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# On Invertibility of Some Operator Sums

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**Abstract**—We study invertibility of some sums of linear bounded operators on Hilbert space (Theorem 1). A criterion on invertibility of sums of projections is found. Some equivalent conditions on invertibility of difference of two projections are obtained. We prove that block projection operators preserve invertibility of positive operators. We present three corollaries from Theorem 1; it is shown for instance that if $A, B \in \mathcal{B}(\mathcal{H})$ are nonnegative and $A - B$ is invertible, then $A + B$ is also invertible.
We also prove the following result: Let $X, Y \in \mathcal{B}(\mathcal{H})$ be self-adjoint operators, $X \geq 0$ and $-X \leq Y \leq X$. If $Y$ is invertible, then $X$ is also invertible. It is shown that for unitary operators $U, V$ the operator $U + V$ is invertible if and only if $||U - V|| < 2$.

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## INTRODUCTION

Let $\mathcal{H}$ be a Hilbert space over complex field $\mathbb{C}$, $\mathcal{B}(\mathcal{H})$ be $*$-algebra of linear bounded operators on $\mathcal{H}$ and $\mathbb{I}$ be identity operator on $\mathcal{H}$. Let $\sigma(X)$ denote the spectrum of $X \in \mathcal{B}(\mathcal{H})$. An operator $T \in \mathcal{B}(\mathcal{H})$ is called *projection*, if $T = T^* = T^2$; *isometry*, if $T^*T = \mathbb{I}$; *unitary*, if $T^*T = TT^* = \mathbb{I}$; *partial isometry*, if $TT^*T = T$. Let $\mathcal{B}(\mathcal{H})^{\mathrm{pr}}$ be the projection lattice in $\mathcal{B}(\mathcal{H})$ and let $P^{\perp} = \mathbb{I} - P$ for $P \in \mathcal{B}(\mathcal{H})^{\mathrm{pr}}$. The cone of nonnegative operators in $\mathcal{B}(\mathcal{H})$ will be denoted by $\mathcal{B}(\mathcal{H})^+$. For $X \in \mathcal{B}(\mathcal{H})$ the absolute value $|X| = \sqrt{X^*X}$ belongs to $\mathcal{B}(\mathcal{H})^+$.

In this paper we investigate invertibility of some sums of linear bounded operators. We generalize one result of [1] in Theorem 1, by using algebraic Lemma 2 from [2]. It is also shown via this Lemma that block projection operators preserve invertibility of positive operators (Proposition 1 and Corollary 5).

We infer three corollaries from Theorem 1, two of these consequences were obtained in [1]. We show that if $A, B \in \mathcal{B}(\mathcal{H})^+$ and $A - B$ is invertible, then $A + B$ is also invertible in Corollary 3.

Let operators $X, Y \in \mathcal{B}(\mathcal{H})$ be self-adjoint, $X \geq 0$ and $-X \leq Y \leq X$. If $Y$ is invertible, then $X$ is also invertible (Corollary 4). Theorem 2 comprises one criterion on invertibility of sums of projections. Some equivalent conditions on invertibility of difference of two projections are obtained (Corollary 6). In Theorem 3 we investigate left invertibility of the sum of two isometries. It is shown that for unitary operators $U, V \in \mathcal{B}(\mathcal{H})$ the operator $U + V$ is invertible if and only if $||U - V|| < 2$ (Corollary 7).

Let $\mathcal{H}$ be a separable infinite-dimensional Hilbert space and $U$ be a unilateral shift operator on $\mathcal{H}$. Then the operator $U + V$ is not left invertible for any unitary operator $V \in \mathcal{B}(\mathcal{H})$ (Corollary 8).

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