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On Independence of Events in Noncommutative Probability Theory

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Abstract—We consider a tracial state φ on a von Neumann algebra $\mathcal{A}$ and assume that projections P, Q of $\mathcal{A}$ are independent if $\varphi(PQ) = \varphi(P)\varphi(Q)$. First we present the general criterion of a projection pair independence. We then give a geometric criterion for independence of different pairs of projections. If atoms P and Q are independent then $\varphi(P) = \varphi(Q)$. Also here we deal with an analog of a "symmetric difference" for a pair of projections P and Q, namely, the projection $R \equiv P \vee Q - P \wedge Q$. If $R \neq 0, I$, the pairs $\{P, R\}$ and $\{Q, R\}$ are independent then $\varphi(P) = \varphi(Q) = 1/2$ and $\varphi(P \wedge Q + P \vee Q) = 1$. If, moreover, P and Q are independent, then $\varphi(P \wedge Q) \leq 1/4$ and $\varphi(P \vee Q) \geq 3/4$. For an atomless von Neumann algebra $\mathcal{A}$ a tracial normal state is determined by its specification of independent events. We clarify our considerations with examples of projection pairs with the differemt mutual independency relations. For the full matrix algebra we give several equivalent conditions for the independence of pairs of projections.

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1. INTRODUCTION

Let φ be a tracial state on a von Neumann algebra $\mathcal{A}$. Projections P, Q of $\mathcal{A}$ are said to be *independent* [1], if $\varphi(PQ) = \varphi(P)\varphi(Q)$. In [1] for this notion of independence, the structure of a finite von Neumann algebra generated by an independent family of subalgebras is determined. As a consequence, the authors obtain extensions of the Kolmogorov and Hewitt–Savage zero-one laws. For any P the projection set $\{Q \in \mathcal{A} :\ P \text{ and } Q \text{ are independent}\}$ comprises a quantum logic and is closed in the strong operator topology [2, Theorem 3.1]. The strong law of large numbers for d-dimensional arrays in von Neumann algebras holds for this notion of independence [2]; for the strong laws of large numbers for positive measurable operators and applications see [4, 5] and [6]. C^*-independence and W^*-independence of von Neumann algebras was investigated in [7]. In [8] complete C^*-independence of operator algebras is introduced. Equivalent characterization is given for C^*-subalgebras to be completely independent in terms of maximal tensor product.

In this paper we present the general criterion of a projection pair independence (Theorem 1). We then give a geometric criterion for the independence of the following pairs of projections: 1) isoclinic (Theorem 3); 2) one of which is an atom of an algebra (Theorem 4). In particular, if atoms P and Q are independent then $\varphi(P) = \varphi(Q)$. The projection $R \equiv P \vee Q - P \wedge Q$ is an analog of a "symmetric difference" for the pair of projections P and Q. If $R \neq 0, I$, the pairs $\{P, R\}$ and $\{Q, R\}$ are independent then $\varphi(P) = \varphi(Q) = 1/2$ and $\varphi(P \wedge Q + P \vee Q) = 1$. If, moreover, P and Q are independent, then $\varphi(P \wedge Q) \leq 1/4$ and $\varphi(P \vee Q) \geq 3/4$ (Theorem 5).

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Originally studied by Gohberg and Krein in [9], the block projection operators admit a natural extension to the setting of quasi-normed ideals and noncommutative integration. Let $n \geq 2$ and projections $P_1, \ldots, P_n \in \mathcal{A}$ be such that $P_1 + \ldots + P_n = I$. Define a block projection operator $\mathcal{P}_n : \mathcal{A} \to \mathcal{A}$ by the formula $\mathcal{P}_n(X) = \sum_{k=1}^{n} P_k X P_k$ for all $X \in \mathcal{A}$. Assume that $P, Q \in \mathcal{A}^{\mathrm{pr}}$ and $\tilde{P} = \mathcal{P}_n(P)$, $\tilde{Q} = \mathcal{P}_n(Q) \in \mathcal{A}^{\mathrm{pr}}$. Then the following conditions are equivalent: (i) P, Q are independent; (ii) $\tilde{P}, \tilde{Q}$ are independent (Theorem 6). Block projection operators on a von Neumann algebras (and on algebras of measurable with respect to semifinite normal traces operators) were investigated in [10, 11]. In this case we established several uniform submajorization inequalities for block projection operators in [12].

Let φ, ψ be tracial normal states on an atomless von Neumann algebra $\mathcal{A}$. If they have identical independent events, then they coincide (Theorem 7).

We give the examples of projection pairs P and Q that are 1) isoclinic independent; 2) non-isoclinic independent; 3) non independent but with independent $\{P, R\}$ and $\{Q, R\}$; 4) independent with independent $\{P, R\}$ and $\{Q, R\}$. For the full matrix algebra we give several equivalent conditions for the independence of pairs of projections (Theorem 2).

2. DEFINITIONS AND NOTATION

Let $\mathcal{A}$ be a von Neumann algebra of operators in a Hilbert space $\mathcal{H}$, $\mathcal{A}^{\mathrm{pr}}$ be the lattice of projections in $\mathcal{A}$, I be the identity operator in $\mathcal{A}$, and $P^{\perp} = I - P$ for $P \in \mathcal{A}^{\mathrm{pr}}$. An operator $A \in \mathcal{A}$ is said to be a commutator if $A = XY - YX$ for some $X, Y \in \mathcal{A}$. For $P, Q \in \mathcal{A}^{\mathrm{pr}}$, we write $P \sim Q$ (Murray–von Neumann equivalence) if $P = U^*U$ and $Q = UU^*$ for some $U \in \mathcal{A}$. The projection $P \wedge Q$ is defined by the equality $(P \wedge Q)\mathcal{H} = P\mathcal{H} \cap Q\mathcal{H}$, and $P \vee Q = (P^{\perp} \wedge Q^{\perp})^{\perp}$ projects onto $\overline{\mathrm{Lin}(P\mathcal{H} \cup Q\mathcal{H})}$. Let $\mathcal{A}^+$ be the cone of positive elements of $\mathcal{A}$. If $A \in \mathcal{A}$ then $|A| = \sqrt{A^*A} \in \mathcal{A}^+$. A positive linear functional φ on $\mathcal{A}$ is called

- faithful if $\varphi(X) > 0$ for all $X \in \mathcal{A}^+$, $X \neq 0$;
- normal if $A_i \nearrow A$ $(A_i, A \in \mathcal{A}^+) \Rightarrow \varphi(A) = \sup\limits_i \varphi(A_i)$;
- tracial if $\varphi(Z^*Z) = \varphi(ZZ^*)$ for all $Z \in \mathcal{A}$;
- a state if $\varphi(I) = 1$.

According to [13], projections P, Q are *in generic position* in Hilbert space $\mathcal{H}$, if

$$P \wedge Q = P \wedge Q^{\perp} = P^{\perp} \wedge Q = P^{\perp} \wedge Q^{\perp} = 0.$$

In this case $\mathcal{H} = \mathcal{K} \oplus \mathcal{K}$ and

$$P = \begin{pmatrix} 1_{\mathcal{K}} & 0 \\ 0 & 0 \end{pmatrix}, \quad Q = \begin{pmatrix} c^2 & sc \\ cs & s^2 \end{pmatrix} \tag{1}$$

for some $s, c \in \mathcal{B}(\mathcal{K})^+$ with $\mathrm{Ker}(s) = \mathrm{Ker}(c) = \{0\}$ and $s^2 + c^2 = 1_{\mathcal{K}}$.

Nonzero projections $P, Q \in \mathcal{A}$ are called *isoclinic* (with an angle $\theta \in (0, \pi/2)$, we denote this by $P \overset{\theta}{\approx} Q$), if

$$PQP = \cos^2\theta P, \quad QPQ = \cos^2\theta Q.$$

A nonzero projection $P \in \mathcal{A}$ is said to be *minimal* (or an *atom* of the algebra $\mathcal{A}$), if any nonzero projection $Q \in \mathcal{A}$, $Q \leq P$, coincides with P.

Let φ be a tracial state on a von Neumann algebra $\mathcal{A}$, $||A||_1 = \varphi(|A|)$ for $A \in \mathcal{A}$. Projections P, Q of $\mathcal{A}$ are called *independent* [1], if $\varphi(PQ) = \varphi(P)\varphi(Q)$. This definition naturally depends on the choice of the state φ.

Let φ, ψ be tracial states on a von Neumann algebra $\mathcal{A}$. We say that they *have identical independent events* if, for any pair of projections P, Q of $\mathcal{A}$, $\varphi(PQ) = \varphi(P)\varphi(Q)$ if and only if $\psi(PQ) = \psi(P)\psi(Q)$.

3. ON INDEPENDENCE OF PROJECTIONS

Lemma 1. *Let φ be a tracial state on a von Neumann algebra $\mathcal{A}$. For $P, Q \in \mathcal{A}^{pr}$ the following conditions are equivalent:*

(i) *P and Q are independent;*

(ii) *Q and P are independent;*

(iii) *P and $Q^{\perp}$ are independent;*

(iv) *$P^{\perp}$ and Q are independent;*

(v) *$P^{\perp}$ and $Q^{\perp}$ are independent.*

The proof is obvious.

Theorem 2. *Let φ be a faithful normal tracial state on a von Neumann algebra $\mathcal{A}$. For $P, Q \in \mathcal{A}^{pr}$ the following conditions are equivalent:*

(i) *P, Q are independent;*

(ii) $||I + z(PQ - \varphi(P)Q)||_1 \geq 1$ *for all $z \in \mathbb{C}$;*

(iii) $||I + z(PQ - \varphi(Q)P)||_1 \geq 1$ *for all $z \in \mathbb{C}$.*

Proof. Due to the linearity of the state φ the states $P, Q \in \mathcal{A}^{\mathrm{pr}}$ meet the equivalences:

$$\varphi(PQ) = \varphi(P)\varphi(Q) \Leftrightarrow \varphi(PQ - \varphi(P)Q) = 0 \Leftrightarrow \varphi(PQ - \varphi(Q)P) = 0. \tag{2}$$

Now apply Theorem 4.8 of [14], according to which any element $A \in \mathcal{A}$ admits the equivalence: $\varphi(A) = 0 \Leftrightarrow ||I + zA||_1 \geq 1$ for all $z \in \mathbb{C}$. Therefore, for all $P, Q \in \mathcal{A}^{\mathrm{pr}}$ we have the equivalences:

$$\varphi(PQ) = \varphi(P)\varphi(Q) \Leftrightarrow \varphi(PQ - \varphi(P)Q) = 0 \Leftrightarrow ||I + z(PQ - \varphi(P)Q)||_1 \geq 1 \quad \text{for all} \quad z \in \mathbb{C}.$$

Theorem is proved. □

According to the proof of [15, Proposition 2.4], for arbitrary $P, Q \in \mathcal{B}(\mathcal{H})^{\mathrm{pr}}$ we can write the direct sum

$$\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3 \oplus \mathcal{H}_4 \oplus \mathcal{H}_5,$$

so

$$P = P_1 \oplus 1 \oplus 1 \oplus 0 \oplus 0, \quad Q = Q_1 \oplus 1 \oplus 0 \oplus 1 \oplus 0,$$

where P_1 and Q_1 are in generic position in $\mathcal{H}_1$. Let $n = \dim \mathcal{H} < +\infty$ and $n_i = \dim \mathcal{H}_i$ for $i = 1, 2, \dots, 5$, and let operator c be as in (1) for Q_1.

Theorem 2. *Consider $\mathcal{A} = \mathbb{M}_n(\mathbb{C})$ and $\varphi = \dfrac{1}{n} tr$, the normalized canonical trace. For $P, Q \in \mathcal{A}^{pr}$ the following conditions are equivalent:*

(i) *P, Q are independent;*

(ii) *$PQ - \varphi(P)Q$ is unitarily equivalent to zero-diagonal matrix;*

(iii) *$PQ - \varphi(Q)P$ is unitarily equivalent to zero-diagonal matrix;*

(iv) *$PQ - \varphi(P)Q$ is a commutator;*

(v) *$PQ - \varphi(Q)P$ is a commutator;*

(vi) $\sum_{k=1}^{n} s_k(I + z(PQ - \varphi(P)Q)) \geq n$ *for all $z \in \mathbb{C}$;*

(vii) $\sum_{k=1}^{n} s_k(I + z(PQ - \varphi(Q)P)) \geq n$ *for all $z \in \mathbb{C}$;*

(viii) $\mathrm{tr}(c^2) + n_2 = \dfrac{1}{n}\left(\dfrac{n_1}{2} + n_2 + n_3\right)\left(\dfrac{n_1}{2} + n_2 + n_4\right)$, *where matrix c is as before.*

Proof. For $A \in \mathcal{A}$ we have the equivalences $\varphi(A) = 0 \Leftrightarrow A$ is unitarily equivalent to zero-diagonal matrix $\Leftrightarrow A$ is a commutator, cf. [16, Problem 182], [17, Chap. II, Problem 209]. Then we apply relation (2) for the equivalence of (i)–(v).

Let us prove the equivalence of (i), (vi) and (vii). Recall that in our case $||A||_1 = \frac{1}{n}\sum_{k=1}^{n} s_k(A)$, here $s_k(A)$ is a k-th singular number of the matrix A, and we may rewrite conditions (ii) and (iii) of Theorem 1 in terms of singular numbers.

Let us show the equivalence of (i) and (viii). We have

$$\varphi(PQ)=\frac{1}{n}\Big(\mathrm{tr}(c^2)+n_2\Big),\quad \varphi(P)\varphi(Q)=\frac{1}{n^2}\Big(\frac{n_1}{2}+n_2+n_3\Big)\Big(\frac{n_1}{2}+n_2+n_4\Big).$$

In particular, if projections P,Q are in generic position, then $n=n_1$ is even and $\varphi(c^2)=1/4$, $\varphi(s^2)=\varphi(P_1-c^2)=1/2-1/4=1/4$ for c, s of (1). This completes the proof. $\square$

Theorem 3. *Let φ be a faithful tracial state on a von Neumann algebra $\mathcal{A}$. Let $P,Q\in\mathcal{A}^{pr}$ be isoclinic with an angle $\theta\in(0,\pi/2)$. Then the following conditions are equivalent:*

(i) *P,Q are independent;*

(ii) $\varphi(P)=\cos^2\theta$.

Proof. If $P\overset{\theta}{\approx}Q$ then $P\sim Q$ by [18, Theorem 10.8]. Hence $\varphi(P)=\varphi(Q)$. By definition of φ we have

$$\begin{aligned}\varphi(PQ)-\varphi(P)\varphi(Q)\varphi(QPQ)-\varphi(P)\varphi(Q)&=\varphi(QPQ-\varphi(Q)P)\\ &=\varphi(\cos^2\theta Q-\varphi(P)Q)=(\cos^2\theta-\varphi(P))\varphi(Q).\end{aligned}$$

This proves the theorem. $\square$

Corollary 1. *Let φ be a faithful tracial state on a von Neumann algebra $\mathcal{A}$. Consider $P,Q\in\mathcal{A}^{pr}$ with $P\wedge Q=0$ and $||P-Q||<1$. Then the following conditions are equivalent:*

(i) *P,Q are independent;*

(ii) $\varphi(P)=\dfrac{1+\sqrt{1-||P-Q||^2}}{2}$.

Proof. By [18, Theorem 10.8] there exists $R\in\mathcal{A}^{\mathrm{pr}}$, $R\le P\vee Q$ and $P\overset{\theta}{\approx}R\overset{\theta}{\approx}Q$ with the angle $\theta=\dfrac{1}{2}\arcsin||P-Q||$. If $P\overset{\theta}{\approx}R$ then $||P-R||=\sin\theta$, $P\sim R$ and $P\wedge R=0$ [18, Theorem 10.5]. Since $P\sim R$ and $R\sim Q$ we have

$$\varphi(P)=\varphi(R)=\varphi(Q)=\cos^2\theta$$

and

$$\begin{aligned}\varphi(PQ)-\varphi(P)\varphi(Q)=\varphi(QPQ)-\varphi(P)\varphi(Q)&=\varphi(\cos^2\theta Q)-\varphi(P)\varphi(Q)\\ &=\varphi(QRQ)-\varphi(R)\varphi(Q)=0\end{aligned}$$

by Theorem 3 and the equality $\cos^2\theta=\dfrac{1+\sqrt{1-||P-Q||^2}}{2}$ due to the well-known trigonometric identity $\cos^2\dfrac{t}{2}=\dfrac{1+\cos t}{2}$. This completes the proof. $\square$

Example 1. Consider $\mathcal{A}=\mathbb{M}_4(\mathbb{C})$ and $\varphi=\dfrac{1}{4}\mathrm{tr}$, $R_x=\begin{pmatrix}x & \pm\sqrt{x-x^2}\\ \pm\sqrt{x-x^2} & 1-x\end{pmatrix}$ for $x\in[0,1]$ and $P=\mathrm{diag}(1,1,0,0)$, $Q=R_a\oplus R_b$ with $a,b\in[0,1]$. Then P,Q are independent and $PQP=QPQ=\mathrm{diag}(R_a,0,0)$ (i.e. $\mathrm{rank}(PQP)=1$), so P, Q are not isoclinic. Consider also

$$Q_1=\begin{pmatrix}\frac12 & 0 & a & b\\ 0 & \frac12 & b & -a\\ a & b & \frac12 & 0\\ b & -a & 0 & \frac12\end{pmatrix}$$

with $a,b\in\mathbb{R}$ and $a^2+b^2=\frac14$. Then P,Q_1 are independent and $PQ_1P=\frac12P$, $Q_1PQ_1=\frac12Q_1$, i.e. P,Q_1 are isoclinic with the angle $\theta=\frac{\pi}{4}$.

Example 2. Consider $\mathcal{A}=\mathbb{M}_3(\mathbb{C})$ and $\varphi=\dfrac{1}{3}\mathrm{tr}$, $P=\mathrm{diag}(1,1,0)$,

$Q = \begin{pmatrix} \frac{9a^2+1}{3} & -3ab & b \\ -3ab & 1-3a^2 & a \\ b & a & \frac{2}{3} \end{pmatrix}$ with $a, b \in \mathbb{R}$ and $a^2 + b^2 = \frac{2}{9}$. Then P, Q are independent and

$PQP = \begin{pmatrix} \frac{9a^2+1}{3} & -3ab & 0 \\ -3ab & 1-3a^2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \neq \lambda P, \forall a, b, \lambda \in \mathbb{R}$, $\mathrm{rank}(PQP) = 2$ and

$QPQ = \begin{pmatrix} 4a^2 + \frac{1}{9} & -4ab & \frac{b}{3} \\ -4ab & 1-4a^2 & \frac{a}{3} \\ \frac{b}{3} & \frac{a}{3} & \frac{2}{9} \end{pmatrix}$.

Theorem 4. *Let φ be a faithful tracial state on a von Neumann algebra $\mathcal{A}$. Consider $P, Q \in \mathcal{A}^{pr}$ such that P is an atom of $\mathcal{A}$. Then the following conditions are equivalent:*

(i) *P, Q are independent;*

(ii) $\varphi(Q) = ||PQ||^2$.

In particular, if atoms $P, Q \in \mathcal{A}^{pr}$ are independent then $\varphi(P) = \varphi(Q)$.

Proof. If P is an atom of $\mathcal{A}$ then the von Neumann algebra $P\mathcal{A}P$ possesses only two projections 0 and P, hence $P\mathcal{A}P = \mathbb{C}P$. We have $PQP = tP$ with $t = ||PQP|| = ||PQ||^2$ and

$$\varphi(PQ) - \varphi(P)\varphi(Q) = \varphi(PQP) - \varphi(P)\varphi(Q) = \varphi(tP) - \varphi(P)\varphi(Q)$$
$$= t\varphi(P) - \varphi(P)\varphi(Q) = (t - \varphi(Q))\varphi(P).$$

The theorem is proved. □

Corollary 2. *Assume that $\mathcal{A} = \mathbb{M}_n(\mathbb{C})$ and $\varphi = \dfrac{1}{n} tr$. Consider $P, Q \in \mathcal{A}^{pr}$ with one-dimensional P. Then the following conditions are equivalent:*

(i) *P, Q are independent;*

(ii) $\varphi(Q) = ||PQ||^2$.

Lemma 2. *We have $P \wedge Q \leq PQP$ for all $P, Q \in \mathcal{B}(\mathcal{H})^{\mathrm{pr}}$.*

Proof. From the inequality $P \wedge Q \leq Q$, we obtain, multiplying both sides by P

$$P \wedge Q = P \cdot P \wedge Q \cdot P \leq PQP.$$

Lemma is proved. □

Remark 1. The projections $P \wedge Q^{\perp} + P^{\perp} \wedge Q$ and $P \vee Q - P \wedge Q$ are analogs of the "symmetric difference" for $P, Q \in \mathcal{A}^{\mathrm{pr}}$. We have $P \wedge Q^{\perp} + P^{\perp} \wedge Q \leq P \vee Q - P \wedge Q$ for all $P, Q \in \mathcal{A}^{\mathrm{pr}}$ [19, Theorem 2].

Theorem 5. *Let φ be a faithful tracial state on a von Neumann algebra $\mathcal{A}$ and projections $P, Q \in \mathcal{A}^{pr}$ be independent with the projection $R = P \vee Q - P \wedge Q \neq 0, I$. Then $\varphi(P) = \varphi(Q) = 1/2$, $\varphi(P \wedge Q + P \vee Q) = 1$ and $\varphi(P \wedge Q) = \varphi(P^{\perp} \wedge Q^{\perp})$, $\varphi(P \vee Q) = \varphi(P^{\perp} \vee Q^{\perp})$.*

If, moreover, P and Q are independent, then $\varphi(P \wedge Q) \leq 1/4$ and $\varphi(P \vee Q) \geq 3/4$.

Proof. DeMorgan's law applied to $P \vee Q - P \wedge Q$ yields $R^{\perp} = P \wedge Q + P^{\perp} \wedge Q^{\perp}$ and

$$\varphi(P - P \wedge Q) = \varphi(RP) = \varphi(P)\varphi(R) = \varphi(P)(\varphi(P \vee Q - Q) + \varphi(Q - P \wedge Q))$$
$$= \varphi(P)(\varphi(P - P \wedge Q) + \varphi(Q - P \wedge Q)) \tag{3}$$

by the equivalence

$$P \vee Q - Q \sim P - P \wedge Q, \tag{4}$$

cf. [20, Chap. V, Proposition 1.6]. Similarly we have

$$\varphi(Q - P \wedge Q) = \varphi(Q)(\varphi(Q - P \wedge Q) + \varphi(P - P \wedge Q)). \tag{5}$$

Summing relations (3) and (5) term by term, we obtain $\varphi(P) + \varphi(Q) = 1$.

Since P and Q are independent with the projection $R^\perp$, $Q^\perp$ is independent with the projection $R^\perp$ by Lemma 1. Again we sum the relations

$$\varphi(R^\perp P) = \varphi(P \wedge Q) = \varphi(R^\perp)\varphi(P), \quad \varphi(R^\perp Q^\perp) = \varphi(P^\perp \wedge Q^\perp) = \varphi(R^\perp)\varphi(Q^\perp),$$

term by term and conclude that $\varphi(P) + \varphi(Q^\perp) = 1$. This combined with the equality $\varphi(P) + \varphi(Q) = 1$ yields $\varphi(P) = 1/2$. Similarly we have $\varphi(Q) = 1/2$.

Now by (4) we obtain $\varphi(P \wedge Q + P \vee Q) = 1$ and again by DeMorgan's law we have

$$\varphi(P \wedge Q) = 1 - \varphi(P \vee Q) = \varphi(I - P \vee Q) = \varphi(P^\perp \wedge Q^\perp).$$

Similarly, $\varphi(P \vee Q) = \varphi(P^\perp \vee Q^\perp)$.

Recall that $P \wedge Q \leq PQP$ for all $P, Q \in \mathcal{A}^{\mathrm{pr}}$ by Lemma 2. If all the projection pairs of the projection triple $\{P, Q, R = P \vee Q - P \wedge Q\}$ are independent, then

$$\varphi(P \wedge Q) \leq \varphi(PQP) = \varphi(PQ) = \varphi(P)\varphi(Q) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$$

and $\varphi(P \vee Q) \geq 3/4$ by the equality $\varphi(P \wedge Q + P \vee Q) = 1$. Thus $\varphi(R) \geq 1/2$. The theorem is proved. $\square$

Remark 2. If $P, Q \in \mathcal{A}^{\mathrm{pr}}$ with $PQ = QP$ and $\varphi(P), \varphi(Q) \in (0, 1)$, then by [22, Proposition 2.5] the following conditions are equivalent:

(i) P, Q, and $R = P \vee Q - P \wedge Q$ are pairwise independent;

(ii) $\varphi(P) = \varphi(Q) = \varphi(R) = 1/2$.

Corollary 3. *Let φ be a faithful tracial state on a von Neumann algebra $\mathcal{A}$ and $P, Q \in \mathcal{A}^{pr}$ be independent with the projection $P \wedge Q^\perp + P^\perp \wedge Q \neq 0, I$. Then $\varphi(P) = \varphi(Q) = 1/2$, $\varphi(P \wedge Q + P \vee Q) = 1$ and $\varphi(P \wedge Q) = \varphi(P^\perp \wedge Q^\perp)$, $\varphi(P \vee Q) = \varphi(P^\perp \vee Q^\perp)$. If, moreover, P and Q are independent, then $\varphi(P^\perp \wedge Q) \leq 1/4$ and $\varphi(P^\perp \vee Q) \geq 3/4$.*

Proof. We have $R^\perp = P^\perp \vee Q - P^\perp \wedge Q$ for $R = P \wedge Q^\perp + P^\perp \wedge Q$ by DeMorgan's law and can apply Lemma 1 and Theorem 5. $\square$

Example 3. Consider $\mathcal{A} = \mathbb{M}_4(\mathbb{C})$ and $\varphi = \dfrac{1}{4}\mathrm{tr}$, $P = \begin{pmatrix} \frac{5}{6} & -\frac{1}{3} & -\frac{1}{6} & 0 \\ -\frac{1}{3} & \frac{1}{3} & -\frac{1}{3} & 0 \\ -\frac{1}{6} & -\frac{1}{3} & \frac{5}{6} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, $Q = \begin{pmatrix} \frac{5}{6} & \frac{1}{3} & -\frac{1}{6} & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ -\frac{1}{6} & \frac{1}{3} & \frac{5}{6} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

Then $\varphi(PQ) = \frac{5}{18} \neq \frac{1}{4} = \varphi(P)\varphi(Q)$, i.e. P and Q are non independent. It is clear that $P \vee Q = \mathrm{diag}(1, 1, 1, 0)$ and $P \wedge Q = \begin{pmatrix} \frac{1}{2} & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$. For $R = P \vee Q - P \wedge Q$ we have

$$\varphi(PR) = 1/4 = \varphi(P)\varphi(R) = \varphi(QR) = \varphi(Q)\varphi(R).$$

Consider also $Q_1 = \begin{pmatrix} \frac{2}{3} & \frac{1}{3} & -\frac{1}{3} & 0 \\ \frac{1}{3} & \frac{2}{3} & \frac{1}{3} & 0 \\ -\frac{1}{3} & \frac{1}{3} & \frac{2}{3} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, then $PQ_1 = P \wedge Q = P \wedge Q_1$. It is clear that $\varphi(PQ_1) = \frac{1}{4} = \varphi(P)\varphi(Q_1)$ and $P \vee Q_1 = \mathrm{diag}(1, 1, 1, 0)$. For $R_1 = P \vee Q_1 - P \wedge Q_1$ we have $\varphi(R_1) = 1/2$ and

$$\varphi(PR_1) = 1/4 = \varphi(P)\varphi(R_1) = \varphi(Q_1 R_1) = \varphi(Q_1)\varphi(R_1).$$

So all the projection pairs of the projection triple $\{P, Q_1, R_1\}$ are independent.

Theorem 6. *Let φ be tracial normal state on a von Neumann algebra $\mathcal{A}$. Let $n \geq 2$ and $P_1, \ldots, P_n \in \mathcal{A}^{pr}$ be with $P_1 + \ldots + P_n = I$, $\mathcal{P}_n(A) = \sum_{k=1}^n P_k A P_k$ for $A \in \mathcal{A}$. Assume that $P, Q \in \mathcal{A}^{pr}$ and $\tilde{P} = \mathcal{P}_n(P)$, $\tilde{Q} = \mathcal{P}_n(Q) \in \mathcal{A}^{pr}$. Then the following conditions are equivalent:*

(i) *P, Q are independent;*

(ii) *$\tilde{P}, \tilde{Q}$ are independent.*

Proof. For all $A \in \mathcal{A}$ and $n \geq 2$ by [11, Lemma 2] we have the representation

$$\mathcal{P}_n(A) = \frac{1}{2^{n-1}} \sum_{k=1}^{2^{n-1}} S_k A S_k,$$

where the unitaries $S_k \in \mathcal{A}^{\mathrm{sa}}$, $k = 1, \ldots, 2^{n-1}$, have the form

$$P_1 \pm P_2 \pm \cdots \pm P_n.$$

Therefore, the projection $\tilde{P}$ (resp., $\tilde{Q}$) is a convex combination of projections $S_k P S_k$ (resp., $S_k Q S_k$), $k = 1, \ldots, 2^{n-1}$. Since $\mathcal{A}^{\mathrm{pr}}$ belongs to the set

$$\mathrm{ext}\{X \in \mathcal{A}^+ : \, ||X|| \leq 1\}$$

of the extreme points of the positive part of the unit ball of $\mathcal{A}$ [21, Chapter 2, 2.8.14], we infer that $\tilde{P} = S_k P S_k$ (resp., $\tilde{Q} = S_k Q S_k$) for all $k = 1, \ldots, 2^{n-1}$.

(i) $\Rightarrow$ (ii). We have

$$\begin{aligned} \varphi(\tilde{P}\tilde{Q}) &= \varphi(S_k P S_k \cdot S_k Q S_k) = \varphi(S_k \cdot PQ \cdot S_k) = \varphi(PQ) \\ &= \varphi(P)\varphi(Q) = \varphi(S_k P S_k)\varphi(S_k Q S_k) = \varphi(\tilde{P})\varphi(\tilde{Q}) \end{aligned}$$

for all $k = 1, \ldots, 2^{n-1}$.

(ii) $\Rightarrow$ (i). We have

$$\begin{aligned} \varphi(PQ) &= \varphi(S_k \cdot PQ \cdot S_k) = \varphi(S_k P S_k \cdot S_k Q S_k) = \varphi(\tilde{P}\tilde{Q}) \\ &= \varphi(\tilde{P})\varphi(\tilde{Q}) = \varphi(S_k P S_k)\varphi(S_k Q S_k) = \varphi(P)\varphi(Q) \end{aligned}$$

for all $k = 1, \ldots, 2^{n-1}$. The theorem is proved. $\square$

4. INDEPENDENCE AND DETERMINATION OF TRACIAL STATES

In this section, we assume that $\mathcal{A}$ is an atomless von Neumann algebra with a faithful, normal, normalized trace φ. Let $S(\mathcal{A})$ be the Murray–von Neumann algebra associated with $\mathcal{A}$. Since φ is finite we have $S(\mathcal{A}) = S(\mathcal{A}, \varphi)$. Let $A \in S(\mathcal{A})$ and let $A = V|A|$ be the polar decomposition of A. Then $l(A) = VV^*$ and $r(A) = V^*V$ are left and right supports of the element A, respectively. The projection $s(A) = l(A) \vee r(A)$ is the support of the element A. It is clear that $r(A) = s(|A|)$ and $l(A) = s(|A^*|)$. We denote by m the linear Lebesgue measure on $\mathbb{R}$.

For every $X \in \mathcal{A}$, the generalised singular value function $\mu(X)$, denoted $t \mapsto \mu(t, X)$ for $t \in [0, 1]$, is defined by the formula (see, e.g., [23, 24])

$$\mu(t, X) = \inf\{||XP|| : P \in \mathcal{A}^{\mathrm{pr}}, \ \varphi(I - P) \leq t\}.$$

For a self-adjoint element $B \in S(\mathcal{A})$ wtih the Jordan decomposition $B = B_+ - B_-$, let $\lambda(B) = \lambda(\cdot, B)$ be the eigenvalue function of B (also known as the spectral scale, see [25, 26] and [27]) defined by

$$\lambda(t, B) = \begin{cases} \mu(t, B_+), \quad 0 < t < \varphi(\mathrm{supp}(B_+)), \\ \lim_{\varepsilon \to 0+} \lambda(1 - t - \varepsilon, B_-), \quad \varphi(\mathrm{supp}(B_+)) \leq t < 1. \end{cases}$$

Theorem 7. *Let φ, ψ be tracial normal states on an atomless von Neumann algebra $\mathcal{A}$. If they have identical independent events, then they coincide.*

Proof. Recall that if $B = B^* \in S(\mathcal{A}, \varphi)$, then there exists an atomless commutative weakly closed $*$-subalgebra $\mathcal{B}$ in $\mathcal{A}$ containing the spectral family of the operator B, and a $*$-isomorphism V acting from $S(\mathcal{B}, \varphi)$ onto $S([0, \varphi(s(B))), m)$ such that $V(B) = \lambda(B)$ and $\lambda(V(f)) = \lambda(f)$ for every $f = f^* \in S(\mathcal{B}, \varphi)$, see Proposition 3.1 in [28]. Then the Abelian algebra $\mathcal{B}$ is $*$-isomorphic to the algebra of multiplicators $L_\infty(\Omega, \mathfrak{A}, \mu)$ on a localized measure space $(\Omega, \mathfrak{A}, \mu)$, acting on a Hilbert space $L_2(\Omega, \mathfrak{A}, \mu)$. Moreover, $(\Omega, \mathfrak{A}, \mu)$ is atomless. If $P, Q \in \mathcal{B}^{\text{pr}}$ then there exist $S, T \in \mathfrak{A}$ such that P and Q are multiplicators by the indicators χ_S and χ_T, respectively. We apply Theorem 1 of [29] to the restrictions $\varphi|_{\mathcal{B}^{\text{pr}}}$, $\psi|_{\mathcal{B}^{\text{pr}}}$ and obtain $\varphi|_{\mathcal{B}^{\text{pr}}} = \psi|_{\mathcal{B}^{\text{pr}}}$. Thus $\varphi(B) = \psi(B)$ for all selfadjoint operators $B \in \mathcal{A}$ by linearity and normality of the states φ, ψ and the Spectral Theorem. Since every operator $A \in \mathcal{A}$ has the form $B + iC$ with the selfadjoint operators $B, C \in \mathcal{A}$ and $i \in \mathbb{C}$ with $i^2 = -1$, we have $\varphi = \psi$ by linearity of φ and ψ. The theorem is proved. □

Corollary 4. *Let φ, ψ be tracial normal states on an atomless von Neumann algebra $\mathcal{A}$. If they have identical mutually favorable events in the sense that, for any $P, Q \in \mathcal{A}^{\text{pr}}$*

$$\varphi(PQ) \geq \varphi(P)\varphi(Q) \Longleftrightarrow \psi(PQ) \geq \psi(P)\psi(Q), \tag{6}$$

then φ and ψ coincide.

Proof. It is clear by complementation (i.e. we consider the pair $P^\perp, Q$) that (6) implies that φ and ψ have identical mutually unfavorable events as well. It follows that φ and ψ have identical independent events, and we apply Theorem 7. □

Theorem 8. *Let φ, ψ be tracial normal states on an atomless von Neumann algebra $\mathcal{A}$. If*

$$\varphi(P) = 1/2 \Longleftrightarrow \psi(P) = 1/2,$$

for any $P \in \mathcal{A}^{pr}$ then $\varphi = \psi$.

Proof. We repeat the proof of Theorem 7 applying Corollary 2 of [29]. □

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