RIEMANN–SCHWARZ REFLECTION PRINCIPLE AND ASYMPTOTICS OF MODULES OF RECTANGULAR FRAMES

S.R. Nasyrov

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Abstract

We investigate asymptotical behavior of the conformal module of a doubly-connected domain which is the difference of two homothetic rectangles under stretching it along the abscissa axis. Thereby, we give the answer to a question put by Prof. M. Vuorinen.

1 Introduction

In recent years, investigation of conformal modules of quadrilaterals, ring domains and capacities of condensers with polygonal boundaries has attracted increasing interest. The conformal modules play an important role in investigations of various problems of mechanics of continuum, electrostatics, tomography, heat conduction, image processing, etc. (see, e. g., [24, 27, 35, 47, 51, 52]). They provide a powerful tool in theory of quasiconformal mappigs (see [1,4,39]), allow to obtain new results in theory of special functions (see, e. g., [22, 23]), etc.

The main object of our study is rectangular frames, i. e., doubly-connected domains which are the difference of two homothetic rectangles, and their modules.

At first, we recall some classical definitions. Consider a plane doubly-connected domain D with nondegenerate boundary components. One of its important characteristics is the *conformal module* $m(D)$. There are several equivalent definitions of $m(D)$; we give some of them.

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If D is conformally equivalent to an annulus $\{r_1 < |z| < r_2\}$, then

$$m(D) := \frac{1}{2\pi} \ln \frac{r_2}{r_1}.$$

On the other hand,

$$m(D) := \lambda(\Gamma),$$

where $\lambda(\Gamma)$ is the extremal length of curve-family (see, e. g., [1]) Γ consisting of all curves joining in D its boundary components. Furthermore,

$$m(D) := 1/\lambda(\Gamma'),$$

Γ' being the family of all curves in D separating its boundary components. At last,

$$m(D) := 1/\operatorname{Cap}(C),$$

where $\operatorname{Cap}(C)$ is the conformal capacity of the condenser C defined by D.

If a doubly-connected domain D is symmetric with respect to one or two of the coordinate axis, then its module can be easily found via module of quadrilateral which is a half or a quarter of D. We recall that a simply-connected Jordan domain Q, with boundary four points A_k, $1 \leq k \leq 4$, given in a positive order, is called a *quadrilateral.* The definition can be extended to non-Jordan domains, if the points A_k are understood as prime ends. We denote the quadrilateral as $Q(A_1, A_2, A_3, A_4)$ or simply Q if it is clear which points A_k are fixed. The parts of ∂Q lying between A_1 and A_2, A_3 and A_4 we call *horizontal sides* of Q, the other two parts of the boundary are *vertical sides.* Let us map conformally $Q(A_1, A_2, A_3, A_4)$ onto a rectangle $[0, a] \times [0, b]$ so that the horizontal sides are mapped onto the horizontal sides of the rectangular. The number

$$m(Q) := \frac{a}{b}$$

is called the *module* of Q.

It is well-known (see, e. g., [1]) that $m(Q)$ is equal to the extremal length $\lambda(\Gamma)$ of the family Γ consisting of curves in D joining its vertical sides. Besides, $m(Q) = 1/\lambda(\Gamma')$ where Γ' is the family of curves in D joining its horizontal sides.

Special consideration is given to studying of behavior of the modules under various deformations of domains, their numerical calculation, and asymptotics at degeneration. In this regard we can note the survey by R. Kühnau [34] and the papers [3, 5, 8–10, 13, 14, 17, 18, 20–22, 26, 36–38, 53, 59].

Finding conformal modules is often associated with determination of conformal mapping of a given doubly connected domain onto an annulus or a given quadrilateral onto a rectangle; however, there are some methods that do not use explicit conformal mappings. As a rule, we can not construct analytically the conformal mapping of the given domain onto canonical one and to calculate its module. Therefore, approximate methods play an important role. There are many papers, monographs and surveys on the topic, see,

e. g., [5, 12, 16, 20, 24, 34, 35, 46, 48, 51, 58, 62]. Not with the aim to conduct an exhaustive analysis of the existing methods in this direction we note some of them.

Methods of boundary value problems and potential theory. As a rule, we find a harmonic in a given domain function satisfying the Dirichlet or Neumann conditions on parts of the boundary. To find the solution of the boundary value problem various methods are used such that the finite elements method with modifications [5, 20], simulation method or method of fundamental solutions [3, 31–33], when the approximate solution is represented as a linear combination of fundamental solutions to the Laplace equation, etc.

Integral equation method. The boundary values of the required conformal mapping between a given domain and a canonic one (a disk, a rectangle, an annulus, etc.) satisfies integral equations that uses an information on the boundary of the domain and the conjugation operator for harmonic functions. There are some well-known equations such as Mikhlin's, Warschawski's, Gershgorin's, Symm's, Theodorsen's integral equations and their modifications for various situations (see, e. g., the survey [62], and [10, 24, 28, 41, 42, 50, 55–57, 62]).

Domain decomposition method. It uses decomposition of a given quadrilateral Q into two or a finite number of smaller quadrilaterals Q_j. The module $m(Q)$ is very close to $\sum m(Q_j)$ for sufficiently long quadrilateral (see [38,46–48]).

Osculation methods. The osculation method (Schmiegungsverfahren) of Koebe [29, 30] approximates the desired conformal mapping by a composition of more elementary maps (see [25]).

Approximate methods based on finding the Laurent coefficients of the desired mapping. We mention here Fornberg's method [15], its generalizations and modifications [10, 40, 60, 61].

Schwarz-Cristoffel integrals. For polygonal boundaries the Schwarz-Cristoffel integrals can be used (see, e. g., [12]). There are softwares to practical using of this method ([11, 26]).

When using these methods for calculating conformal modules for regions with angles and for elongated ones, problems arise. Therefore, for these cases asymptotic formulas and estimates are very useful.

Now we describe the main problem which is investigated in the paper. It is well-known that modules doubly-connected domains and quadrilaterals are invariant under conformal mappings and quasiinvariant under quasiconformal ones (see, e. g., [1]): if f is an H-quasiconformal mapping of D onto $\widetilde{D}$, then

$$\frac{1}{H}\, m(D) \le m(\widetilde{D}) \le H\, m(D).$$

One of the simplest H-quasiconformal mappings is the stretching along the abscissa axis $f_H : x+iy \mapsto Hx+iy$. M. Vuorinen states the following problem[1]: Investigate how the module $m(D)$ is deformed under f_H for sufficiently large H. In particular, which is asymptotical behavior of $m(D)$ if D is the difference of two homothetic squares?

[1]The problem was formulated in a private talk during the conference 'Geometric Analysis and Its Applications', Volgograd, 2004.

The main result of the paper is

Theorem 1.1 *If $D_1 = D_1^\sigma := [-1,1]^2 \setminus [-\sigma,\sigma]^2$, $\sigma \in (0,1)$, $D_H = D_H^\sigma := f_H(D_1)$, then*

$$m(D_H^\sigma) \sim \frac{1-\sigma}{4\sigma H}, \quad H \to \infty. \tag{1}$$

Theorem 1.1 gives a good approximate formula for the module for sufficiently elongated rectangular frames and some L-shaped regions that are $(1/4)$ of the rectangular frames considered here. The L-shaped regions are standard domains considered, including, for computation of their conformal modules by many authors, see [16, 21]. In [45] we suggested an algorithm for some L-shaped domains which are stretched polyominoes, i. e., figures consisting of a finite number of disjoint uniform rectangles. We should note that recently D. Dautova [9] solved a similar problem and found an asymptotic formula for modules of diamond-shaped domains.

Now we overview the content of the paper. In Section 2 we give a solution to the problem for $\sigma = 1/2$, in addition, we deduce an explicit formula for $m(D_H)$ via elliptic integrals. It should be noted that when $H = 1$ an explicit formula for $m(D_H)$ is well-known, see Remark 1.3 below. In Section 3 we establish continuity of module of quadrilateral under kernel convergence in the sense of Carathéodory. In Section 4 the general case is considered. The results of Sections 2 and 4 were announced in [6] and [44].

Let $E := \{|z| < 1\}$, $U := \{\operatorname{Im} z > 0\}$, $E^+ := E \cap U$, $S_{\gamma\delta} := \{e^{i\varphi} \mid \gamma < \varphi < \delta\}$, $0 < \delta - \gamma < 2\pi$. We denote by $[a,b]$ the segment with endpoints a, $b \in \mathbb{C}$.

The elliptic integral of the first kind

$$K(r) := \int_0^1 \frac{d\xi}{\sqrt{(1-\xi^2)(1-r^2\xi^2)}}.$$

It is known (see, e. g., [2, 4]) that

$$\lim_{r\to 0}\left(K(r') - \ln\frac{4}{r}\right) = 0, \tag{2}$$

where as usual $r' = \sqrt{1-r^2}$. From (2) it follows that $K(r) \sim \ln\frac{4}{r'}$ as $r \to 1$. Therefore,

$$K(r) \sim \frac{1}{2}\ln\frac{1}{1-r}, \quad \frac{K(r)}{K(r')} \sim \frac{1}{\pi}\ln\frac{1}{1-r}, \quad r \to 1. \tag{3}$$

From (2) we also obtain that

$$\frac{K(r')}{K(r)} \sim \frac{2}{\pi}\ln\frac{1}{r}, \quad r \to 0. \tag{4}$$

Remark 1.2 *The ring domain consisting of the unit disk minus a radial slit from 0 to r, $0 < r < 1$, is usually the Grötzsch ring and its modulus is denoted*

$$\mu(r) = \frac{\pi}{2}\frac{K(r')}{K(r)},$$

see [34]. The asymptotic formula (4) can be refined by using of the results from [4], Theorem 5.13.

Remark 1.3 *When $H = 1$ an explicit formula for $m(D_H^\sigma)$ is well-known (see, e. g., [7]):*

$$m(D_1^\sigma) = \mu\left(\left(\frac{l-l'}{l+l'}\right)^2\right), \quad l = \mu^{-1}\left(\frac{2}{\pi}\frac{1-\sigma}{1+\sigma}\right), \quad l' = \sqrt{1-l^2}.$$

2 The case $\sigma = 1/2$

Consider the part Q_H of D_H lying in the first quarter of the plane. It is the union of three rectangles of the same size. Let us map conformally one of the rectangles with vertices at the points $(H+i)/2$, $H+i/2$, $H/2+i$, and $H+i$ onto the quarter of the unit disk $U_1 := \{z \mid |z| < 1, \operatorname{Re} z > 0, \operatorname{Im} z > 0\}$ by the mapping f so that $f((H+i)/2) = 0$, $f(H+i/2) = 1$, and $f(H/2+i) = i$. Let $e^{i\kappa} = f(H+i)$.

By the Riemann-Schwarz reflection principle f could be extended up to the conformal mapping of the rectangle $[0,H]\times[0,1]$ onto the unit disk E; we will designate the extension also through f. Then f maps conformally Q_H onto the domain which is three quarters of the unit disk, besides, $f(H/2) = -i$, $f(H) = e^{-i\kappa}$, $f(i/2) = -1$, and $f(i) = -e^{-i\kappa}$.

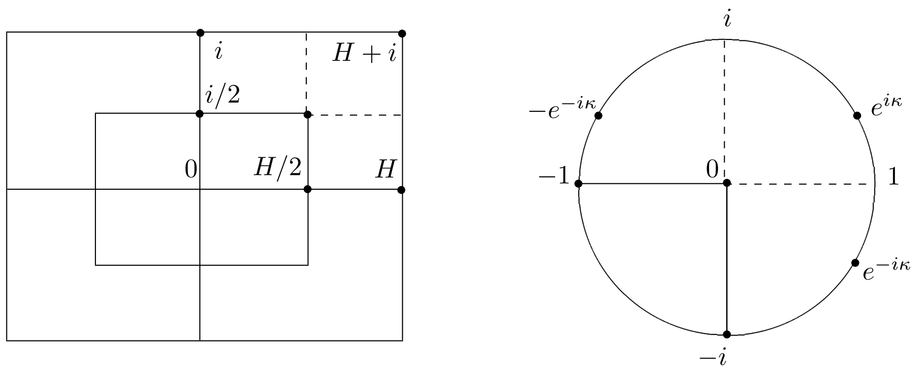

Fig. 1

The function $g(\zeta) := (if(\zeta))^{2/3}$ maps conformally Q_H onto E^+ at an appropriate choice of branch of the power function. We have $g(H/2) = 1$, $g(H) = e^{i\alpha}$, $f(i/2) = -1$, $f(i) = -e^{-i\beta}$ where

$$\alpha = (\pi - 2\kappa)/3, \quad \beta = \pi - 2\kappa/3. \tag{5}$$

The module of D_H, by the symmetry principle for quasiconformal mappings (see, e. g., [1]), is equal

$$m(D_H) = 1/(4\lambda(\Gamma)) \tag{6}$$

where Γ is the family of all curves in Q_H which join $[H/2, H]$ and $[i/2, i]$. Because of conformal invariance of the module we obtain

$$\lambda(\Gamma) = \lambda(\Gamma'), \tag{7}$$

where Γ' is a family of all curves in E^+ connecting $S_{0\alpha}$ to $S_{\beta\pi}$. By the symmetry principle,

$$\lambda(\Gamma') = 2\lambda(\widetilde{\Gamma}) \tag{8}$$

where $\widetilde{\Gamma}$ is the family of all curves which join $S_{-\alpha,\alpha}$ and $S_{\beta,2\pi-\beta}$ in E^+.

Let us map conformally the unit disk E onto the upper half-plane U so that the points $e^{i\beta}$, $e^{-i\beta}$, $e^{-i\alpha}$, and $e^{i\alpha}$ are mapped on $-1/l$, -1, 1, and $1/l$, $l > 1$. Now we express l through α and β. Equating the cross-ratios

$$\frac{-1/l+1}{-1/l-1} \cdot \frac{1/l-1}{1/l+1} = \frac{e^{i\beta}-e^{-i\beta}}{e^{i\beta}-e^{-i\alpha}} \cdot \frac{e^{i\alpha}-e^{-i\alpha}}{e^{i\alpha}-e^{-i\beta}}$$

we obtain

$$l = \frac{\sqrt{1-\cos(\alpha+\beta)} - \sqrt{2\sin\beta\sin\alpha}}{\sqrt{1-\cos(\alpha+\beta)} + \sqrt{2\sin\beta\sin\alpha}}\,. \tag{9}$$

Besides,

$$\lambda(\widetilde{\Gamma}) = \frac{2K(l)}{K(l')}\,. \tag{10}$$

From (6), (7), (8), and (10) we have

$$m(D_H) = \frac{K(l')}{16K(l)}\,. \tag{11}$$

Now we find the relation between κ and H. For this purpose we map conformally E onto the upper half-plane U by a function φ so that $-e^{-i\kappa}$, $-e^{i\kappa}$, $e^{-i\kappa}$, and $e^{i\kappa}$ are mapped on $-1/k$, -1, 1, and $1/k$. We note that k satisfies the condition

$$\frac{2K(k)}{K(k')} = H \tag{12}$$

because the quadrilateral, which is the upper half-plane U with fixed points $-1/k$, -1, 1, and $1/k$, is conformally equivalent to the rectangle of length H and height 1 under the mapping $\varphi \circ f$. From the equality of cross-ratios

$$\frac{-1/k+1}{-1/k-1} \cdot \frac{1/k-1}{1/k+1} = \frac{-e^{-i\kappa}+e^{i\kappa}}{-e^{-i\kappa}-e^{i\kappa}} \cdot \frac{e^{i\kappa}-e^{-i\kappa}}{e^{i\kappa}-e^{-i\kappa}}$$

we have

$$\kappa = \arcsin\frac{1-k}{1+k}\,. \tag{13}$$

Therefore, we prove

Theorem 2.1 *For $\sigma = 1/2$ the module of $D_H = D_H^\sigma$ is defined by (11) where l is found from (9) taking into account (5), (12), and (13).*

Corollary 2.2 *We have*

$$m(D_H) \sim \frac{1}{4H}, \quad H \to \infty.$$

Actually, from (13) it follows that $\kappa \sim (1-k)/2$ as $H \to \infty$. Now, taking into account (9) and (5), we obtain

$$1 - l \simeq \sqrt{\sin \beta} = \sqrt{\sin \frac{2\kappa}{3}} \simeq \sqrt{1-k}.$$

Thus, using (11), (12), and (3), we have

$$m(D_H) = \frac{K(l')}{16K(l)} \sim \frac{\pi}{16 \ln[(1-l)^{-1}]} \sim \frac{\pi}{8 \ln[(1-k)^{-1}]} \sim \frac{1}{4H}, \quad H \to \infty.$$

3 Convergence of domains and their modules

At first, we recall some results of the theory of prime ends of a plane domain and of a sequence of plane domains converging to a kernel ([54], Ch. IV) in convenient for us notations.

Consider a sequence of simply-connected domains G_n on the Riemann sphere $\overline{\mathbb{C}}$ converging to a kernel G with respect to a fixed point $S_0 \in \overline{\mathbb{C}}$ (see, e. g., [19], Ch. II, § 5). We assume that the boundaries of G_n and G are nondegenerate, i. e., each of them contains more than one point. Further we provide G_n and G with metrics induced from the sphere; in the case when all G_n are contained in a fixed Euclidean disk it is possible to change the spherical metrics by the Euclidean one.

Consider a crosscut γ of G, i. e., a Jordan arc in G with endpoints on ∂G (see, e. g., [49], Sect. 9.2, or [19], Ch. II, § 3). Without loss of generality, we may assume that γ has distinct endpoints and does not pass through S_0. Every crosscut γ subdivide G into two subdomains; we denote by $\operatorname{Int}\gamma$ the subdomain not containing S_0.

A sequence (γ_m) of crosscuts of G is a *null-chain* of G if for every $m \geq 1$ we have $\operatorname{Int}\gamma_{m+1} \subset \operatorname{Int}\gamma_m$, γ_m and γ_{m+1} have a positive distance relative to G, and $\operatorname{diam}\gamma_m \to 0$, $m \to \infty$. There is the following equivalence relation between null-chains of G: two null-chains (γ_m) and (β_m) are *equivalent* if for every j there exists k such that $\operatorname{Int}\beta_k \subset \operatorname{Int}\gamma_j$ and $\operatorname{Int}\beta_j \subset \operatorname{Int}\gamma_k$. An equivalence class of null-chains is called a *prime end* of G. We call the set $|P| := \cap_{m\geq 1}\overline{\operatorname{Int}\gamma_m}$ the *impression* of P; it does not depend on choice of the null-chain (γ_m).

If G is a Jordan domain, then for every prime end P the impression $|P|$ consists of a unique point and we can identify P with the point. Therefore, in the case, the set $PE(G)$ of all prime ends of G coincides with the boundary of G, and the set $G \cup PE(G)$ is identified with the closure of G in the spherical metrics.

Let $F : E \to G$ be a conformal mapping from the unit disk E onto G. The classical Carathéodory theorem (see, e. g., [49], Theorem 9.6, or [19], p. 41, Theorem 2) states that there exists a one-to one correspondence between the set of boundary points of E and the set $PE(G)$ of prime ends of G; moreover, the correspondence lets us extend f up to a homeomorphism from $\overline{E}$ onto $\overline{G} := G \cup PE(G)$ if we introduce a topology in $\overline{G}$ by a natural way.

Let γ^n be a crosscut of G_n, $n \geq 1$. We say that the sequence (γ^n) is *a crosscut of the sequence* $\widetilde{G} := (G_n)$ *lying over* γ (see [54], p. 64) if the following conditions are fulfilled:

1. For every neighborhood U of γ there exists n_0 such that γ^n lies in U for any $n \geq n_0$;
2. if points p_1, $p_2 \in G$ are separated by γ in D, then there exists n_1 such that p_1 and p_2 are separated by γ^n in G_n for every $n \geq n_1$.

In [54], p. 65, it is proved

Lemma 3.1 *For any crosscut γ of G there exists a crosscut $\widetilde{\gamma} := (\gamma^n)$ of $\widetilde{G}$ lying over γ.*

Let (γ_m) be a null-chain of G and $\widetilde{\gamma}_m := (\gamma_m^n)$ for every m be a crosscut of $\widetilde{G}$ lying over γ_m. It is possible to introduce a relation of equivalence on the set of such sequences $(\widetilde{\gamma}_m)$. The sequences $(\widetilde{\gamma}_m)$ and $(\widetilde{\beta}_m)$ of crosscuts $\widetilde{\gamma}_m := (\gamma_m^n)$ and $\widetilde{\beta}_m := (\beta_m^n)$ of $\widetilde{G}$ are called *equivalent* if for every j there exist k and n_0 such that for every $n \geq n_0$ we have $\operatorname{Int}\gamma_k^n \subset \operatorname{Int}\beta_j^n$ and $\operatorname{Int}\beta_k^n \subset \operatorname{Int}\gamma_j^n$. The classes of equivalence $\widetilde{P}$ of such sequences are called the *prime ends* of $\widetilde{G}$.

If a sequence of crosscuts (γ_m) defines a prime end P of G and prime end $\widetilde{P}$ of $\widetilde{G}$ contains $(\widetilde{\gamma}_m)$, where $\widetilde{\gamma}_m$ is a crosscut of $\widetilde{G}$ lying over γ_m, then we say that $\widetilde{P}$ is *the prime end of* $\widetilde{G}$ *corresponding to the prime end* P of G. Theorems 5 from [54], Ch. 4, § 2, actually states that the described correspondence $\Phi : P \mapsto \widetilde{P}$ is a bijection between the set of prime ends of G and the set of prime ends of $\widetilde{G}$.

Now consider a sequence (P_n) where P_n is an inner point or a prime end of G_n. Let $\widetilde{P} = \Phi(P)$ be the prime end of $\widetilde{G}$ corresponding to a prime end P of G and P be defined by a sequence (γ_m). Let $\widetilde{P}$ be defined by $(\widetilde{\gamma}_m)$ where $\widetilde{\gamma}_m$ lies over γ_m for every m, and $\widetilde{\gamma}_m := (\gamma_m^n)$. We say that *the sequence* (P_n) *converges to* $\widetilde{P}$ if for every m there exists n_0 such that γ_m^n separates P_n from S_0 in G_n for every $n \geq n_0$; if P_n is a prime end of G_n, defined by a sequence of crosscuts (β_m^n), then this condition means that for $n \geq n_0$ the crosscut γ_m^n separates β_j^n from S_0 in G_n for sufficiently large j.

Let f_n and f be conformal mappings of E onto G_n and G, extended to $\overline{E}$ up to homeomorphisms of the closures of the domains in topology connected with prime ends. Let P_n be a sequence consisting of points or prime ends of G_n, and let P be a boundary prime end of G. Denote $\zeta_n = f_n^{-1}(P_n)$, $\zeta_0 = f^{-1}(P)$.

In [54], p. 75, the following statement is proved.

Theorem 3.2 *A sequence (P_n) converges to the prime end $\widetilde{P}$ of $\widetilde{G}$ corresponding to P if and only if $\zeta_n \to \zeta_0$.*

From Theorem 3.2 we deduce the following result on continuity of the module of quadrilateral under kernel convergence of domains.

Theorem 3.3 *Let A_n, B_n, C_n, and D_n be distinct boundary prime ends of G_n. Let the sequences (A_n), (B_n), (C_n), and (D_n) converge to the prime ends $\widetilde{A}$, $\widetilde{B}$, $\widetilde{C}$ and $\widetilde{D}$ of $\widetilde{G}$ corresponding to distinct prime ends A, B, C, and D of G. Then $m(G_n(A_n, B_n, C_n, D_n)) \to m(G(A, B, C, D))$.*

Actually, taking into account conformal invariance of the module of quadrilateral, by Theorem 3.2 we have

$$m(G_n(A_n, B_n, C_n, D_n)) = m(E(a_n, b_n, c_n, d_n)) \to$$

$$\to E(a, b, c, d) = m(G(A, B, C, D))$$

where a_n, b_n, c_n, and d_n are preimages of A_n, B_n, C_n, and D_n under the map f_n, and a, b, c, and d are preimages of A, B, C, and D under the map f.

Remark 3.4 *The statement of Theorem 3.3 is valid not only for univalent domains, it is true for p-valent ones (Riemann surfaces) as well (see, e. g., [43]).*

4 General case

As in the case $\sigma = 1/2$, consider the part Q_H of D_H lying in the first quarter of the plane. Let us shift Q_H to the left on the value σH; as a result we receive the domain $\widetilde{Q}_H$. By the symmetry principle, the module of D_H is equal to

$$m(D_H) = \frac{1}{4\lambda(\Gamma)} \tag{14}$$

where Γ is the family of curves joining $[0, (1-\sigma)H]$ and $[-\sigma H + i\sigma, -\sigma H + i]$ in the quadrilateral $\widetilde{Q}_H$.

Now consider the domain

$$\widetilde{Q} := \cup_{H>0} \widetilde{Q}_H = (\mathbb{R} \times (0, 1)) \setminus ((-\infty, 0] \times (0, \sigma]).$$

Let us map conformally $\widetilde{Q}$ onto the horizontal strip $\mathbb{R} \times (0, 1)$ with keeping the infinite prime ends and the origin. For this purpose we map conformally the upper half-plane U onto $\widetilde{Q}$ and $G = \{0 < \operatorname{Im} \omega < 1\}$ by the functions

$$z = C \int_1^\zeta \sqrt{\frac{\zeta - s}{\zeta - 1}} \, \frac{d\zeta}{\zeta}, \quad \omega = \frac{\ln \zeta}{\pi},$$

where $C > 0$, $0 < s < 1$.

Near $\zeta = 0$ we have

$$z = C\sqrt{s} \ln \zeta + \sum_{k=0}^{\infty} \sigma_k \zeta^k.$$

Since the intersection of the domain $\widetilde{Q}$ and the left half-plane is a half-strip of width $(1-\sigma)/\pi$, we have $C\sqrt{s} = (1-\sigma)/\pi$. Near $\zeta = \infty$

$$z = C \ln \zeta + \sum_{k=0}^{\infty} \frac{\beta_k}{\zeta^k},$$

therefore, similarly we obtain $C = 1/\pi$. Then $s = (1-\sigma)^2$ and

$$z = \frac{1}{\pi} \int_1^{\zeta} \sqrt{\frac{\zeta - (1-\sigma)^2}{\zeta - 1}}.$$

Considering it we conclude that for sufficiently large $M > 0$ we have

$$z = (1-\sigma)\omega + \sum_{k=0}^{\infty} \sigma_k e^{k\pi\omega} \tag{15}$$

in the half-plane $\Pi_\mu^- := \{\operatorname{Re}\omega < -M,\, 0 < \operatorname{Im}\omega < 1\}$, and

$$z = \omega + \sum_{k=0}^{\infty} \beta_k e^{-k\pi\omega} \tag{16}$$

in the half-plane $\Pi_\mu^+ := \{\operatorname{Re}\omega > M,\, 0 < \operatorname{Im}\omega < 1\}$.

From rectilinearity of the boundary arcs of the domains and the Riemann-Schwarz reflection principle it follows that convergence of the series (15) and (16) is uniform in the closed half-planes $\overline{\Pi_\mu^-}$ and $\overline{\Pi_\mu^+}$.

From (15) and (16) we deduce that on the vertical segments in $\widetilde{Q}$, lying on the lines $\operatorname{Re}\omega = -\widetilde{\sigma}\, H$, where

$$\widetilde{\sigma} = \frac{\sigma}{1-\sigma},$$

we have

$$\operatorname{Re} z(\omega) = -\widetilde{\sigma}\, H + O(1), \quad H \to \infty.$$

In the same way, on the segments, lying on the lines $\operatorname{Re}\omega = (1-\sigma)H$,

$$\operatorname{Re} z = (1-\sigma)H + O(1), \quad H \to \infty,$$

Therefore,

$$\lambda(\Gamma) \sim m(\widetilde{P}) \tag{17}$$

where $\widetilde{P}$ is the quadrilateral which is the rectangle

$$P := [-\widetilde{\sigma}\, H, (1-\sigma)H] \times [0,1]$$

with the segments $[-\widetilde{\sigma}\, H, -\widetilde{\sigma}\, H + i]$ and $[0, (1-\sigma)H]$ as vertical sides.

Let us map U onto P by the function

$$z = C \int_0^{\zeta} \frac{d\xi}{\sqrt{(1-\xi^2)(1-k^2\xi^2)}} + C_1$$

where

$$C_1 = \frac{(1-\sigma)^2 - \sigma}{1-\sigma}\,H, \quad C = \frac{(1-\sigma)^2 + \sigma}{2K(k)(1-\sigma)},$$

and $k \in (0,1)$ is defined by the relation

$$\frac{2K(k)}{K(k')} = \frac{(1-\sigma)^2 + \sigma}{1-\sigma}\,H. \tag{18}$$

The mapping takes the points $-\widetilde{\sigma}\,H + i$, $\widetilde{\sigma}\,H$, 0, and $(1-\sigma)H$, i. e., the vertices of the quadrilateral P, into $(-1/k)$, (-1), a, and 1, where

$$a = \operatorname{sn}\left[\frac{\sigma - (1-\sigma)^2}{\sigma + (1-\sigma)^2}\,K(k), k\right]. \tag{19}$$

Here $\operatorname{sn}[\,\cdot\,, k]$ is the Jacobi elliptic sine corresponding to the parameter k (see, e. g., [2]).

Now we map the upper half-plane U onto itself conformally so that the points $-1/k$, -1, a, and 1 are mapped onto $-1/\nu$, -1, 1, and $1/\nu$ $(0 < \nu < 1)$. Then the module of the quadrilateral $\widetilde{P}$ is equal to

$$m(\widetilde{P}) = \frac{2K(\nu)}{K(\nu\,')} \tag{20}$$

where ν is defined by the equality of the cross-ratios:

$$\frac{1/\nu - 1}{1/\nu + 1}\cdot\frac{-1/\nu + 1}{-1/\nu - 1} = \frac{1-a}{1+1}\cdot\frac{-1/k + 1}{-1/k - a}$$

or

$$\frac{1-\nu}{1+\nu} = \sqrt{\frac{1-a}{1+ka}}\cdot\sqrt{\frac{1-k}{2}}. \tag{21}$$

Therefore, for finding the asymptotics of $m(P)$ we need to know the asymptotic behavior of a as $H \to \infty$. It is possible to do using (19), but we prefer to apply geometric considerations which are based on rectilinearity of the boundary arcs and the reflection principle. Let us prove the following auxiliary statement.

Lemma 4.1 *Let Q be a quadrilateral which is the square $[0,1]^2$ with vertices at the points c, 1, $1+i$, and i, where $c \in (0,1)$. Denote $Q_H = f_H(Q)$. Then*

$$m(Q_H) \sim (1-c)H, \quad H \to \infty.$$

Proof. Let $\widetilde{Q}_H = (1/H)Q_H$. Since $m(\widetilde{Q}_H)$ is a monotonic function of H, it is sufficient to consider the sequence $H_n = 2n$ and to prove that

$$m(Q_{H_n}) \sim (1-c)H_n, \quad n \to \infty.$$

For short we will write Q_n instead of Q_{H_n}.

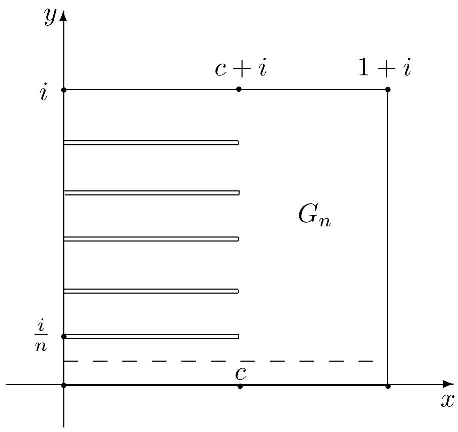

Fig. 2

Consider the domains $Q_n^1, Q_n^2, \ldots, Q_n^{2n}$, where

$$Q_n^k = [0,1] \times \left[\frac{k-1}{2n}, \frac{k}{2n}\right].$$

Let us glue Q_n^k and Q_n^{k+1} along $\{(x,y) \mid 0 \le x \le 1,\ y = j/(2n)\}$ for odd k, and along $\{(x,y) \mid c \le x \le 1,\ y = j/(2n)\}$ for even k. As a result, we obtain the domain G_n which is the unit square with $(n-1)$ horizontal slits (Fig. 2).

We will consider G_n as a quadrilateral with vertices c, 1, $1+i$, and $c+i$. By the symmetry principle, $m(G_n) = m(Q_n)/(2n)$. The domains G_n converge to the rectangle $G := [c,1] \times [0,1]$ as $n \to \infty$, and the sequences of their vertices converge to four distinct prime ends of $\widetilde{G} = (G_n)$ corresponding to the vertices of G. Actually, let us take $(1/4)$ of concentric circles with radius $r_m \to 0$ as crosscuts γ_m which define a prime end P being a vertex of G. Denote by P_n the prime end of G_n which has the same impression as P. Let γ_m^n be a crosscut of G_n which is the union of γ_m and, if it is necessary, a segment connecting one of its endpoint to one of the nearest points of ∂G_n. Let us denote $\widetilde{\gamma}_m := (\gamma_m^n)$. Then the crosscut $(\widetilde{\gamma}_m)$ defines the prime end of $\widetilde{G} := (G_n)$ corresponding to P and γ_m^n for sufficiently large m and n separates the corresponding vertex of G_n from any fixed point of the kernel G. By Theorem 3.3, $m(G_n) \to m(G) = 1-c$, and Lemma 4.1 is proved.

Remark 4.2 *The quadrilateral Q_H could be considered as a generalized long quadrilateral. Asymptotics of the modules of long quadrilaterals were investigated in [14, 17, 18, 37, 38], and other papers where various methods for computing the modules were suggested.*

Consider the quadrilateral P^* which is the rectangle P with the segments $[0, (1-\sigma)H]$ and $[-\widetilde{\sigma}\, H+i, (1-\sigma)H+i]$ as horizontal sides. Taking into account conformal invariance of the module, by Lemma 4.1 we have

$$m(P^*) \sim (1-\sigma)H, \quad H \to \infty. \tag{22}$$

Now we can describe the behavior of a as $H \to \infty$. Let us map P^* conformally onto U such that the points $(1-\sigma)H + i$, $-\widetilde{\sigma}H + i$, 0, and $(1-\sigma)H$ are mapped into $-1/\mu$, -1, 1, and $1/\mu$, $0 < \mu < 1$. Then

$$m(P^*) = \frac{K(\mu')}{2K(\mu)}, \tag{23}$$

We should note that because of $m(P^*) \to \infty$ as $H \to \infty$, by (23) and (4), we have $\mu \to 0$, $H \to \infty$. Comparing the cross-ratios of the points in ∂U, corresponding to each other under conformal automorphism, we obtain

$$\frac{1-1/\mu}{1+1} \cdot \frac{-1/\mu+1}{-1/\mu-1/\mu} = \frac{a-1}{a+1/k} \cdot \frac{1/k+1/k}{1/k-1}$$

or

$$\frac{1-a}{1+ak} \cdot \frac{2k}{1-k} = \frac{(\mu-1)^2}{4\mu}.$$

Therefore, taking into account that a, $k \to 1$ as $H \to \infty$ we have

$$1 - a \sim \frac{1-k}{\mu}. \tag{24}$$

By (22), (23), and (3),

$$\frac{1}{(1-\sigma)H} \sim \frac{2K(\mu)}{K(\mu')} \sim \frac{4}{\pi} \ln \frac{1}{\mu}\,. \tag{25}$$

Now we use the asymptotic behavior (3) of the elliptic integrals. With use of that and by (18)

$$\ln \frac{1}{1-k} \sim \pi \frac{K(k)}{K(k')} \sim \frac{\pi}{2} \cdot \frac{(1-\sigma)^2 + \sigma}{1-\sigma} H.$$

Now from (3), (20), (21), (24), and (25) we obtain

$$m(\widetilde{P}) = \frac{2K(\nu)}{K(\nu')} \sim \frac{2}{\pi} \ln \frac{1}{1-\nu} \sim \frac{2}{\pi} \ln \frac{1+ak}{1-a} + \frac{1}{\pi} \ln \frac{1}{1-k}$$

$$\sim \frac{2}{\pi} \ln \frac{1}{1-k} - \frac{1}{\pi} \ln \frac{1}{\mu} \sim (1 - \sigma + \widetilde{\sigma})H - (1-\sigma)H = \widetilde{\sigma}H.$$

Because of (14) and (17) it completes the proof of (1).

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Institute of mathematics and mechanics, Kazan (Volga Region) Federal University, e-mail: snasyrov@kpfu.ru