linear Lebesgue measure ν, consider the faithful normal semifinite trace $\tau(f) = \int_{\mathbb{R}^+} f\,\mathrm{d}\nu$ and set

$$f_n = n\chi_{[n,2n]}, \qquad n \in \mathbb{N}.$$

Then $f_n \xrightarrow{\tau\mathrm{l}} 0$ as $n \to \infty$ and for τ-compact g and h, $g = h$, given by the function $\min\{1, x^{-1/2}\}$, $x \in \mathbb{R}^+$, we have $\mu(gf_nh;t) \ge \chi_{(0,n]}(t)/2 \not\to 0$ as $n \to \infty$ for every $t > 0$. Therefore, $gf_nh \stackrel{\tau}{\not\to} 0$ as $n \to \infty$ by Lemma 2.1(viii).

Theorem 4.8. *If $A, A_\alpha \in S(\mathcal{M},\tau)^{\mathrm{nor}}$ and $A_\alpha \xrightarrow{\tau\mathrm{l}} A$, then $A_\alpha^* \xrightarrow{\tau\mathrm{l}} A^*$.*

Proof. *Step* 1. We have $PA_\alpha^* \xrightarrow{\tau} PA^*$ for every $P \in \mathcal{M}_\tau^{\mathrm{pr}}$ by the t_τ-continuity of the involution in $S(\mathcal{M},\tau)$. Therefore, by the t_τ-continuity of the product in $S(\mathcal{M},\tau)$, we obtain

$$PA_\alpha \cdot A_\alpha^*P = PA_\alpha^* \cdot A_\alpha P \xrightarrow{\tau} PA^* \cdot AP = PA \cdot A^*P$$

for every $P \in \mathcal{M}_\tau^{\mathrm{pr}}$. Thus, $\mu(PA_\alpha A_\alpha^*P - PAA^*P;t) \to 0$ for every $t > 0$ by Lemma 2.1(viii).

Step 2. For every $P \in \mathcal{M}_\tau^{\mathrm{pr}}$ and $t > 0$ we estimate

$$\begin{aligned}
\mu(A_\alpha^* P - A^* P; t)^2 &= \mu(|A_\alpha^* P - A^* P|; t)^2 = \mu(|A_\alpha^* P - A^* P|^2; t) \\
&= \mu\big((PA_\alpha - PA)(A_\alpha^* P - A^* P); t\big) \\
&= \mu\big(PA_\alpha A_\alpha^* P + PAA^* P - PA_\alpha A^* P - PAA_\alpha^* P; t\big) \\
&= \mu\big(PA_\alpha^* A_\alpha P + PA^* AP - 2\,\mathrm{Re}(PA_\alpha A^* P); t\big) \\
&= \mu\big(PA_\alpha^* A_\alpha P + PA^* AP - 2\,\mathrm{Re}(PA^* AP) - 2\,\mathrm{Re}(PA_\alpha A^* P - PA^* AP); t\big) \\
&= \mu\big(PA_\alpha^* A_\alpha P - PA^* AP - 2\,\mathrm{Re}(PA_\alpha A^* P - PA^* AP); t\big) \\
&\le \mu\Big(PA_\alpha^* A_\alpha P - PA^* AP; \frac{t}{2}\Big) + 2\mu\Big(\mathrm{Re}(PA_\alpha A^* P - PA^* AP); \frac{t}{2}\Big)
\end{aligned} \tag{4.3}$$

by Lemma 2.1(vi), (vii). According to step 1 we have

$$\mu\Big(PA_\alpha^* A_\alpha P - PA^* AP; \frac{t}{2}\Big) \to 0$$

for every $P \in \mathcal{M}_\tau^{\mathrm{pr}}$ and $t > 0$. Let us estimate the second term in the last inequality in (4.3):

$$\begin{aligned}
2\mu\Big(\mathrm{Re}\big(PA_\alpha A^* P - PA^* AP\big); \frac{t}{2}\Big) &= 2\mu\Big(P\big(\mathrm{Re}(A_\alpha A^* - A^* A)P\big); \frac{t}{2}\Big) \\
&= \mu\Big(P(A_\alpha - A)A^* P + PA(A_\alpha^* - A^*)P; \frac{t}{2}\Big) \\
&\le \mu\Big(P(A_\alpha - A)A^* P; \frac{t}{4}\Big) + \mu\Big(PA(A_\alpha^* - A^*)P; \frac{t}{4}\Big) \\
&\le \|P\|\,\mu\Big((A_\alpha - A)A^* P; \frac{t}{4}\Big) + \|P\|\,\mu\Big(PA(A_\alpha^* - A^*); \frac{t}{4}\Big) \\
&= \mu\Big((A_\alpha - A)A^* P; \frac{t}{4}\Big) + \mu\Big((PA(A_\alpha^* - A^*))^*; \frac{t}{4}\Big) \\
&= 2\mu\Big((A_\alpha - A)A^* P; \frac{t}{4}\Big) \to 0
\end{aligned}$$

by Lemma 2.1(vi), (ix), (viii) and the t_τ-continuity of multiplication by the operator A^* on the left (see Lemma 4.1). Thus, $\mu(A_\alpha^* P - A^* P; t) \to 0$ for arbitrary $t > 0$ and $P \in \mathcal{M}_\tau^{\mathrm{pr}}$. This completes the proof of the theorem. □

For an operator $A \in S(\mathcal{M}, \tau)$, let $R_\lambda(A)$ denote its resolvent.

Lemma 4.9. *If a net $\{A_\alpha\}$ in the *-algebra $S(\mathcal{M}, \tau)^{\mathrm{h}}$ converges to A in the topology $t_{\tau\mathrm{l}}$, then $R_\lambda(A_\alpha) \xrightarrow{\tau\mathrm{l}} R_\lambda(A)$ for every $\lambda \in \mathbb{C} \setminus \mathbb{R}$.*

Proof. As is well known,

$$R_\lambda(A) - R_\lambda(A_\alpha) = R_\lambda(A_\alpha)(A_\alpha - A)R_\lambda(A).$$

Take a $Q \in \mathcal{M}_\tau^{\mathrm{pr}}$. Since $A_\alpha - A \xrightarrow{\tau\mathrm{l}} 0$, we have $(A_\alpha - A)R_\lambda(A) \xrightarrow{\tau\mathrm{l}} 0$ by Lemma 4.1. Therefore, $(A_\alpha - A)R_\lambda(A)Q \xrightarrow{\tau} 0$, i.e., $\mu((A_\alpha - A)R_\lambda(A)Q; s) \to 0$ for any $s > 0$. Now, using

$$\begin{aligned}
\mu\big(R_\lambda(A_\alpha)(A_\alpha - A)R_\lambda(A)Q; s\big) &\le \|R_\lambda(A_\alpha)\|\,\mu\big((A_\alpha - A)R_\lambda(A)Q; s\big) \\
&\le |\mathrm{Im}\,\lambda|^{-1}\mu\big((A_\alpha - A)R_\lambda(A)Q; s\big),
\end{aligned}$$

we obtain $\mu(R_\lambda(A_\alpha)(A_\alpha - A)R_\lambda(A)Q; s) \to 0$.

Thus, $(R_\lambda(A) - R_\lambda(A_\alpha))Q \xrightarrow{\tau} 0$ for every $Q \in \mathcal{M}_\tau^{\mathrm{pr}}$; i.e., $R_\lambda(A_\alpha) \xrightarrow{\tau\mathrm{l}} R_\lambda(A)$. □

Lemma 4.10. *Let f and g be two continuous functions from $\mathbb{R}$ (or $\mathbb{C}$) to $\mathbb{C}$, and let g be bounded. If the operator functions f and g are $t_{\tau 1}$-continuous on $S(\mathcal{M},\tau)^{\mathrm{h}}$ (on $S(\mathcal{M},\tau)^{\mathrm{nor}}$), then the operator function fg is also $t_{\tau 1}$-continuous on $S(\mathcal{M},\tau)^{\mathrm{h}}$ (on $S(\mathcal{M},\tau)^{\mathrm{nor}}$, respectively).*

Proof. Let $A_\alpha \xrightarrow{\tau 1} A$. We can write

$$(gf)(A)-(gf)(A_\alpha)=(g(A)-g(A_\alpha))f(A)+g(A_\alpha)(f(A)-f(A_\alpha)).$$

Lemma 4.1 implies that $(g(A)-g(A_\alpha))f(A)\xrightarrow{\tau 1} 0$. Using the estimate

$$\mu\big(g(A_\alpha)(f(A)-f(A_\alpha))Q;s\big)\le |g|\,\mu\big((f(A)-f(A_\alpha))Q;s\big)$$

for $Q\in\mathcal{M}_\tau^{\mathrm{pr}}$, we find that $g(A_\alpha)(f(A)-f(A_\alpha))\xrightarrow{\tau 1} 0$. Thus, the operator function gf is $t_{\tau 1}$-continuous. □

Lemma 4.11. *Let a sequence (f_n) of continuous functions acting from $\mathbb{R}$ (or $\mathbb{C}$) to $\mathbb{C}$ converge to a function f uniformly on $\mathbb{R}$ (on $\mathbb{C}$, respectively). If the operator functions f_n are $t_{\tau 1}$-continuous on $S(\mathcal{M},\tau)^{\mathrm{h}}$ (on $S(\mathcal{M},\tau)^{\mathrm{nor}}$), then the operator function f is also $t_{\tau 1}$ -continuous on $S(\mathcal{M},\tau)^{\mathrm{h}}$ (on $S(\mathcal{M},\tau)^{\mathrm{nor}}$, respectively).*

Proof. We will give a proof for functions on $\mathbb{C}$.

Take an $\varepsilon>0$ and choose n_0 such that $\sup_{x\in\mathbb{C}}|f(x)-f_{n_0}(x)|\le\varepsilon/3$. Let $A_\alpha\xrightarrow{\tau 1}A$ and $Q\in\mathcal{M}_\tau^{\mathrm{pr}}$. For $s>0$, by Lemma 2.1(iv)–(vi), (ix) we have

$$\begin{aligned}
&\mu\big((f(A)-f(A_\alpha))Q;s\big)\\
&\quad=\mu\Big(\big(f(A)-f_{n_0}(A)\big)Q+\big(f_{n_0}(A)-f_{n_0}(A_\alpha)\big)Q+\big(f_{n_0}(A_\alpha)-f(A_\alpha)\big)Q;s\Big)\\
&\quad\le\mu\Big((f-f_{n_0})(A)Q;\frac{s}{3}\Big)+\mu\Big(\big(f_{n_0}(A)-f_{n_0}(A_\alpha)\big)Q;\frac{s}{3}\Big)+\mu\Big((f-f_{n_0})(A_\alpha)Q;\frac{s}{3}\Big)\\
&\quad\le\|(f-f_{n_0})(A)Q\|+\mu\Big(\big(f_{n_0}(A)-f_{n_0}(A_\alpha)\big)Q;\frac{s}{3}\Big)+\|(f-f_{n_0})(A_\alpha)Q\|\\
&\quad\le\frac{2\varepsilon}{3}+\mu\Big(\big(f_{n_0}(A)-f_{n_0}(A_\alpha)\big)Q;\frac{s}{3}\Big).
\end{aligned}$$

Since the operator function f_{n_0} is $t_{\tau 1}$-continuous, the second term in the last expression does not exceed $\varepsilon/3$ for sufficiently large values of α. □

Proposition 4.12. *Let $f\colon\mathbb{R}\to\mathbb{R}$ be a continuous function on $\mathbb{R}$ with $f(x)=O(x)$ as $x\to\infty$. Then the operator function f is $t_{\tau 1}$-continuous on $S(\mathcal{M},\tau)^{\mathrm{h}}$.*

Proof. Let us first consider the case where $f(x)\to 0$ as $x\to\infty$. If $p(x)$ and $q(x)$ are real polynomials such that the degree of $p(x)$ is less than the degree of $q(x)$ and $q(x)$ has no real roots, then the rational function $r(x)=p(x)/q(x)$ can be represented as a finite linear combination of functions of the form $(x-\lambda)^{-n}$ ($\lambda\in\mathbb{C}\setminus\mathbb{R}$, $n\in\mathbb{N}$). By Lemmas 4.9 and 4.10, we conclude that the operator function r is $t_{\tau 1}$-continuous. By Stone's theorem, $f(x)$ can be uniformly approximated on $\mathbb{R}$ by a sequence of rational functions $r(x)$ of the form considered above, and then Lemma 4.11 implies the $t_{\tau 1}$-continuity of the operator function f.

Now we turn to the general case. Let us represent f in the form

$$f(x)=f(x)\frac{1}{1+x^2}+f(x)\frac{x^2}{1+x^2}.$$

Denote the first term by $f_1(x)$ and the second by $f_2(x)$. Then $f_1(x)\to 0$ as $x\to\infty$; therefore, the operator function f_1 is $t_{\tau 1}$-continuous on $S(\mathcal{M},\tau)^{\mathrm{h}}$. Successive application of Lemma 4.10 to the functions $xf_1(x)$ and $x(xf_1(x))$ shows that the operator function f_2 is $t_{\tau 1}$-continuous on $S(\mathcal{M},\tau)^{\mathrm{h}}$. □

Lemma 4.13. *The maps $A\mapsto\operatorname{Re}A$ and $A\mapsto\operatorname{Im}A$ are $t_{\tau 1}$-continuous on $S(\mathcal{M},\tau)^{\mathrm{nor}}$.*

Proof. This follows from Theorem 4.8. □

Lemma 4.14. *The map $A \mapsto I + |\mathrm{Re}\, A| + |\mathrm{Im}\, A|$ is $t_{\tau l}$-continuous on $S(\mathcal{M},\tau)^{\mathrm{nor}}$.*

Proof. Apply Lemma 4.13 and Proposition 4.12. □

Proposition 4.15. *Let a continuous function $f\colon \mathbb{C} \to \mathbb{C}$ be $O(|z|)$ as $z \to \infty$. Then the corresponding operator function f is $t_{\tau l}$-continuous on $S(\mathcal{M},\tau)^{\mathrm{nor}}$.*

Proof. Let $\mathcal{A}$ be the algebra of $\mathbb{R}$-valued functions on $\mathbb{C}$ generated by functions of the form $f(\mathrm{Re}\, z)$ and $g(\mathrm{Im}\, z)$ for continuous $\mathbb{R}$-valued functions $f(x)$ and $g(x)$ on $\mathbb{R}$ that tend to zero as $x \to \infty$. By Stone's theorem, the algebra $\mathcal{A}$ is uniformly dense in the algebra $\overline{\mathcal{A}}$ of all continuous functions from $\mathbb{C}$ to $\mathbb{R}$ that tend to zero at infinity. From Lemma 4.13, Proposition 4.12, and Lemma 4.10, it follows that the operator functions corresponding to the functions in $\mathcal{A}$ are $t_{\tau l}$-continuous, and by Lemma 4.11 the same holds for the functions in $\overline{\mathcal{A}}$.

Now let a continuous function $f\colon \mathbb{C} \to \mathbb{R}$ be $O(|z|)$ as $z \to \infty$. We can write

$$f(z) = (1 + |\mathrm{Re}\, z| + |\mathrm{Im}\, z|)^2 \frac{f(z)}{(1 + |\mathrm{Re}\, z| + |\mathrm{Im}\, z|)^2}$$

and apply Lemmas 4.14 and 4.10 twice.

Finally, let a continuous function $f\colon \mathbb{C} \to \mathbb{C}$ be $O(|z|)$ as $z \to \infty$. Then the functions $\mathrm{Re}\, f(z)$ and $\mathrm{Im}\, f(z)$ act continuously from $\mathbb{C}$ to $\mathbb{R}$ and are $O(|z|)$ as $z \to \infty$. The corresponding operator functions $\mathrm{Re}\, f(A)$ and $\mathrm{Im}\, f(A)$ are $t_{\tau l}$-continuous on $S(\mathcal{M},\tau)^{\mathrm{nor}}$, and hence the operator function $f(A) = \mathrm{Re}\, f(A) + \mathrm{i}\ \mathrm{Im}\, f(A)$ is $t_{\tau l}$-continuous on $S(\mathcal{M},\tau)^{\mathrm{nor}}$. □

Corollary 4.16. *The map $A \mapsto |A|$ is $t_{\tau l}$-continuous on $S(\mathcal{M},\tau)^{\mathrm{nor}}$.*

The following result is a corollary to [3, Lemma 3.4].

Lemma 4.17. *Let $\{A_\alpha\}$ and $\{B_\alpha\}$ be two nets in $\mathcal{M}^+$ such that $\{A_\alpha\}$ is uniformly bounded, $A_\alpha \xrightarrow{\tau l} 0$, and $B_\alpha \le A_\alpha$ for all α. Then $B_\alpha \xrightarrow{\tau l} 0$.*

Next, for $\Omega \subset \mathbb{C}$ we set $S(\mathcal{M},\tau)^{\mathrm{nor}}_\Omega = \{A \in S(\mathcal{M},\tau)^{\mathrm{nor}}\colon \mathrm{Sp}(A) \subset \Omega\}$, where $\mathrm{Sp}(A)$ is the spectrum of the operator A.

Theorem 4.18. *Let $\Omega \subset \mathbb{C}$ and $A \in S(\mathcal{M},\tau)^{\mathrm{nor}}_\Omega$. Let $f\colon \Omega \to \mathbb{C}$ be a function such that its restriction to any bounded subset of Ω that is closed in $\mathbb{C}$ is Borel measurable,* $\sup\{|f(z)|/(1+|z|)\colon z \in \Omega\} < \infty$, *and f is continuous at every point in* $\mathrm{Sp}(A)$. *If a net $\{A_\alpha\}$ of operators in $S(\mathcal{M},\tau)^{\mathrm{nor}}_\Omega$ converges to A in the topology $t_{\tau l}$, then $f(A_\alpha) \xrightarrow{\tau l} f(A)$.*

Proof. It suffices to prove the theorem for a real-valued function f, and we will do this in two stages.

1. Suppose that f is bounded on Ω, say $|f| \le 1$, and is continuous at all points in $\mathrm{Sp}(A)$. By the Tietze–Urysohn theorem, there exists a continuous function $g\colon \mathbb{C} \to [-1,1]$ that coincides with f on $\mathrm{Sp}(A)$. By Proposition 4.15,

$$g(A_\alpha) \xrightarrow{\tau l} g(A) = f(A).$$

According to [20, Lemma 2], there exists a bounded continuous function h on Ω such that $h = 0$ on $\mathrm{Sp}(A)$ and $|f - g| \le h$ on Ω.

1a. Now assume additionally that $\Omega = \mathbb{C}$. Then $h(A_\alpha) \xrightarrow{\tau} h(A) = 0$ by Proposition 4.15. Since $0 \le (f - g + h)(A_\alpha) \le 2h(A_\alpha)$, we have $(f - g + h)(A_\alpha) \xrightarrow{\tau l} 0$ by Lemma 4.17. Hence,

$$f(A_\alpha) = (f - g + h)(A_\alpha) + g(A_\alpha) - h(A_\alpha) \xrightarrow{\tau l} f(A).$$

1b. If $\Omega \neq \mathbb{C}$, then according to [20, Lemma 1] we construct a bounded function $k\colon \mathbb{C} \to \mathbb{R}$ that extends h, is continuous at all points in Ω, and is upper semicontinuous on $\mathbb{C}$ (so it is Borel measurable). Using case 1a, we obtain

$$h(A_\alpha) = k(A_\alpha) \xrightarrow{\tau l} k(A) = h(A) = 0,$$

and it remains to repeat verbatim the last two sentences of the previous paragraph.

2. In the general case, we set $g(z) = f(z)/(1+|z|)$. Then $g(A_\alpha) \xrightarrow{\tau l} g(A)$ by the above and $I + |A_\alpha| \xrightarrow{\tau l} I + |A|$ by Proposition 4.15. Hence by [1, Theorem 3] we obtain

$$f(A_\alpha) = g(A_\alpha)(I + |A_\alpha|) \xrightarrow{\tau l} g(A)(I + |A|) = f(A). \qquad \square$$

Corollary 4.19. *Let $\Omega \subset \mathbb{C}$ and $f\colon \Omega \to \mathbb{C}$ be a continuous function on Ω such that $\sup\{|f(z)|/(1+|z|)\colon z \in \Omega\} < \infty$. Then the corresponding operator function is continuous on $S(\mathcal{M},\tau)^{\mathrm{nor}}_{\Omega}$ in the topology $t_{\tau l}$.*

FUNDING

This work was supported within the Strategic Academic Leadership Program of the Kazan Federal University "Priority-2030."

CONFLICT OF INTEREST

The authors of this work declare that they have no conflicts of interest.

REFERENCES

1. A. M. Bikchentaev, "The continuity of multiplication for two topologies associated with a semifinite trace on von Neumann algebra," Lobachevskii J. Math. **14**, 17–24 (2004).
2. A. M. Bikchentaev, "Minimality of convergence in measure topologies on finite von Neumann algebras," Math. Notes **75** (3), 315–321 (2004) [transl. from Mat. Zametki **75** (3), 342–349 (2004)].
3. A. M. Bikchentaev, "Local convergence in measure on semifinite von Neumann algebras," Proc. Steklov Inst. Math. **255**, 35–48 (2006) [transl. from Tr. Mat. Inst. Steklova **255**, 41–54 (2006)].
4. A. M. Bikchentaev, "Local convergence in measure on semifinite von Neumann algebras. II," Math. Notes **82** (5–6), 703–707 (2007) [transl. from Mat. Zametki **82** (5), 783–786 (2007)].
5. A. Bikchentaev and F. Sukochev, "When weak and local measure convergence implies norm convergence," J. Math. Anal. Appl. **473** (2), 1414–1431 (2019).
6. A. M. Bikchentaev, "Convergence in measure and τ-compactness of τ-measurable operators, affiliated with a semifinite von Neumann algebra," Russ. Math. **64** (5), 79–82 (2020) [transl. from Izv. Vyssh. Uchebn. Zaved., Mat., No. 5, 89–93 (2020)].
7. A. M. Bikchentaev, "The topologies of local convergence in measure on the algebras of measurable operators," Sib. Math. J. **64** (1), 13–21 (2023) [transl. from Sib. Mat. Zh. **64** (1), 17–27 (2023)].
8. L. J. Ciach, "Some remarks on the convergence in measure and on a dominated sequence of operators measurable with respect to a semifinite von Neumann algebra," Colloq. Math. **55** (1), 109–121 (1988).
9. E. B. Davies, "A generalisation of Kaplansky's theorem," J. London Math. Soc., Ser. 2, **4** (3), 435–436 (1972).
10. P. G. Dodds, T. K.-Y. Dodds, and B. de Pagter, "Noncommutative Köthe duality," Trans. Am. Math. Soc. **339** (2), 717–750 (1993).
11. P. G. Dodds, T. K. Dodds, F. A. Sukochev, and O. Ye. Tikhonov, "A non-commutative Yosida–Hewitt theorem and convex sets of measurable operators closed locally in measure," Positivity **9** (3), 457–484 (2005).
12. T. Fack and H. Kosaki, "Generalized s-numbers of τ-measurable operators," Pac. J. Math. **123** (2), 269–300 (1986).
13. R. V. Kadison, "Strong continuity of operator functions," Pac. J. Math. **26** (1), 121–129 (1968).
14. M. A. Muratov and V. I. Chilin, "Topological algebras of measurable and locally measurable operators," J. Math. Sci. **239** (5), 654–705 (2019) [transl. from Sovrem. Mat., Fundam. Napr. **61**, 115–163 (2016)].
15. G. Sh. Skvortsova, "Weak sequential completeness of factor spaces of the space of integrable operators," Russ. Math. **46** (9), 68–71 (2002) [transl. from Izv. Vyssh. Uchebn. Zaved., Mat., No. 9, 71–74 (2002)]. Math-Net
16. G. Sh. Skvortsova and O. E. Tikhonov, "Convex sets in noncommutative L_1-spaces, closed in the topology of local convergence in measure," Russ. Math. **42** (8), 46–52 (1998) [transl. from Izv. Vyssh. Uchebn. Zaved., Mat., No. 8, 48–55 (1998)]. Math-Net

17. M. Takesaki, *Theory of Operator Algebras. I* (Springer, Berlin, 2002), Encycl. Math. Sci. **124**: Operator Algebras and Non-commutative Geometry **5**.
18. M. Takesaki, *Theory of Operator Algebras. II* (Springer, Berlin, 2003), Encycl. Math. Sci. **125**: Operator Algebras and Non-commutative Geometry **6**.
19. O. E. Tikhonov, "Continuity of operator functions in topologies connected to a trace on a von Neumann algebra," Sov. Math. **31** (1), 110–114 (1987) [transl. from Izv. Vyssh. Uchebn. Zaved., Mat., No. 1, 77–79 (1987)]. Math-Net
20. O. E. Tikhonov, "On convergence of functions of normal operators in strong operator topology," Funct. Anal. Appl. **41** (3), 245–246 (2007) [transl. from Funkts. Anal. Prilozh. **41** (3), 93–95 (2007)].

Translated by K. Shubik

Publisher's note. Pleiades Publishing remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.