The Riemann Boundary Value Problem on Non-rectifiable Curves and Fractal Dimensions

Boris A. Kats

Abstract. The aim of this work is to solve the Riemann boundary value problem on non-rectifiable curve. Its solvability depends on certain metric characteristics of the curve. We introduce new metric characteristics of dimensional type and new sharp conditions of solvability of the problem. In addition, we introduce and study a version of the Cauchy integral over non-rectifiable paths.

Mathematics Subject Classification (2000). Primary 30E25; secondary 30E20.

Keywords. Riemann boundary value problem, jump problem, non-rectifiable arc, metric dimension, Cauchy transform.

Introduction

We consider the following boundary value problem for holomorphic functions. Let Γ be a closed Jordan curve on the complex plane $\mathbb{C}$ bounding finite domain D^+, and $D^- = \overline{\mathbb{C}} \setminus \overline{D^+}$. Find a holomorphic in $\overline{\mathbb{C}} \setminus \Gamma$ function $\Phi(z)$ such that $\Phi(\infty) = 0$, the boundary values $lim_{D^+ \ni z \to t} \Phi(z) \equiv \Phi^+(t)$ and $lim_{D^- \ni z \to t} \Phi(z) \equiv \Phi^-(t)$ exist for any $t \in \Gamma$, and

$$\Phi^+(t) = G(t)\Phi^-(t) + g(t), \quad t \in \Gamma. \tag{1}$$

This boundary value problem is called the Riemann problem. It is well known and has numerous traditional applications in elasticity theory, hydro and aerodynamics and so on (see [1, 2]). Recently a number of authors explored its connections with theory of random matrices, non-classical estimates for orthogonal polynomials and so on (see, for instance, [3, 4]).

If $G(t) \equiv 1$, then the Riemann boundary value problem turns to so-called jump problem:

$$\Phi^+(t) - \Phi^-(t) = g(t), \quad t \in \Gamma. \tag{2}$$

This work was completed with the support of Russian Fund for Basic Researches, grants 09-01-12188-ofi-m and 10-01-00076-a.

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Boris A. Kats
Kazan State University of Architecture and Civil Engineering
Kazan, Zelenaya str. 1, 420043, Russia
e-mail: `katsboris877@gmail.com`