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Abstract Theory of Hybridizable Discontinuous Galerkin Methods for Second-Order Quasilinear Elliptic Problems

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Abstract—An abstract theory for discretizations of second-order quasilinear elliptic problems based on the mixed-hybrid discontinuous Galerkin method. Discrete schemes are formulated in terms of approximations of the solution to the problem, its gradient, flux, and the trace of the solution on the interelement boundaries. Stability and optimal error estimates are obtained under minimal assumptions on the approximating space. It is shown that the schemes admit an efficient numerical implementation.

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1. INTRODUCTION

Below, a family of methods from the class of hybridizable discontinuous Galerkin (HDG) finite element methods (FEM) intended for the approximate solution of second-order elliptic equations is proposed and analyzed in terms of an abstract theory. HDG schemes are hybridizable versions of discontinuous Galerkin schemes (DG or DGFEM schemes) and occupy an intermediate position between the finite-volume method and FEM (for more detail on DG schemes, see, e.g., [1, 2]). HDG schemes were introduced in [3]. They have much in common with mixed hybridizable FEM schemes [4] and are also formulated in terms of approximations to u, q, and λ, where u is the solution of the problem, q is the flux vector, and λ is the trace of u on the interelement boundaries. The characteristic features of HDG schemes are as follows: (a) they are based on discontinuous finite element spaces (of arbitrary accuracy), (b) locally (elementwise) conservative, and (c) admit efficient implementation. For example, when such methods are applied to linear problems, the unknowns corresponding to u and q can be elementwise eliminated from the scheme to obtain a system of algebraic equations for determining only the unknown corresponding to λ. For problems with a symmetric positive definite operator, the matrix of the system is symmetric and positive definite as well. After λ is found, the other unknowns are also recovered elementwise. Note also that HDG schemes do no face difficulties related to the approximation of the main and natural boundary conditions.

The theory of HDG methods being developing at present is associated with linear problems. In contrast, we consider the nonlinear problem

$$-\nabla \cdot k(x, u, \nabla u) + k_0(x, u, \nabla u) = f, \quad x \in \Omega, \quad u = 0, \quad x \in \partial\Omega, \tag{1}$$

where Ω is a bounded polyhedron in R^d with boundary $\partial\Omega$, $f \in L_2(\Omega)$, and $d \geq 2$. The homogeneous Dirichlet problem is used only for methodological reasons. The proposed approximate method and its analysis can be straightforwardly extended to inhomogeneous boundary conditions, including mixed ones.

Assume that the coefficients $k(x, \xi) = (k_1(x, \xi), k_2(x, \xi), \ldots, k_d(x, \xi))$ and $k_0(x, \xi)$ are continuous functions of $x \in \bar{\Omega}$ for any $\xi \in R^{d+1}$ and $k_i(\cdot, 0) = 0$, $i = 0, \ldots, d$. Moreover, for any $x \in \Omega$, we assume that

$$\left|\sum_{i=0}^{d}(k_i(x, \xi) - k_i(x, \eta))v_i\right| \leq \beta|\xi - \eta||v| \quad \forall \xi, \eta, v \in R^{d+1}, \quad \beta > 0, \tag{2}$$

$$\sum_{i=0}^{d}(k_i(x,\xi)-k_i(x,\eta))(\xi_i-\eta_i)\ge\alpha\sum_{i=1}^{d}(\xi_i-\eta_i)^2\quad\forall\xi,\eta\in R^{d+1},\quad\alpha>0. \tag{3}$$

To construct a discrete scheme, Eq. (1) is written as a system of first-order equations involving, in addition to u and q, a new unknown σ:

$$\sigma = \nabla u,\quad q = k(x,u,\sigma),\quad -\nabla\cdot q+k_0(x,u,\sigma) = f. \tag{4}$$

According to the theory of monotone operators, the above conditions ensure that problem (1) has a unique weak solution in the Sobolev space $H^1(\Omega)$ (see, e.g., [5]); moreover, in (4), $\sigma\in[L_2(\Omega)]^d$, $u\in H_0^1(\Omega)$, $q\in H(\mathrm{div};\Omega)=\{q\in[L_2(\Omega)]^d:\nabla\cdot q\in L_2(\Omega)\}$.

The new vector unknown σ introduced into the schemes under consideration is a feature in which they differ from known HDG schemes and mixed hybridizable FEM schemes. As a result, the complexity of the method increases insignificantly, since the unknown can elementwise eliminated from the scheme.

This paper is organized as follows. In Section 2, we define a triangulation $\mathcal{T}_h$ of Ω and abstract spaces of discontinuous finite elements. In Section 3, a technique for the design of DG schemes is used to define the initial family of HDG schemes. In Section 4, an equivalent formulation of the discrete problem is derived by eliminating the variables corresponding to σ and q from the scheme. This formulation is called basic. It uses the concept of a discrete gradient, whose properties are examined in Section 5. Simultaneously, we introduce two basic constraints on the finite element spaces ((H_1) and (H_2)), which ensure the derivation of a stability estimate (Section 5) and an error estimate for the method (Sections 6, 7). The accuracy of the scheme is estimated in terms of the errors of the orthogonal L_2-projections of u, q, and λ onto the corresponding finite element spaces. Optimal error estimates are obtained in Theorem 7 under two conditions (H_3) and (H_4) added to (H_1) and (H_2). In Section 8, we discuss consequences of the abstract conditions and give examples of finite element spaces satisfying them. Finally, in Section 9, we propose and examine a solution method for systems of algebraic equations in the case of a linear original problem and an iterative method for nonlinear problems. The condition number of the arising matrices is estimated.

2. SPACES OF DISCONTINUOUS FINITE ELEMENTS

Let h be a small positive parameter, and let $K_1, K_2, \ldots, K_{N(h)}$ be polyhedral domains in R^d generating a partition of Ω (triangulation of Ω) into subdomains of maximum diameter h (which are referred to as finite elements); i.e., (T_1) $h=\max_i\mathrm{diam}(K_i)$, ($T_2$) $K_i\cap K_j=\emptyset$ for $i\neq j$, and (T_3) $\bar{\Omega}=\bigcup_{i=1}^{N(h)}\bar{K}_i$. Furthermore, let $\mathcal{T}_h$ denote the set of all finite elements and K be an arbitrary element of $\mathcal{T}_h$. By the faces of K, we mean its $(d-1)$-dimensional faces. The triangulation $\mathcal{T}_h$ is assumed to be conformal; i.e., (T_4) any face of K is either a subset of the boundary $\partial\Omega$ or a face of a neighboring element $L\in\mathcal{T}_h$. Additionally, $\mathcal{T}_h$ is assumed to be affinely equivalent [6]; i.e., (T_5) every $K\in\mathcal{T}_h$ is the image of some $\hat{K}\in\hat{\mathcal{T}}$ under a invertible affine transformation: $K=F_K(\hat{K})$, $F_K(x)=B_K\hat{x}+b_K$, where $\hat{\mathcal{T}}$ is a finite set of domains of various shapes with a unit diameter, which are referred to as basis elements. Assume also that (T_6) the triangulation is regular [6]; i.e., $\|B_K\|\sim h$ and $\|B_K^{-1}\|\sim h^{-1}$ for any $K\in\mathcal{T}_h$. For arbitrary functions f and g of h and, possibly, of K, e, $\mathcal{T}_h$, and $\bar{\mathcal{E}}_h$, the expression $f\sim g$ ($f(v)\sim g(v)$) means that there are constants such that $g\le f\le Cg$($cg(v)\le f(v)\le Cg(v)$ for all admissible v). Here and below, c and C possibly with indices denote various positive constants independent of h, K, e, $\mathcal{T}_h$, or $\bar{\mathcal{E}}_h$.

By an inner face of elements from $\mathcal{T}_h$, we mean a face e shared by two elements, say, K and L : $e=\bar{K}\cap\bar{L}$. Otherwise, e is said to be a boundary face. Let $\mathcal{E}_h$ and $\mathcal{E}_h^{\partial}$ denote the sets of all inner and boundary faces, respectively, and $\bar{\mathcal{E}}_h=\mathcal{E}_h\cup\mathcal{E}_h^{\partial}$ be the set of all faces. Let e be an arbitrary face from $\bar{\mathcal{E}}_h$ and n_K be the field of unit normals on ∂K directed outward with respect to K. By ∂K, we mean both the set of faces K and the boundary of K, and $|K|$ and $|e|$ stand for the d- and $(d-1)$-dimensional measures of K and e, respectively. The restriction (trace) of u onto D is designated as $u|_D$. Similarly, for an arbitrary set of functions $H(D)$, let $H(D)|_S = \{u|_S, u \in H(D)\}$ for $S \subset \overline{D}$.

The conventional notation $H^s(D)$ is used for the Sobolev spaces of order s equipped with the standard norm $\|\cdot\|_{s, D}$ and seminorms $|\cdot|_{k, D}$. By $\|\cdot\|_{0, D}$ and $(\cdot, \cdot)_D$, we mean the norm and the inner product in both the inner space $L_2(D)$ and the vector space $[L_2(D)]^d$, while $\|\cdot\|_{s, D}$ and $|\cdot|_{k, D}$ denote the standard norm and seminorm in $[H^s(D)]^d$. We use the following notation:

$$(w, \eta)_{\mathcal{T}_h} = \sum_{K \in \mathcal{T}_h} (w, \eta)_K, \quad (w, \eta)_{\partial\mathcal{T}_h} = \sum_{K \in \mathcal{T}_h} (w, \eta)_{\partial K}, \quad (w \cdot n, \eta)_{\partial\mathcal{T}_h} = \sum_{K \in \mathcal{T}_h} (w \cdot n_K, \eta)_{\partial K}.$$

The space $H^s(\mathcal{T}_h)$ is the set of functions defined on Ω whose restrictions to an arbitrary element $K \in \mathcal{T}_h$ belong to $H^s(K)$. This space is equipped with the norm $\|\cdot\|^2_{s, \mathcal{T}_h} = \sum_{K \in \mathcal{T}_h} \|\cdot\|^2_{s, K}$. Let $H(\text{div}; \mathcal{T}_h) = \{w \in [L_2(\Omega)]^d : w|_K \in H(\text{div}; K), K \in \mathcal{T}_h\}$.

Every element $K \in \mathcal{T}_h$ and every face $e \in \overline{\mathscr{E}}_h$ are assigned finite-dimensional spaces

$$(H_0) \quad V(K) \subset H^1(K), \quad W(K) \subset [H^1(K)]^d, \quad \Lambda(e) \subset L_2(e)$$

such that

(H_0') $V(K)$, $W(K)$, and $\Lambda(e)$ contain constant functions.

Throughout this paper, conditions (H_0) and (H_0') are assumed to hold. Define the finite-dimensional spaces of functions

$$V_h = \{v \in L_2(\Omega) : v|_K \in V(K), K \in \mathcal{T}_h\}, \quad W_h = \{w \in [L_2(\Omega)]^d : w|_K \in W(K), K \in \mathcal{T}_h\},$$

$$\Lambda_h = \{\lambda \in L_2(\overline{\mathscr{E}}_h) : \lambda|_e \in \Lambda(e), e \in \mathscr{E}_h; \lambda|_e = 0, e \in \mathscr{E}_h^\partial\}.$$

Here, V_h is used for approximating the solution u; W_h, for approximating σ and q; and Λ_h, for approximating the trace of u on the interelement boundaries. Note that functions from V_h and W_h do not need to be continuous on Ω and no boundary conditions are imposed on them. Thus, $V_h \subset H^1(\mathcal{T}_h)$ and $W_h \subset H(\text{div}; \mathcal{T}_h)$. Moreover, $u|_e, w \cdot n|_e \in L_2(e)$ for $u \in V_h$ and $w \in W_h$.

Additionally, π_V, π_W, and π_Λ denote local L_2-orthoprojectors in V_h, W_h, and Λ_h, respectively; i.e., $(\pi_V u - u, v)_K = 0\ \forall v \in V(K)$, $(\pi_W q - q, w)_K = 0\ \forall w \in W(K)$, and $(\pi_\Lambda \lambda - \lambda, \mu)_e = 0\ \forall \mu \in \Lambda(e)$.

3. FAMILY OF HDG SCHEMES

First, we formulate auxiliary relations for the unknowns in Eqs. (4) and for the new unknown $\lambda = u|_{\overline{\mathscr{E}}_h}$. Multiplying the restriction of the first equation in (4) to an element K by $w \in W(K)$ and integrating the result by parts, we obtain $(\sigma, w)_K + (u, \nabla \cdot w)_K - (u, w \cdot n_K)_{\partial K} = 0$. Replacing u by λ in the third term yields

$$(\sigma, w)_K + (u, \nabla \cdot w)_K - (\lambda, w \cdot n_K)_{\partial K} = 0 \quad \forall w \in W(K).$$

The following relations are straightforward:

$$\begin{aligned} (q - k(\cdot, u, \sigma), w)_K &= 0 \quad \forall w \in W(K), \\ -(\nabla \cdot q, v)_K + (k_0(\cdot, u, \sigma), v)_K &= (f, v)_K \quad \forall v \in V(K). \end{aligned} \tag{5}$$

Since $q \in H(\text{div}; \Omega)$, we conclude that, under the assumption $q|_{\overline{\mathscr{E}}_h} \cdot n \in L_2(\overline{\mathscr{E}}_h)$, it is true that

$$(q \cdot n, \mu)_{\partial\mathcal{T}_h} = 0 \quad \forall \mu \in \Lambda_h.$$

These relations for the sought variables (σ, u, q, λ) are used to set up an approximate problem. Substituting the unknown functions by their approximation produces the following discrete problem for determining $(\sigma_h, q_h, u_h, \lambda_h) \in W_h \times W_h \times V_h \times \Lambda_h$:

$$(\sigma_h, w)_K + (u_h, \nabla \cdot w)_K - (\lambda_h, w \cdot n_K)_{\partial K} = 0 \quad \forall w \in W(K), \quad K \in \mathcal{T}_h, \tag{6}$$

$$(q_h - k(\cdot, u_h, \sigma_h), w)_K = 0 \quad \forall w \in W(K), \quad K \in \mathcal{T}_h, \tag{7}$$

$$-(\nabla \cdot q_h, v)_K + (k_0(\cdot, u_h, \sigma_h), v)_K = (f, v)_K \quad \forall v \in V(K), \quad K \in \mathcal{T}_h, \tag{8}$$

$$(q_h \cdot n, \mu)_{\partial \mathcal{T}_h} = 0 \quad \forall \mu \in \Lambda_h. \tag{9}$$

Note that scheme (6)–(9) is locally (elementwise) conservative. Indeed, let D be the union of an arbitrary set of finite elements and $v|_D = 1$. After summation (8) over all $K \in D$, we obtain the relation

$$\int_D (f - k_0(\cdot, u_h, \sigma_h))dx + \int_{\partial D} q_h \cdot n_D ds = 0,$$

since, by virtue of (9), $\int_e (q_h|_K - q_h|_L) \cdot n_K ds = 0$ for $e = \bar{K} \cap \bar{L}$.

The method used to construct a discrete scheme is characteristic of finite-volume methods or DG schemes. It can also be noted that, in the special case when Eq. (1) coincides with the Poisson equation $(q = k(x, u, \sigma) = \sigma, k_0(x, u, \sigma) = 0)$, scheme (6)–(9) coincides with the hybridizable mixed FEM scheme with suitably defined finite element spaces (see [4; 7, pp. 178–181]).

Equations (6)–(9) define the desired family of schemes. A particular scheme is obtained by specifying the spaces $\{V(K), W(K), \Lambda(e)\}$. The main objective of the subsequent study is to identify the conditions on the families of spaces in assumption (H_0) that are responsible for the stability and accuracy of the scheme.

4. BASIC FORMULATION OF THE SCHEME

We introduce additional definitions and notation. Note that $\sigma_h|_K$ is uniquely determined by Eq. (6) if $u_h|_K$ and $\lambda_h|_{\partial K}$ are given. By definition, $\mathbf{u}_h|_K = (u_h|_K, \lambda_h|_{\partial K})$ is an approximation of $\mathbf{u}|_K = (u|_K, u|_{\partial K})$, and the function σ_h is an approximation of $\sigma = \nabla u$. In this context, the operator $(u_h, \lambda_h) \longrightarrow \sigma_h$ defined elementwise by (6) can be treated as a discrete analogue of the gradient. Denote it by ∇_h, so $\sigma_h = \nabla_h \mathbf{u}_h$. This can be formalized as follows.

Let $\Lambda(\partial K) = \{\mu \in L_2(\partial K) : \mu|_e \in \Lambda(e), e \in \partial K\}$. We introduce the space $\mathbf{V}(K) = V(K) \times \Lambda(\partial K)$ with a common element $\mathbf{v} = (v, \mu)$ and define the operator $\nabla_K : \mathbf{V}(K) \longrightarrow W(K)$ according to the rule

$$(\nabla_K \mathbf{v}, w)_K = -(v, \nabla \cdot w)_K + (\mu, w \cdot n_K)_{\partial K} \quad \forall w \in W(K). \tag{10}$$

The operator ∇_K will be referred to as a local discrete gradient. Setting $\mathbf{V}_h = V_h \times L_h = \{\mathbf{v} = (v, \mu) : \mathbf{v}|_K = (v|_K, \mu|_{\partial K}) \in \mathbf{V}(K)\}$, we define the discrete gradient operator $\nabla_h : \mathbf{V}_h \longrightarrow W_h$ by the relations $(\nabla_h \mathbf{v})|_K = \nabla_K(\mathbf{v}|_K)$; i.e.,

$$(\nabla_h \mathbf{v}, w)_K = -(v, \nabla \cdot w)_K + (\mu, w \cdot n_K)_{\partial K} \quad \forall w \in W_h, \quad K \in \mathcal{T}_h. \tag{11}$$

Let us obtain the basic formulation of scheme (6)–(9). Let $\mathbf{v}_h = (v_h, \mu_h) \in \mathbf{V}_h$. For $w = q_h$ in (11), it is true that $-(v_h, \nabla \cdot q_h)_K = (\nabla_h \mathbf{v}_h, q_h)_K - (\mu_h, q_h \cdot n_K)_K$. Combining this equality with (8) yields

$$(q_h, \nabla_h \mathbf{v}_h)_K + (k_0(\cdot, u_h, \sigma_h), v_h)_K - (q_h \cdot n_K, \mu_h)_{\partial K} = (f, v_h)_K. \tag{12}$$

Adding (12) to (7) with $w = -\nabla_h \mathbf{v}_h$, we derive the identity

$$(k(\cdot, u_h, \sigma_h), \nabla_h \mathbf{v}_h)_K + (k_0(\cdot, u_h, \sigma_h), v_h)_K - (q_h \cdot n_K, \mu_h)_{\partial K} = (f, v_h)_K. \tag{13}$$

Note that $\sigma_h = \nabla_h \mathbf{u}_h$ according to (6). Summing up (13) over all K and using (9), we obtain an equivalent problem for determining $\mathbf{u}_h = (u_h, \lambda_h) \in \mathbf{V}_h$, namely,

$$\mathbf{a}_h(\mathbf{u}_h, \mathbf{v}_h) = \mathbf{f}_h(\mathbf{v}_h) \quad \forall \mathbf{v}_h = (v_h, \mu_h) \in \mathbf{V}_h, \tag{14}$$

where $\mathbf{a}_h(\mathbf{u}_h, \mathbf{v}_h) = (k(\cdot, u_h, \nabla_h\mathbf{u}_h), \nabla_h\mathbf{v}_h)_{\mathcal{T}_h} + (k_0(\cdot, u_h, \nabla_h\mathbf{u}_h), v_h)_{\mathcal{T}_h}$ and $\mathbf{f}_h(\mathbf{v}_h) = (f, v_h)_{\mathcal{T}_h}$. Then $\sigma_h = \nabla_h\mathbf{u}_h$ and $q_h = \pi_W(k(\cdot, u_h, \sigma_h))$ are elementwise determined using the found solution.

Clearly, the properties of the form $\mathbf{a}_h$ depend on the properties of the discrete gradient. Specifically, condition (3) implies the estimate

$$\mathbf{a}_h(\mathbf{u}_1, \mathbf{u}_1 - \mathbf{u}_2) - \mathbf{a}_h(\mathbf{u}_2, \mathbf{u}_1 - \mathbf{u}_2) \ge \alpha\|\nabla_h(\mathbf{u}_1 - \mathbf{u}_2)\|^2_{0,\Omega} \quad \forall \mathbf{u}_1, \mathbf{u}_2 \in \mathbf{V}_h. \tag{15}$$

If the functional $\mathbf{v} \longrightarrow \|\nabla_h\mathbf{v}\|_{0,\Omega}$ determines the norm in $\mathbf{V}_h$, then (15) means that $\mathbf{a}_h$ is strongly monotone. In this case, the unique solvability of the problem is easy to prove by relying on the theory of monotone operators. Clearly, this property of the discrete gradient is critical to the unique solvability and stability of the scheme. In what follows, we are interested only in the case where problem (14) is uniquely solvable. Obviously, the solution $(u_h, \lambda_h, \sigma_h, q_h)$ is a unique solution of scheme (6)–(9).

5. STABILITY OF THE SCHEME

Let $V(\partial K) = (v|_{\partial K}, v \in V(K)\}$ and $\mathbf{V}_0(K) = \{(v, \mu) \in \mathbf{V}(K) : (v, \mu) = (c, c), c \in R\}$ be a one-dimensional subspace of $\mathbf{V}(K)$ (of constant functions).

The basic constraints on the choice of the spaces $V(K)$, $W(K)$, and $\Lambda(e)$ are as follows.

(H_1) The following relation holds only for $(v, \mu) \in \mathbf{V}_0(K)$:

$$(v, \mu) \in V(K) \times \Lambda(\partial K) : (v, \nabla \cdot w)_K - (\mu, w \cdot n_K)_{\partial K} = 0 \quad \forall w \in W(K).$$

(H_2) $\{V(K)\}_{K \in \mathcal{T}_h}$, $\{W(K)\}_{K \in \mathcal{T}_h}$, and $\{\Lambda(e)\}_{e \in \mathcal{E}_h}$ are affinely equivalent families. (If $\hat{K}$ is the basis element corresponding to K, then, for example, $V(K)$ is the image of some $V(\hat{K})$ under an affine transformation (see [6]). An important example is given by spaces of polynomials. For particular spaces, condition (H_1) suffices to verify only on $\hat{K}$ by virtue of (H_2).)

As compared with (H_0), (H_0'), and (H_2), condition (H_1) is more restrictive: it relates all three spaces (V, W, Λ) and, according to it, the local discrete gradient must vanish only on constant functions. In view of condition (H_2), in what follows, we will use the standard FEM technique for estimating the approximation error and reverse inequalities of the type $|w|_{1,K} \le ch^{-1}|w|_{0,K}$, which are based on norm equivalence in finite-dimensional spaces.

On $\mathbf{V}(K)$ we define the seminorm

$$|\mathbf{v}|^2_{1,K} = \|\nabla v\|^2_{0,K} + h^{-1}\|v - \mu\|^2_{0,\partial K}, \quad \mathbf{v} = (v, \mu).$$

It is easy to see that this is the norm on the quotient space $\mathbf{V}(K)/\mathbf{V}_0(K)$.

Theorem 1. *Assume that conditions* $(H_{1,2})$ *hold; i.e.,* (H_1) *and* (H_2) *are both satisfied.* (*The notation* $(H_{1,2,3,\ldots})$ *is understood in a similar manner.*) *Then, for* $\mathbf{v} \in \mathbf{V}(K)$,

(a) $\nabla_K\mathbf{v} = 0$ *is equivalent to* $\mathbf{v} \in \mathbf{V}_0(K)$;

(b) $\|\nabla_K\mathbf{v}\|_{0,K} \sim \sup_{w \in W(K)\setminus\{0\}}|(\nabla_K\mathbf{v}, w)_K|/\|w\|_{0,K} \sim |\mathbf{v}|_{1,K}$.

Proof. Assertion (a) follows directly from (10) and (H_1). Let us prove (b). According to (10), we have

$$(\nabla_K\mathbf{v}, w)_K = -(v, \nabla \cdot w)_K + (\mu, w \cdot n_K)_{\partial K} = J_K(w) \quad \forall w \in W(K).$$

Let $J_K = \sup_{w \in W(K)}|J_K(w)_K|/\|w\|_{0,K}$. In this definition, we pass from the element K to the corresponding basis element $\hat{K} \in \hat{\mathcal{T}}$. Let $x = F_K(\hat{x}) = B_K\hat{x} + b_K$ be the transformation of $\hat{K}$ at K and $|B_K| = \det(B_K)$. The notation $\hat{v}(\hat{x}) = v(x)$ is used for the image of a scalar function v under the transformation $x = F_K(\hat{x})$, while $\hat{w}(\hat{x}) = |B_K|B_K^{-1}w(x)$ denotes the image of a vector function w (under the Piola transform).

We have (see, e.g., [7, p. 98])

$$J_K(w) = \int_{\hat{K}} \hat{v}\hat{\nabla} \cdot \hat{w}d\hat{x} + \int_{\partial\hat{K}} \hat{\mu}\hat{w} \cdot \hat{n}d\hat{s} = \hat{J}(\hat{\mathbf{v}}; \hat{w}),$$

$$\|w\|^2_{0,K} = \int_{\hat{K}} |B_K|^{-1}(B_K^{\mathrm{T}}B_K)\hat{w}\cdot\hat{w}d\hat{s},$$

where $\hat{\nabla}\cdot\hat{w} = \partial\hat{w}_1/\partial\hat{x}_1 + \ldots + \partial\hat{w}_d/\partial\hat{x}_d$ and $\hat{n}$ is the unit outward normal vector to $\partial\hat{K}$. Denote by λ_K^2 and Λ_K^2 the minimal and maximal eigenvalues of the matrix $|B_K|^{-1}(B_K^{\mathrm{T}}B_K)$. Then

$$\Lambda_K^{-1}\hat{J}(\hat{\mathbf{v}}) \le J_K \le \lambda_K^{-1}\hat{J}(\hat{\mathbf{v}}), \quad \text{where} \quad \hat{J}(\hat{\mathbf{v}}) = \sup_{\hat{w}\in W(K)} |\hat{J}(\hat{\mathbf{v}};\hat{w})|/\|\hat{w}\|_{0,\hat{K}}. \tag{16}$$

The functional $\hat{J}(\hat{\mathbf{v}})$ defines a seminorm on $\mathbf{V}(\hat{K})$. According to (H_1), the equality $\hat{J}(\hat{\mathbf{v}}) = 0$ implies that $\hat{\mathbf{v}}\in\mathbf{V}_0(\hat{K})$; i.e., the norm $\hat{J}$ on the finite-dimensional space $\mathbf{V}(\hat{K})/\mathbf{V}_0(\hat{K})$ is equivalent to $|\hat{\mathbf{v}}|_1^2 = \|\hat{\nabla}\hat{v}\|^2_{0,\hat{K}} + \|\hat{v}-\hat{\mu}\|^2_{0,\partial\hat{K}}$. Thus, relation (16) implies the estimates

$$c\Lambda_K^{-1}|\hat{\mathbf{v}}|_1 \le J_K \le C\lambda_K^{-1}|\hat{\mathbf{v}}|_1. \tag{17}$$

We change variables back to the original element in (17) and use the following well-known estimates (see, e.g., [6]):

$$|B_k|^{1/2}\|B_K\|^{-1} \le \Lambda_K^{-1}, \quad \lambda_K^{-1} \le |B_K|^{1/2}\|B_K^{-1}\|, \quad \|\hat{v}\|_{0,\hat{K}} = |B_K|^{-1/2}\|v\|_{0,K},$$

$$|B_K|^{-1/2}\|B_K^{-1}\|^{-1}\|\nabla v\|_{0,K} \le \|\hat{\nabla}\hat{v}\|_{0,\hat{K}} \le |B_K|^{-1/2}\|B_K\|\|\nabla v\|_{0,K},$$

$$|B_K|^{-1/2}\|B_K^{-1}\|^{-1/2}\|g\|_{0,\partial K} \le \|\hat{g}\|_{0,\partial\hat{K}} \le |B_K|^{-1/2}\|B_K\|^{1/2}\|g\|_{0,\partial K}.$$

Since $\mathcal{T}_h$ is regular, in these estimates, we have $\|B_K\| \sim h$ and $\|B_K^{-1}\| \sim h^{-1}$. Therefore, as is easy to see, (17) implies (b). The theorem is proved.

Let us equip the above-introduced spaces $H^1(\mathcal{T}_h)$, V_h, Λ_h, $\mathbf{V}_h$, and W_h with norms. Recently, various norms of $H^s(\mathcal{T}_h)$ have been extensively studied in the context of the theory of nonconformal finite elements and discontinuous Galerkin methods (see, e.g., [8–10] for $s = 1$ and [11] for $s = 2$). For example, $H^1(\mathcal{T}_h)$ can be equipped with the norm

$$|[v]|^2_{H^1(\mathcal{T}_h)} = \|\nabla v\|^2_{0,\mathcal{T}_h} + h^{-1}\|\langle[v]\rangle\|^2_{0,\bar{\mathcal{E}}_h}, \quad \langle v\rangle|_e = |e|^{-1}(v,1)_e, \quad \|\cdot\|_{0,\bar{\mathcal{E}}_h} = \|\cdot\|_{L_2(\bar{\mathcal{E}}_h)},$$

where $[v]$ is the jump in v on $\bar{\mathcal{E}}_h$ and $[v] = v$ on $\mathcal{E}_h^{\partial}$ (in what follows, we use only the modulus $[v]$). An interesting feature of this norm is that it satisfies the following analogue of a Poincare-type inequality [9]:

$$\|v\|_{0,\Omega} \le c|[v]|_{H^1(\mathcal{T}_h)}.$$

Therefore,

$$\|v\|_{1,\mathcal{T}_h} \le c|[v]|_{H^1(\mathcal{T}_h)} \quad \forall v\in H^1(\mathcal{T}_h). \tag{18}$$

On V_h, we introduce the norm

$$\|v\|^2_{H^1(\mathcal{T}_h)} = \|\nabla v\|^2_{0,\mathcal{T}_h} + h^{-1}\|[v]\|^2_{0,\bar{\mathcal{E}}_h}.$$

Clearly, $|[v]|_{H^1(\mathcal{T}_h)} \le \|v\|_{H^1(\mathcal{T}_h)}$ and

$$\|v\|_{0,\Omega} \le c\|v\|_{H^1(\mathcal{T}_h)} \quad \forall v\in V_h. \tag{19}$$

We introduce the norm $\|\cdot\|_{0,\Omega}$ on W_h and $\|\cdot\|_{0,\mathscr{E}_h}$ on Λ_h. The space $\mathbf{V}_h$ is equipped with the norm

$$\|\mathbf{v}\|_{1,h}^2 = \sum_{K\in\mathscr{T}_h}|\mathbf{v}|_{1,K}^2 = \sum_{K\in\mathscr{T}_h}(\|\nabla v\|_{0,K}^2 + h^{-1}\|v-\mu\|_{0,\partial K}^2).$$

The fact that this functional defines a norm follows from the condition $\mu = 0$ on $\mathscr{E}_h^{\partial}$, $\mu \in \Lambda_h$.

Theorem 2. *Let conditions* $(H_{1,2})$ *hold. Then, for any* $\mathbf{v} = (v, \mu) \in \mathbf{V}_h$,

$$\text{(a)}\quad \|\nabla_h\mathbf{v}\|_{0,\Omega} \sim \|\mathbf{v}\|_{1,h}, \quad \text{(b)}\quad \|\mathbf{v}\|_{1,h} \geq c\|v\|_{H^1(\mathscr{T}_h)} \geq c\|v\|_{0,\Omega}.$$

Proof. The first assertion follows from Theorem 1(b). To prove (b), we define the functional

$$F(\mu) = \sum_{K\in\mathscr{T}_h}\|\mu - v\|_{0,\partial K}^2 = \sum_{e\in\mathscr{E}_h}\int_e((\mu - v|_K)^2 + (\mu - v|_L)^2)ds + \int_{\mathscr{E}_h^{\partial}} v^2 ds, \quad e \in \bar{K}\cap\bar{L}$$

and estimate it on Λ_h from below. Since $(\mu - v|_K)^2 + (\mu - v|_L)^2 = 2(\mu - (v|_K + v|_L)/2)^2 + 0.5(v|_K - v|_L)^2$, the definition of $[v]$ implies that $F(\mu) \geq 0.5\|[v]\|_{0,\bar{\mathscr{E}}_h}^2$. Therefore, by virtue of estimate (19), we obtain

$$\|\mathbf{v}\|_{1,h}^2 = \|\nabla v\|_{0,\mathscr{T}_h}^2 + h^{-1}F(\mu) \geq \|\nabla v\|_{0,\mathscr{T}_h}^2 + 0.5h^{-1}\|[v]\|_{0,\bar{\mathscr{E}}_h}^2 \geq 0.5\|v\|_{H^1(\mathscr{T}_h)}^2 \geq c\|v\|_{0,\Omega}^2.$$

The theorem is proved.

Lemma 1. *Let conditions* $(H_{1,2})$ *hold and* $\langle v\rangle_K = |K|^{-1}(v, 1)_K$. *Then*

$$\|\nabla_h\mathbf{v}\|_{0,\Omega}^2 \sim |[\mathbf{v}]|_{1,h}^2 = \sum_{K\in\mathscr{T}_h}(\|\nabla v\|_{0,K}^2 + h^{-1}\|\mu - \langle v\rangle_K\|_{0,\partial K}^2).$$

Proof. In a similar manner to the proof of Theorem 1, we show that $\|\nabla_K\mathbf{v}\|_{0,K}^2 \sim \|\nabla v\|_{0,K}^2 + h^{-1}\|\mu - \langle v\rangle_K\|_{0,\partial K}^2$, which yields the assertion of the lemma.

Lemma 2. *Under conditions* (2), (3), *and* $(H_{1,2})$, *the form* $\mathbf{a}_h$ *is strongly monotone, Lipschitz-continuous, and coercive; i.e., for* $\mathbf{u}_1, \mathbf{u}_2, \mathbf{v} \in \mathbf{V}_h$, *we have the estimates*

$$\mathbf{a}_h(\mathbf{u}_1, \mathbf{u}_1 - \mathbf{u}_2) - \mathbf{a}_h(\mathbf{u}_2, \mathbf{u}_1 - \mathbf{u}_2) \geq c_0\|\mathbf{u}_1 - \mathbf{u}_2\|_{1,h}^2,$$

$$|\mathbf{a}_h(\mathbf{u}_1, \mathbf{v}) - \mathbf{a}_h(\mathbf{u}_2, \mathbf{v})| \leq C_0\|\mathbf{u}_1 - \mathbf{u}_2\|_{1,h}\|\mathbf{v}\|_{1,h}, \quad \mathbf{a}_h(\mathbf{v}, \mathbf{v}) \geq c_0\|\mathbf{v}\|_{1,h}^2.$$

Proof. The first assertion follows directly from estimate (15) and Theorem 2. Theorem 2 also implies that $\|\nabla_h\mathbf{v}\|_{0,\Omega} + \|v\|_{0,\Omega} \leq c\|\mathbf{v}\|_{1,h}$ for any $\mathbf{v} = (v, \mu) \in \mathbf{V}_h$. Now, the second estimate in the lemma is easily derived from condition (2). To verify coercivity, it is sufficient to set $\mathbf{u}_1 = \mathbf{v}$ and $\mathbf{u}_2 = 0$ in the first estimate.

Theorem 3. *Let conditions* (2), (3), *and* $(H_{1,2})$ *hold. Then discrete problem* (6)–(9) *has a unique solution and this solution satisfies the stability estimate*

$$\|\mathbf{u}_{1,h} - \mathbf{u}_{2,h}\|_{1,h} + \|\sigma_{1,h} - \sigma_{2,h}\|_{0,\Omega} + \|q_{1,h} - q_{2,h}\|_{0,\Omega} \leq c\|f_1 - f_2\|_{0,\Omega},$$

where $\sigma_{i,h}, q_{i,h}, \mathbf{u}_{i,h} = (u_{i,h}, \lambda_{i,h})$ *are the solution of scheme* (6)–(9) *with right-hand sides* f_i, $i = 1, 2$.

Proof. Consider problem (14). The linear functional $\mathbf{f}_h$ is bounded in $\mathbf{V}_h$ uniformly in h, since $|\mathbf{f}_h(\mathbf{v})| \leq \|f\|_{0,\Omega}\|v\|_{0,\Omega} \leq c\|\mathbf{v}\|_{1,h}$ according to Theorem 2. Then Lemma 2 and the theory of monotone operators imply both the existence of a unique solution and the stability estimate $\|\mathbf{u}_{1,h} - \mathbf{u}_{2,h}\|_{1,h} \leq c\|f_1 - f_2\|_{0,\Omega}$. Since $\sigma_{i,h} = \nabla_h\mathbf{u}_{i,h}$, $q_{i,h} = \pi_W(k(\cdot, u_{i,h}, \sigma_{i,h}))$, Theorem 2 implies that

$$\|\sigma_{1,h} - \sigma_{2,h}\|_{0,\Omega} = \|\nabla_h(\mathbf{u}_{1,h} - \mathbf{u}_{2,h})\|_{0,\Omega} \leq c\|\mathbf{u}_{1,h} - \mathbf{u}_{2,h}\|_{1,h},$$

while the Lipschitz continuity of k and assertion (b) in Theorem 2 imply that

$$\|q_{1,h} - q_{2,h}\|_{0,\Omega} \leq c(\|u_{1,h} - u_{2,h}\|_{0,\Omega} + \|\sigma_{1,h} - \sigma_{2,h}\|_{0,\Omega}) \leq c\|\mathbf{u}_{1,h} - \mathbf{u}_{2,h}\|_{1,h}.$$

The assertion of the theorem follows from these estimates.

6. BASIC PROJECTORS

In what follows, we need projectors onto W_h and $\mathbf{V}_h$, respectively. Their definition is based on the properties of the operators ∇_K and ∇_h and the corresponding orthogonal decompositions of discrete spaces.

Projector Π_W. The spaces $W(K)$, $V(K)$, and $\mathbf{V}(K)$ are equipped with inner products by setting $(\cdot,\cdot)_{W(K)} = (\cdot,\cdot)_K$, $(\cdot,\cdot)_{V(K)} = (\cdot,\cdot)_K$, and $(\mathbf{u},\mathbf{v})_{\mathbf{V}(K)} = (u, v)_K + h(\lambda,\mu)_{\partial K}$. Let $\nabla_K^* : W(K) \longrightarrow \mathbf{V}(K)$ be the adjoint of the operator ∇_K:

$$(\nabla_K^* w, \mathbf{v})_{\mathbf{V}(K)} = -(\nabla\cdot w, v)_K + (w\cdot n_K, \mu)_{\partial K} \quad \forall \mathbf{v} = (v,\mu)\in \mathbf{V}(K).$$

Let $\Phi(K) = \ker(\nabla_K^*) = \{w\in W(K) : (w, \nabla_K\mathbf{v})_K = 0\ \forall\mathbf{v}\in\mathbf{V}(K)\}$, $\Phi(K)^\perp = \mathrm{Im}(\nabla_K)$, $\mathbf{V}_0(K) = \ker(\nabla_K)$, and $\mathbf{V}_0(K)^\perp = \mathrm{Im}(\nabla_K^*)$. Note that $\dim\Phi(K)^\perp = \dim\mathbf{V}_0(K)^\perp$ and

$$W(K) = \Phi(K)\oplus\Phi(K)^\perp, \quad \mathbf{V}(K) = \mathbf{V}_0(K)\oplus\mathbf{V}_0(K)^\perp. \tag{20}$$

Note also that ∇_K induces an isomorphism between the spaces $\mathbf{V}_0(K)^\perp$ and $\Phi(K)^\perp$.

Theorem 4. *Under condition (H_1), there exists a projector $\Pi_W : [H^1(\mathcal{T}_h)]^d \longrightarrow W_h$ and, for any $q\in [H^1(\mathcal{T}_h)]^d$, it is defined by the relations*

$$(\nabla\cdot\Pi_W q, v)_K = (\nabla\cdot q, v)_K \quad \forall v\in V(K), \tag{21}$$

$$(\Pi_W q\cdot n_K, \mu)_{\partial K} = (q\cdot n_K,\mu)_{\partial K} \quad \forall\mu\in\Lambda(\partial K), \tag{22}$$

$$(\Pi_W q, w)_K = (q, w)_K \quad \forall w\in\Phi(K). \tag{23}$$

Moreover, under condition (H_2), it satisfies the estimate

$$\|\Pi_W q\|_{0,K} \le c(|q|_{0,K} + h|q|_{1,K}) \quad \forall q\in [H^1(\mathcal{T}_h)]^d. \tag{24}$$

Proof. Since $W(K) = \Phi(K)\oplus\Phi(K)^\perp$ and the operator ∇_K is an isomorphism between $\mathbf{V}_0(K)^\perp$ and $\Phi(K)^\perp$, we conclude that the linear functional

$$g(w) = \begin{cases} -(\nabla\cdot q, v)_K + (q\cdot n_K,\mu)_{\partial K}, & w = \nabla_K\mathbf{v}\in\Phi(K)^\perp,\\ (q,w)_K, & w\in\Phi(K)\end{cases}$$

is defined on $W(K)$. To estimate its norm, we use the embedding inequality

$$\|u\|_{0,\partial K} \le ch^{-1/2}(h|u|_{1,K} + |u|_{0,K}) \quad \forall u\in H^1(K)$$

and Theorem 1. (The dependence of the constants on h can be shown in a standard manner: on a basis element, $\|\hat{u}\|_{0,\partial\hat{K}} \le c(|\hat{u}|_{1,\hat{K}} + |\hat{u}|_{0,\hat{K}})$ by virtue of the embedding $H^1(\hat{K})\subset L_2(\partial\hat{K})$. From this, the required estimate is derived by passing to K via an affine transformation and taking into account the regularity of $\mathcal{T}_h$.) We have $g|_{\Phi(K)^\perp}(w) = (q,\nabla v)_K + (q\cdot n_K, \mu - v)_{\partial K}$ and

$$\begin{aligned}\left|g|_{\Phi(K)^\perp}(w)\right| &\le c(|q|_{0,K}\|\nabla v\|_{0,K} + \|q\|_{0,\partial K}\|\mu - v\|_{0,\partial K})\\ &\le c(|q|_{0,K} + h^{1/2}\|q\|_{0,\partial K})(\|\nabla v\|_{0,K} + h^{-1/2}\|\mu - v\|_{0,\partial K})\\ &\le c(|q|_{0,K} + h|q|_{1,K})\|\nabla_K\mathbf{v}\|_{0,K} = c(|q|_{0,K} + h|q|_{1,K})\|w\|_{0,K}.\end{aligned}$$

Since $|g|_{\Phi(K)}(w)| \le |q|_{0,K}\|w\|_{0,K}$, it follows that

$$|g(w)| \le c(|q|_{0,K} + h|q|_{1,K})\|w\|_{0,K} \quad \forall w\in W(K). \tag{25}$$

Now $\Pi_W q|_K \in W(K)$ is defined as the solution of the equation

$$(\Pi_W q, w)_K = g(w) \quad \forall w\in W(K). \tag{26}$$

Clearly, the solution of Eq. (26) exists and estimate (24) follows from (25). If $q|_K \in W(K)$, then, by definition of ∇_K, we have $g(w) = (g, \nabla_K \mathbf{v})_K = (q, w)_K$ for $w \in \Phi(K)^\perp$. Since $g(w) = (g, w)_K$ for $w \in \Phi(K)$, it holds that $(\Pi_W q - q, w)_K = 0$ for any $w \in W(K)$; i.e., $\Pi_W|_{[H^1(K)]^d}$ is a projector onto $W(K)$.

Let us obtain a characterization of Π_W. Choosing $w \in \Phi(K)$ in (26), we have $(\Pi_W q - q, w)_K = 0$, i.e., (23). According to (20), $\mathbf{V}(K) = \mathbf{V}_0(K) \oplus \mathbf{V}_0(K)^\perp$. Let $\mathbf{v} \in \mathbf{V}_0(K)^\perp$. Then $w = \nabla_K \mathbf{v} \in \Phi(K)^\perp$ and

$$(\Pi_W q, \nabla_K \mathbf{v})_K = -(\nabla \cdot q, v)_K + (q \cdot n_K, \mu)_{\partial K}. \tag{27}$$

Clearly, this relation also holds for $\mathbf{v} \in \mathbf{V}_0(K)$, i.e., for all $\mathbf{v} \in \mathbf{V}(K)$. By the definition of ∇_K, the left-hand side of (27) is equal to $-(\nabla \cdot \Pi_W q, v)_K + (\Pi_W q \cdot n_K, \mu)_{\partial K}$. Therefore,

$$-(\nabla \cdot \Pi_W q, v)_K + (\Pi_W q \cdot n_K, \mu)_{\partial K} = -(\nabla \cdot q, v)_K + (q \cdot n_K, \mu)_{\partial K} \quad \forall (v, \mu) \in V(K) \times \Lambda(\partial K).$$

This relation yields (21) for $\mu = 0$ and (22) for $v = 0$. The theorem is proved.

In what follows, $P_m(D)$ denotes the restriction of the set of polynomials of degrees at most $m \geq 0$ in all variables to the domain D.

Lemma 3. *Let conditions* $(H_{1,2})$ *hold. Then, for any* $q \in [H^1(\mathcal{T}_h)]^d$,

$$\|q - \Pi_W q\|_{0,K} \leq c(|q - \pi_W q|_{0,K} + h|q - \pi_W q|_{1,K}), \quad \|\Pi_W q\|_{1,K} \leq c\|q\|_{1,K}.$$

Proof. By virtue of (H_0'), we have $W(K) \supseteq [P_0(K)]^d$. Therefore, $|q - \pi_W q|_{0,K} \leq ch|q|_{1,K}$. The first estimate in the lemma follows from the stability of Π_W, i.e., from estimate (24):

$$|q - \Pi_W q|_{0,K} = |(q - \pi_W q) - \Pi_W(q - \pi_W q)|_{0,K} \leq |q - \pi_W q|_{0,K} + |\Pi_W(q - \pi_W q)|_{0,K}$$
$$\leq |q - \pi_W q|_{0,K} + c(|q - \pi_W q|_{0,K} + h|q - \pi_W q|_{1,K}) \leq c(|q - \pi_W q|_{0,K} + h|q - \pi_W q|_{1,K}).$$

This also implies the estimate $|q - \Pi_W q|_{0,K} \leq ch|q|_{1,K}$, since $|\pi_W q|_{1,K} \leq |q|_{1,K}$. We use the reverse inequality in the following standard estimates:

$$|\Pi_W q|_{1,K} \leq |\Pi_W q - \pi_W q|_{1,K} + |\pi_W q|_{1,K} \leq ch^{-1}|\Pi_W q - \pi_W q|_{0,K} + c|q|_{1,K}$$
$$\leq ch^{-1}(|\Pi_W q - q|_{0,K} + |q - \pi_W q|_{0,K}) + c|q|_{1,K} \leq c|q|_{1,K}.$$

Combining this with (24), we derive the final estimate of the lemma. The lemma is proved.

Projector $\Pi_\mathbf{V}$. Let $L_2^0(\overline{\mathscr{E}}_h) = \{\lambda \in L_2(\overline{\mathscr{E}}_h) : \lambda|_{\mathscr{E}_h^\partial} = 0\}$. We treat $\mathbf{V}_h$ as a subspace of $\mathbf{H} = L_2(\mathcal{T}_h) \times L_2^0(\overline{\mathscr{E}}_h)$. We introduce the following constraints on the chosen finite-element spaces:

$$(H_3) \quad \dim V(K) = \dim \nabla \cdot W(K), \quad \nabla \cdot W(K) = \{\nabla \cdot w : w \in W(K)\};$$

$$(H_4) \quad W_n(e, K) = W_n(e, L) \quad \text{for} \quad e = \overline{K} \cap \overline{L} \in \mathscr{E}_h; \quad \dim \Lambda(e) = \dim W_n(e, K),$$

where $W_h(e, K) = \{w \cdot n_K|_e, w \in W(K)\}$ for $e \in \partial K$.

To define the projector $\Pi_\mathbf{V} : \mathbf{H} \longrightarrow \mathbf{V}_h$, we introduce the operators $\nabla_K^0 : V(K) \longrightarrow W(K)$ and $\nabla_{\partial K} : \Lambda(\partial K) \longrightarrow W(K)$ by setting

$$(\nabla_K^0 v, w)_{W(K)} = -(v, \nabla \cdot w)_K, \quad (\nabla_{\partial K}\mu, w)_{W(K)} = (\mu, w \cdot n_K)_{\partial K}.$$

According to (11), we have $\nabla_K \mathbf{v} = \nabla_K^0 v + \nabla_{\partial K}\mu$. Condition (H_1) implies that both ∇_K^0 and $\nabla_{\partial K}$ have zero kernels. Let $\Phi_0(K)^\perp = \operatorname{Im}(\nabla_K^0)$. Then $W(K) = \Phi_0(K) \oplus \Phi_0(K)^\perp$ and $V(K) = \operatorname{Im}((\nabla_K^0)^*)$.

Lemma 4. *Let conditions* $(H_{1,2})$ *hold. Then* $\|\nabla_K^0 v\|_{0,K} \sim h^{-1}\|v\|_{0,K}$ *and*

$$\|\nabla_{\partial K}\mu\|_{0,K} \sim h^{-1/2}\|\mu\|_{0,\partial K}; \quad \|(\nabla_K^0)^* w\|_{0,K} \sim h^{-1}\|w\|_{0,K}, \quad w \in \Phi_0(K)^\perp.$$

The proof is similar to that of Theorem 1.

Lemma 5. *Let conditions* $(H_{1,2,3})$ *hold. Then there exists a projector* $\mathcal{P}_V : L_2(\Omega) \longrightarrow V_h$ *such that, for any* $w \in W_h$ *and* $u \in L_2(\Omega)$

$$(\mathcal{P}_V u - u, \nabla \cdot w)_{\mathcal{T}_h} = \|\mathcal{P}_V u\|_{0,\Omega} \leq c\|u\|_{0,\Omega}; \quad \|\mathcal{P}_V u - u\|_{0,\Omega} \leq c\|\pi_V u - u\|_{0,\Omega}. \tag{28}$$

Under conditions $(H_{1,2,4})$ *there exists a projector* $\mathcal{P}_\Lambda : L_2(\overline{\mathcal{E}}_h) \longrightarrow \Lambda_h$ *such that*

$$(\lambda - \mathcal{P}_\Lambda\lambda, w \cdot n)_{\partial\mathcal{T}_h} = 0; \quad \|\mathcal{P}_\Lambda\lambda\|_{0,\overline{\mathcal{E}}_h} \le c\|\lambda\|_{0,\overline{\mathcal{E}}_h}; \quad \|\mathcal{P}_\Lambda\lambda - \lambda\|_{0,\overline{\mathcal{E}}_h} \le c\|\pi_\Lambda\lambda - \lambda\|_{0,\overline{\mathcal{E}}_h} \tag{29}$$

for any $w \in W_h$ *and* $\lambda \in L_2^0(\overline{\mathcal{E}}_h)$.

Proof. For an element $K \in \mathcal{T}_h$ we define an operator $\mathcal{P}_V^K : L_2(K) \longrightarrow V(K)$ such that $\mathcal{P}_V^K u$ for $u \in L_2(K)$ is the solution of the equation $(\mathcal{P}_V^K u - u, \nabla \cdot w)_K = 0$ $\forall w \in W(K)$. We write it in the form $(\nabla_K^0 \mathcal{P}_V u, w)_K = -(u, \nabla \cdot w)_K$ $\forall w \in W(K)$. Then, by Lemma 4, we have

$$ch^{-1}\|\mathcal{P}_V^K u\|_{0,K} \le \|\nabla_K^0\mathcal{P}_V^K u\|_{0,K} \sim \sup_{w \in W(K)\setminus\{0\}} |(\nabla_K^0\mathcal{P}_V^K u, w)_K| / \|w\|_{0,K}$$
$$= \sup_{w \in W(K)\setminus\{0\}} |(u, \nabla \cdot w)_K| / \|w\|_{0,K} \le ch^{-1}\|u\|_{0,K}.$$

This implies the unique solvability of the equation under condition (H_3) and also the estimate $\|\mathcal{P}_V^K u\|_{0,K} \le c\|u\|_{0,K}$. Thus, defining $\mathcal{P}_V$ as $(\mathcal{P}_V u)|_K = \mathcal{P}_V^K(u|_K)$, we obtain the first two relations in (28). The third estimate in (28) follows from the second one, since both $\mathcal{P}_V^K$ and $\mathcal{P}_V$ are projectors.

Let us prove the second part of the lemma. Each face is associated with the operator $\mathcal{P}_\Lambda^e : L_2(e) \longrightarrow \Lambda(e)$. Let $\lambda \in L_2(e)$. On the face $e \in \mathcal{E}_h^\partial$ we set $\mathcal{P}_\Lambda^e\lambda = 0$. On the internal face $e = \overline{K} \cap \overline{L}$ the function $\mathcal{P}_\Lambda^e\lambda$ is defined as the solution of the equation $(\mathcal{P}_\Lambda^e\lambda - \lambda, w_n)_e = 0$ $\forall w_n \in W_n(e, K)$. Condition (H_1) implies that, for $\mu \in \Lambda(e)$, the relation $(\mu, w_n)_e = 0$ $\forall w_n \in W_n(e, K)$ yields $\mu = 0$. Therefore, by the second condition in (H_4) the equation is uniquely solvable and, by using the transition to a basis element and back, it is easy to obtain

$$\|\mu\|_{0,e} \sim \sup_{w \in W(e)} |(\mu, w_n)_e| / \|w\|_{0,e},$$

where $W(e) = W_n(e, K)\setminus\{0\}$. Consequently,

$$\|\mathcal{P}_\Lambda^e\lambda\|_{0,e} \sim \sup_{w \in W(e)} |(\mathcal{P}_\Lambda^e\lambda, w_n)_e| / \|w\|_{0,e} = \sup_{w \in W(e)} |(\lambda, w_n)_e| / \|w\|_{0,e} \le c\|\lambda\|_{0,e}. \tag{30}$$

Clearly, $\mathcal{P}_\Lambda^e$ is a projector on $\Lambda(e)$. The projector $\mathcal{P}_\Lambda$ is defined so that $(\mathcal{P}_\Lambda\lambda)|_e = \mathcal{P}_\Lambda^e(\lambda|_e)$. Then summing estimates (30), we obtain the second estimate in (29), which implies the third estimate. Let us prove the first relation in (29).

Let $w \in W_h$. On each face $e = \overline{K} \cap \overline{L}$ the function $w_h = (w_K - w_L) \cdot n_K|_e$ belongs to $W_n(e, K)$ by virtue of (H_4). Therefore, $(\mathcal{P}_\Lambda\lambda - \lambda, (w_K - w_L) \cdot n_K)_e = 0$. Summing up these equalities over all $e \in \mathcal{E}_h$, we obtain the first relation in (29). The lemma is proved.

Corollary 1. Let the projector $\Pi_\mathbf{V} = (\Pi_V, \Pi_\Lambda) : \mathbf{H} \longrightarrow \mathbf{V}_h$ be defined so that, for $\mathbf{u} = (u, \lambda) \in \mathbf{H}$: (a) $\Pi_V\mathbf{u} = \mathcal{P}_V u$ if condition (H_3) holds; otherwise, $\Pi_V\mathbf{u} = \pi_V u$; (b) $\Pi_\Lambda\mathbf{u} = \mathcal{P}_\Lambda\lambda$ if condition (H_4) holds; otherwise, $\Pi_\Lambda\mathbf{u} = \pi_\Lambda\lambda$.

Then $\varepsilon_\mathbf{H}(u) = \|u - \Pi_V\mathbf{u}\|_{0,\Omega} + h^{1/2}\|u - \Pi_\Lambda\mathbf{u}\|_{0,\overline{\mathcal{E}}_h}$ satisfies the estimate

$$\varepsilon_\mathbf{H}(u) \le c\|u - \pi_V u\|_{0,\mathcal{T}_h} + ch^{1/2}\|u - \pi_\Lambda u\|_{0,\overline{\mathcal{E}}_h}.$$

Lemma 6. *Let* $u \in H_0^1(\Omega)$, $\mathbf{u} = (u|_{\mathcal{T}_h}, u|_{\overline{\mathcal{E}}_h})$, *the projector* $\Pi_\mathbf{V}$ *be defined in Corollary* 1, *and* $\varepsilon_\mathbf{V}(u) = \|\nabla u - \nabla_h(\Pi_\mathbf{V}\mathbf{u})\|_{0,\mathcal{T}_h}$. *Then* $\varepsilon_\mathbf{V} = \|\nabla u - \pi_W(\nabla u)\|_{0,\mathcal{T}_h}$ *under conditions* $(H_{1,2,3,4})$

$$\varepsilon_\mathbf{V}(u) \le \|\nabla u - \pi_W(\nabla u)\|_{0,\mathcal{T}_h} + C\begin{cases} h^{-1}\|u - \pi_V u\|_{0,\mathcal{T}_h} + h^{-1/2}\|u - \pi_\Lambda u\|_{0,\mathcal{E}_h} & \text{for} \quad (H_{1,2}), \\ h^{-1}\|u - \pi_V u\|_{0,\mathcal{T}_h} & \text{for} \quad (H_{1,2,4}), \\ h^{-1/2}\|u - \pi_\Lambda u\|_{0,\mathcal{E}_h} & \text{for} \quad (H_{1,2,3}). \end{cases}$$

Proof. In view of the definition of ∇_h, it is easy to see that, for $w \in W_h$

$$(\nabla u - \nabla_h(\Pi_{\mathbf{V}}\mathbf{u}), w)_{\mathcal{T}_h} = -(u - \Pi_V\mathbf{u}, \nabla \cdot w)_{\mathcal{T}_h} + (u - \Pi_\Lambda\mathbf{u}, w \cdot n)_{\partial\mathcal{T}_h} = I_1 + I_2.$$

By the definition of $\Pi_{\mathbf{V}}$, the term I_1 vanishes under condition (H_3). If condition (H_4) holds, then $I_2 = 0$. Therefore, $(\nabla u - \nabla_h(\Pi_{\mathbf{V}}\mathbf{u}), w)_{\mathcal{T}_h} = 0\ \forall w \in W_h$ under conditions (H_{1-4}). By the definition of π_W, it follows that $\nabla_h(\Pi_{\mathbf{V}}\mathbf{u}) = \pi_W(\nabla u)$. This proves the first assertion in the lemma.

According to the reverse inequalities, $\|\nabla \cdot w\|_{0,\Omega} \le ch^{-1}\|w\|_{0,\Omega}$ and $\|w\|_{0,\bar{\mathcal{E}}_h} \le ch^{-1/2}\|w\|_{0,\Omega}$ for any $w \in W_h$. Therefore, under conditions $(H_{1,2})$, we have

$$|(\nabla u - \nabla_h(\Pi_{\mathbf{V}}\mathbf{u}), w)_{\mathcal{T}_h} \le c(h^{-1}\|u - \pi_V u\|_{0,\mathcal{T}_h} + h^{-1/2}\|u - \pi_\Lambda u\|_{0,\mathcal{E}_h})\|w\|_{0,\Omega}.$$

This implies the estimate for $\varepsilon_{\mathbf{V}}(u)$ under conditions $(H_{1,2})$. The other two estimates are derived in a similar manner. The lemma is proved.

The subspace $\overline{W}_h$ of the space W_h:

$$\overline{W}_h = \{w \in W_h : (w \cdot n, \mu)_{\partial\mathcal{T}_h} = 0\ \forall \mu \in \Lambda_h\}. \tag{31}$$

Choosing $\mu = 0$ outside an arbitrary internal face $e = \overline{K} \cap \overline{L}$, we obtain

$$(w_n, \mu)_e = 0 \quad \forall \mu \in \Lambda(e), \tag{32}$$

where $w_n = (w_K - w_L) \cdot n_K|_e$. If condition (H_4) holds, then $w_n \in W_n(e, K)$ and the dimensions of the spaces $\Lambda(e)$ and $W_n(e, K)$ coincide. Since the adjoint equation $(\mu, w_n)_e = 0\ \forall w_n \in W_n(e, K)$, according to condition (H_1), has only the trivial solution, we conclude that, under conditions $(H_{1,4})$ solution (32) is unique and $w_n = 0$; i.e., the normal components of w are continuous on $\mathcal{E}_h$. This means that $\overline{W}_h \subset H(\mathrm{div}; \Omega)$ under conditions $(H_{1,4})$.

7. ERROR ESTIMATE FOR THE SCHEME

In this section, u is the solution of the original problem (1); the functions q, k_0, and σ are defined according to (4); $\mathbf{u} = (u, \lambda)$, where $\lambda = u|_{\bar{\mathcal{E}}_h}$; and $\mathbf{u}_h = (u_h, \lambda_h)$ is an approximate solution.

First, we estimate the error of the scheme in discrete norms. Let $\Pi_{\mathbf{V}}\mathbf{u} = (\Pi_V\mathbf{u}, \Pi_\Lambda\mathbf{u})$ be the projection of the solution $\mathbf{u}$ in $\mathbf{V}_h$ as defined in Corollary 1. Then the function

$$\mathbf{e} = \mathbf{u}_h - \Pi_{\mathbf{V}}\mathbf{u} = (e_u, e_\lambda) = (u_h - \Pi_V\mathbf{u}, \lambda_h - \Pi_\Lambda\mathbf{u})$$

characterizes the error of the scheme in $\mathbf{V}_h$. Define the following measures of the error:

$$\epsilon_h = \varepsilon_{\mathbf{H}} + \varepsilon_{\mathbf{V}} + \varepsilon_W, \quad \varepsilon_{\mathbf{H}} = \|u - \Pi_V\mathbf{u}\|_{0,\Omega} + h^{1/2}\|u - \Pi_\Lambda\mathbf{u}\|_{0,\partial\mathcal{T}_h},$$

$$\varepsilon_{\mathbf{V}} = \|\nabla u - \nabla_h(\Pi_{\mathbf{V}}\mathbf{u})\|_{0,\mathcal{T}_h}, \quad \varepsilon_W = \|q - \pi_W q\|_{0,\Omega} + \|\sigma - \pi_W\sigma\|_{0,\Omega} + \|q - \Pi_W q\|_{0,\Omega}.$$

Theorem 5. *Let conditions* (2), (3), *and* $(H_{1,2})$ *hold and* $q \in [H^1(\mathcal{T}_h)]^d$. *Then*

$$\|\nabla_h(\mathbf{u}_h - \Pi_{\mathbf{V}}\mathbf{u})\|_{0,\Omega} \le c\epsilon_h.$$

Proof. It follows from (5) that $-(\nabla \cdot q, e_u)_K + (k_0, e_u)_K = (f, e_u)_K$. Combining this equality with identity (27) yields

$$(\Pi_W q, \nabla_h\mathbf{e})_K - (e_\lambda, q \cdot n_K)_{\partial K} + (k_0, e_u)_K = (f, e_u)_K.$$

Summing up this identity over all K and taking into account that the second term vanishes when summed, since the normal components of $q \in H(\mathrm{div}; \Omega)$ are continuous on $\mathcal{E}_h$ and e_λ vanishes on $\mathcal{E}_h^\partial$, we obtain

$$(q, \nabla_h\mathbf{e})_{\mathcal{T}_h} + (k_0, e_u)_{\mathcal{T}_h} + (\Pi_W q - q, \nabla_h\mathbf{e})_{\mathcal{T}_h} = (f, e_u)_{\mathcal{T}_h}. \tag{33}$$

Combining estimate (15) and the Lipschitz continuity of k and k_0 with (33) produces

$$\begin{aligned} c_0\|\nabla_h\mathbf{e}\|_{0,\Omega}^2 &\le \mathbf{a}_h(\mathbf{u}_h, \mathbf{e}) - \mathbf{a}_h(\Pi_{\mathbf{V}}\mathbf{u}, \mathbf{e}) = \mathbf{f}_h(\mathbf{e}) - \mathbf{a}_h(\Pi_{\mathbf{V}}\mathbf{u}, \mathbf{e}) \\ &= (f, e_u)_{\mathcal{T}_h} - \mathbf{a}_h(\Pi_{\mathbf{V}}\mathbf{u}, \mathbf{e}) = (k(\cdot, u, \sigma) - k(\cdot, \Pi_V\mathbf{u}, \nabla_h(\Pi_{\mathbf{V}}\mathbf{u})), \nabla_h\mathbf{e})_{\mathcal{T}_h} \\ &+ (k_0(\cdot, u, \sigma) - k_0(\cdot, \Pi_V\mathbf{u}, \nabla_h(\Pi_{\mathbf{V}}\mathbf{u})), e_u)_{\mathcal{T}_h} + (\Pi_W q - q, \nabla_h\mathbf{e})_{\mathcal{T}_h} \\ &\le c(\|u - \Pi_V\mathbf{u}\|_{0,\Omega} + \|\sigma - \nabla_h(\Pi_{\mathbf{V}}\mathbf{u})\|_{0,\Omega} + \|\Pi_W q - q\|_{0,\Omega})(\|\nabla_h\mathbf{e}\|_{0,\Omega} + \|e_u\|_{0,\Omega}). \end{aligned}$$

It remains to be noted that $\|e_u\|_{0,\Omega} \le c\|\nabla_h \mathbf{e}\|_{0,\Omega}$ according to Theorem 2. The theorem is proved.

Theorem 6. *Let the assumptions of Theorem* 5 *hold and* $\sigma_h = \nabla_h \mathbf{u}_h$. *Then*

$$\|\sigma_h - \pi_W\sigma\|_{0,\Omega} + \|e_u\|_{0,\Omega} + \|\nabla e_u\|_{0,\mathcal{T}_h} + h^{-1/2}\|[e_u]\|_{0,\bar{\mathcal{E}}_h} \le c\epsilon_h,$$

$$\|e_\lambda\|_{0,\partial\mathcal{T}_h} \le ch^{-1/2}\|e_u\|_{0,\mathcal{T}_h} + h^{1/2}\epsilon_h.$$

Proof. According to Theorems 2 and 5, we have

$$c\epsilon_h \ge \|\nabla_h \mathbf{e}\|_{0,\Omega} \sim \|\mathbf{e}\|_{1,h} \sim \|\nabla e_u\|_{0,\mathcal{T}_h} + h^{-1/2}\|e_\lambda - e_u\|_{0,\partial\mathcal{T}_h} \ge c\|e_u\|_{H^1(\mathcal{T}_h)} \ge C\|e_u\|_{0,\Omega},$$

where $\|e_u\|_{H^1(\mathcal{T}_h)} \sim \|\nabla e_u\|_{0,\mathcal{T}_h} + h^{-1/2}\|[e_u]\|_{0,\bar{\mathcal{E}}_h}$. This yields the first estimate. Since $\|e_u\|_{0,\partial K} \le ch^{-1/2}\|e_u\|_{0,K}$, it follows that $\|e_\lambda\|_{0,\partial\mathcal{T}_h} \le \|e_u\|_{0,\partial\mathcal{T}_h} + \|e_\lambda - e_u\|_{0,\partial\mathcal{T}_h} \le ch^{-1/2}\|e_u\|_{0,\mathcal{T}_h} + ch^{1/2}\epsilon_h$, which implies the second estimate. The theorem is proved.

Theorem 7. *Let the assumptions of Theorems* 5 *and* 6 *hold and* $q_h = \pi_W(k(\cdot, u_h, \sigma_h))$. *Then*

$$\|\sigma - \sigma_h\|_{0,\Omega} + \|u - u_h\|_{0,\Omega} + \|q - q_h\|_{0,\Omega} \le c\epsilon_h, \quad \|\lambda - \lambda_h\|_{0,\partial\mathcal{T}_h} \le ch^{-1/2}\epsilon_h.$$

Under the additional condition

$$(H_3') \quad V(K) = \nabla \cdot W(K),$$

it is also true that $\|\nabla \cdot (q - q_h)\|_{0,\mathcal{T}_h} \le \|\nabla \cdot q - \pi_V(\nabla \cdot q)\|_{0,\mathcal{T}_h} + c\epsilon_h$.

Proof. Theorem 6 implies $\|\sigma - \sigma_h\|_{0,\Omega} + \|u - u_h\|_{0,\Omega} \le c\epsilon_h$ and the second estimate of the lemma. Using the Lipschitz continuity of k, we obtain

$$\|q - q_h\|_{0,\Omega} \le \|q - \pi_W q\|_{0,\Omega} + \|\pi_W(k(\cdot, u, \sigma) - k(\cdot, u_h, \sigma_h))\|_{0,\Omega}$$
$$\le \|q - \pi_W q\|_{0,\Omega} + c(\|u - u_h\|_{0,\Omega} + \|\sigma - \sigma_h\|_{0,\Omega}) \le c\epsilon_h.$$

To prove the last inequality, we use the properties of the projector Π_W and apply Eqs. (5) and (8). For $v \in V_h$, we obtain $(\nabla \cdot (\Pi_W q - q_h), v)_{\mathcal{T}_h} = (k_0(\cdot, u, \sigma) - k_0(\cdot, u_h, \sigma_h), v)_\Omega$. From this, by virtue of condition (H_3') and the Lipschitz continuity of k_0, it follows that $\|\nabla \cdot (\Pi_W q - q_h)\|_{0,\mathcal{T}_h} \le c\epsilon_h$. Therefore, $\|\nabla \cdot (q - q_h)\|_{0,\mathcal{T}_h} \le \|\nabla \cdot (q - \Pi_W q)\|_{0,\mathcal{T}_h} + \|\nabla \cdot (q_h - \Pi_W q)\|_{0,\mathcal{T}_h} \le \|\nabla \cdot q - \pi_V(\nabla \cdot q)\|_{0,\mathcal{T}_h} + c\epsilon_h$. The theorem is proved.

Theorems 6 and 7 imply that the error of the scheme is determined by ϵ_h. Consider its components. According to Lemma 3, ε_W satisfies the estimate

$$\varepsilon_W \le c(|\sigma - \pi_W\sigma|_{0,\mathcal{T}_h} + |q - \pi_W q|_{0,\mathcal{T}_h} + h|q - \pi_W q|_{1,\mathcal{T}_h}).$$

Here, the terms are matched in the order of h. Thus, the estimate fir ε_W follows from the error estimates for the local L_2-orthoprojector in W_h, which are well known in the theory of FEM. The errors of local L_2-orthoprojectors can also be used to estimate $\varepsilon_{\mathbf{H}}$. By Corollary 1, $\varepsilon_{\mathbf{H}} \le c\|u - \pi_V u\|_{0,\mathcal{T}_h} + ch^{1/2}\|u - \pi_\Lambda u\|_{0,\partial\mathcal{T}_h}$. Note that the above estimates for ε_W and $\varepsilon_{\mathbf{H}}$ hold under conditions $(H_{1,2})$. The estimate for $\varepsilon_{\mathbf{V}}$ is given in Lemma 6. If conditions $(H_{1,2,3,4})$ are satisfied, we have $\varepsilon_{\mathbf{V}} = \|\sigma - \pi_W\sigma\|_{0,\mathcal{T}_h}$. Otherwise, the error estimate is, generally speaking, worsened.

8. EXAMPLES OF FINITE ELEMENT SPACES

Note some consequences of conditions $(H_{1,2,3,4})$. Consider the space $\overline{W}_h$. Under condition (H_1), it is nontrivial; i.e., $\overline{W}_h \ne \{0\}$, since, according to Eq. (9), it includes, for example, the component q_h of the solution to scheme (6)–(9). Moreover, it is easy to see that this space also includes the range of Π_W. It was shown above that, under conditions $(H_{1,4})$, we have $\overline{W}_h \subset H(\mathrm{div}; \Omega)$. Hence, $-q_h \in H(\mathrm{div}; \Omega)$. This means that the family of spaces $\{W(K)\}$ can be used to construct a conformal approximation of $H(\text{div}; \Omega)$. More precisely, we can define a set of linear functionals $\Sigma(K)$ on $W(K)$ (degrees of freedom) and, accordingly, families of finite elements $\{(K, W(K), \Sigma(K)), K \in \mathcal{T}_h\}$ consistent with the continuity condition for the normal components of the approximations on the interelement boundaries. The degrees of freedom $\Sigma(K)$ of the element K can be chosen according to the definition of Π_W in Lemma 4. Furthermore, condition (H_1) guarantees the existence of Π_W, while condition (H_2) ensures its stability in $H^1(\mathcal{T}_h)$. As a result, we can prove the Ladyzhenskaya–Babuska–Brezzi condition (LBB or inf-sup condition; see, for example, [7, Section II.2.3, p. 57]). More specifically, the following result is true.

Lemma 8. *Let conditions* $(H_{1,2})$ *hold. Then*

$$\sup_{w \in \overline{W}_h} |(v, \nabla \cdot w)_{\mathcal{T}_h}| / \|w\|_{1, \mathcal{T}_h} \geq c\|v\|_{0, \Omega} \quad \forall v \in V_h.$$

The proof relies on the well-known continuous analogue of this inequality and makes use of Fortin's technique and the projector Π_W (see, e.g., [7, p. 58, Proposition 2.8]).

Therefore, conditions $(H_{1,2,4})$ not only guarantee the inclusion $\overline{W}_h \subset H(\text{div}; \Omega)$ but also ensure that the LBB condition is satisfied for the pair of spaces (V_h, W_h). As is known, such a pair of spaces can be used to construct mixed FEM schemes. Finally, according to what was shown above, conditions $(H_{1,2,3,4})$ guarantee the "optimal" error estimates for the scheme under consideration.

Below are examples of spaces $V(K)$, $W(K)$, and $\Lambda(e)$ satisfying conditions $(H_{1,2,3,4})$. All of them are well known in the theory of mixed FEM; they are polynomial and, as a result, satisfy condition (H_2).

Example 1. If K is a d-simplex, we can use the family of Raviart–Thomas spaces RT_k defined as

$$W(K) = [P_k(K)]^d \oplus xP_k(K), \quad V(K) = P_k(K), \quad \Lambda(e) = P_k(e), \quad k \geq 0. \tag{34}$$

The properties of this family are well known (see [12] and, for example, [7, pp. 116–130]). Let $R_k(\partial K) = \{v \in L_2(\partial K) : v|_e \in P_k(e), e \in \partial K\}$. Then $\Lambda(\partial K) = R_k(\partial K)$ and

$$\nabla \cdot w \in V(K), \quad w \cdot n_K|_{\partial K} \in \Lambda(\partial K), \quad w \in W(K).$$

It follows that conditions $(H_{3,4})$ and (H_3') are satisfied. It remains to check condition (H_1). For this purpose, we use the following characterization of spaces (34): the relations

$$w \in W(K) : (w, \sigma)_K = 0 \quad \forall \sigma \in [P_{k-1}(K)]^d, \quad (w \cdot n_K, g)_{\partial K} = 0 \quad \forall g \in R_k(\partial K) \tag{35}$$

imply that $w = 0$. Relations (35) are equivalent to a homogeneous system of linear algebraic equations of dimension $N \times M$ for the expansion coefficients w over some basis; here, $M = \dim W(K)$ and $N = d\dim P_{k-1}(K) + \dim R_k(\partial K)$. According to (35), the matrix A of this system has a full column rank. It is well known (and is easy to verify) that A is a square matrix. Therefore, the relations

$$\sigma \in [P_{k-1}(K)]^d, \quad g \in R_k(\partial K) : (\sigma, w)_K + (g, w \cdot n_K)_{\partial K} = 0 \quad \forall w \in W(K) \tag{36}$$

imply that $\sigma = 0$ and $g = 0$. In (36), we set $\sigma = \nabla v$ and $g = \lambda - v$, where $v \in V(K)$ and $\lambda \in \Lambda(\partial K)$. After integration by parts, the relations

$$-(v, \nabla \cdot w)_K + (\lambda, w \cdot n_K)_{\partial K} = 0 \quad \forall w \in W(K)$$

imply $v = \lambda = \text{const}$. Therefore, condition (H_1) also holds.

If scheme (6)–(9) is constructed using spaces (34), then error estimates for the scheme depending on the smoothness of the solution follow from Theorems 6 and 7. A remark has to be made only about the estimate of $\varepsilon_\Lambda = h^{1/2}\|u - \pi_\Lambda u\|_{0, \partial\mathcal{T}_h}$ in $\varepsilon_{\mathbf{H}}$. We have

$$\varepsilon_\Lambda^2 = h \sum_{K \in \mathcal{T}_h} \|u - \pi_\Lambda u\|_{0, \partial K}^2 \leq ch \sum_{K \in \mathcal{T}_h} h^{2k+1} \|u\|_{k+1/2, \partial K}^2 \leq ch^{2k+2} \sum_{K \in \mathcal{T}_h} \|u\|_{k+1, K}^2 = ch^{2k+2}\|u\|_{k+1, \mathcal{T}_h}^2.$$

For sufficiently smooth solutions, $\epsilon_h \leq ch^{k+1}$.

Example 2. If K is a d-simplex, we can use the spaces BDM_k defined as

$$W(K) = [P_k(K)]^d, \quad V(K) = P_{k-1}(K), \quad \Lambda(e) = P_k(e), \quad k \geq 1. \tag{37}$$

The properties of this family of spaces are well known (see [13] and, for example, [7, pp. 116–130]). Obviously, conditions (H_3') and ($H_{3,4}$) hold for this family. To verify condition (H_1), we use the following characterization of spaces (37): if $v \in P_{k-1}(K)$ and $q \in R_k(\partial K) = \Lambda(\partial K)$, then the relations $(\nabla v, w)_K + (g, w \cdot n_K)_{\partial K} = 0$ $\forall w \in W(K)$, imply that $v = \text{const}$ and $g = 0$ (see, e.g., [7, p. 114, Lemma 3.1]; generally speaking, the proof in [7] is given for $d = 2, 3$, but it can easily be extended to $d > 3$). Setting $g = \lambda - v$ with $\lambda \in \Lambda(\partial K)$, after integration by parts, as before, we see that condition (H_1) holds. If scheme (6)–(9) is constructed using spaces (37), then error estimates for the scheme depending on the smoothness of the solution follow from Theorems 6 and 7. For sufficiently smooth solutions, $\epsilon_h \le ch^k$.

Example 3. If K is affinely equivalent to $\hat{K} = [-1, 1]^d$, $d = 2, 3$, we can use the spaces $BDM_{[k]}$ defined as

$$W(K) = [P_k(K) \oplus B(k, d)]^d, \quad V(K) = P_{k-1}(K), \quad \Lambda(e) = P_k(e), \quad k \ge 1. \tag{38}$$

For the definition of $B(k, d)$ and the properties of this family of spaces, see, for example, [7, pp. 116–130]. The structure of $W(K)$ is such that, for $w \in W(K)$, we have $\nabla \cdot w \in P_{k-1}(K)$ and $w \cdot n_K|_e \in P_k(e)$, which mean that conditions (H_3') and ($H_{3,4}$) hold. The satisfaction of condition (H_1) is checked in the same manner as for the family RT_k with the use of the following characterization of spaces (38): the relations

$$w \in W(K) : (w, \sigma)_K = 0 \quad \forall \sigma \in [P_{k-2}(K)]^d, \quad (w \cdot n_K, g)_{\partial K} = 0 \quad \forall g \in \Lambda(\partial K) = R_k(\partial K)$$

imply that $w = 0$ (see, e.g., [7, p. 121, Proposition 3.5]). By analogy with the case of BDM_k, the estimate $\epsilon_h \le ch^k$ holds for sufficiently smooth solutions.

Note the families of spaces $RT_{[k]}$ and $BDFM_{[k+1]}$, which also satisfy conditions ($H_{1,2,3,4}$) (for more detail on these spaces, see, e.g., [7, pp. 119–125]).

9. ITERATIVE METHOD

Fix some bases in the spaces $V(\hat{K})$, $W(\hat{K})$, and $\Lambda(\hat{e})$. No conditions are imposed on the choice of the bases: they can be of the node type (Lagrangian basis) and of the spectral type. Relying on condition (H_2), we can use affine transformations $\hat{K} \longrightarrow K$ and $\hat{e} \longrightarrow e$ to define suitable bases in $V(K)$, $W(K)$, $K \in \mathcal{T}_h$ and $\Lambda(e)$, $e \in \overline{\mathcal{E}}_h$ and, hence, in V_h, W_h, and Λ_h. Assume in what follows that a basis in each of these spaces is chosen as described above. The vectors of expansion coefficient of $v_h \in V_h$, $w_h \in W_h$, and $\mu_h \in \Lambda_h$ in terms of the corresponding bases are denoted by v, w, and μ, while the spaces of vectors of corresponding dimensions are designated as V, W, and Λ. The notation v_K, w_K, and μ_K is used for the vectors of expansion coefficients of $v_h|_K$, $w_h|_K$ and $\mu_h|_{\partial K}$, while the spaces of vectors of corresponding dimension are denoted by V_K, W_K, and Λ_K. To be definite, assume that $v = (v_{K_1}, v_{K_2}, \dots, v_{K_{N(h)}})$, $w = (w_{K_1}, w_{K_2}, \dots, w_{K_{N(h)}})$, and $\mathcal{T}_h = \{K_1, K_2, \dots, K_{N(h)}\}$. Using the notation $a \cdot b$ for the inner product in an arbitrary R^m, we define a nonlinear vector function $\mathbf{A}_h$ by the relation

$$\mathbf{A}_h(\mathbf{u}) \cdot \mathbf{v} = \mathbf{a}_h(\mathbf{u}_h, \mathbf{v}_h) = (k(\cdot, u_h, \nabla_h \mathbf{u}_h), \nabla_h \mathbf{v}_h)_{\mathcal{T}_h} + (k_0(\cdot, u_h, \nabla_h \mathbf{u}_h), v_h)_{\mathcal{T}_h}, \tag{39}$$

where $\mathbf{v} = (v, \mu) \in \mathbf{V}$ is the vector of expansion coefficients of $\mathbf{v}_h = (v_h, \mu_h) \in \mathbf{V}_h$. Let $\mathbf{F}_h \cdot \mathbf{v} = \mathbf{f}_h(\mathbf{v}_h) = (f, v_h)_{\mathcal{T}_h}$ for any $\mathbf{v}_h \in \mathbf{V}_h$. Then scheme (14) is equivalent to the solution of a system of algebraic, generally speaking, nonlinear equations of the form

$$\mathbf{A}_h(\mathbf{u}) = \mathbf{F}_h, \quad \mathbf{u} \in \mathbf{V}. \tag{40}$$

A preliminarily remark has to be made concerning linear problems. If the original equation (1) is linear and its operator is symmetric, i.e.,

$$k(x, u, \sigma) = A(x)\sigma + b(x)u, \quad k_0(x, u, \sigma) = c(x) \cdot \sigma + d(x)u,$$

$$k(x, u, \sigma) \cdot \eta + k_0(x, u, \sigma)v = k(x, v, \eta) \cdot \sigma + k_0(x, v, \eta)u,$$

then it follows directly from (39) that $\mathbf{A}_h$ is a symmetric positive definite matrix.

Consider system (40) with $\mathbf{F}_h = (f, 0)$ and $k_0 = 0$. We use the operators with zero kernels introduced before Lemma 4: $\nabla^0_K : V(K) \longrightarrow W(K)$ and $\nabla_{\partial K}: \Lambda(\partial K) \longrightarrow W(K)$, so that $\nabla_K \mathbf{v}_h = \nabla^0_K v_h + \nabla_{\partial K}\mu_h$. We have

$$\mathbf{A}_h\mathbf{u}\cdot\mathbf{v} = \sum_{K \in \mathcal{T}_h} (A(\nabla^0_K u_h + \nabla_{\partial K}\lambda_h) + bu_h, \nabla^0_K v_h + \nabla_{\partial K}\mu_h)_K = \mathbf{F}_h \cdot \mathbf{v}.$$

Setting $\mathbf{v} = (v, 0)$ with $v = (0, \ldots, v_K, \ldots., 0)$ and $\mathbf{v} = (0, \mu)$ yields

$$(A(\nabla^0_K u_h + \nabla_{\partial K}\lambda_h) + bu_h, \nabla^0_K v_h)_K = f_K \cdot v_K, \quad K \in \mathcal{T}_h, \tag{41}$$

$$\sum_{K \in \mathcal{T}_h} (A(\nabla^0_K u_h + \nabla_{\partial K}\lambda_h) + bu_h, \nabla_{\partial K}\mu_h)_K = 0. \tag{42}$$

For fixed λ_h, it is easy to see that $u_h|_K$ can be elementwise determined from Eqs. (41) and can then be used in (42). As a result, we obtain a system of algebraic equations for determining only the unknown λ. Below, this procedure is described in more detail as applied to the matrix $\mathbf{B}_h$ defined by the relation

$$\mathbf{B}_h\mathbf{u}\cdot\mathbf{v} = (\nabla_h\mathbf{u}_h, \nabla_h\mathbf{v}_h)_{\mathcal{T}_h}.$$

Let $\|\mathbf{v}\|^2_{\mathbf{B}_h} = \mathbf{B}_h\mathbf{v}\cdot\mathbf{v}$. Then the estimates from Lemma 2 become

$$(\mathbf{A}_h(\mathbf{u}_1) - \mathbf{A}_h(\mathbf{u}_2))\cdot(\mathbf{u}_1 - \mathbf{u}_2) \ge c_0\|\mathbf{u}_1 - \mathbf{u}_2\|^2_{\mathbf{B}_h} \quad \forall \mathbf{u}_1, \mathbf{u}_2 \in \mathbf{V},$$

$$|(\mathbf{A}_h(\mathbf{u}_1) - \mathbf{A}_h(\mathbf{u}_2))\cdot\mathbf{v}| \le C_0\|\mathbf{u}_1 - \mathbf{u}_2\|_{\mathbf{B}_h}\|\mathbf{v}\|_{\mathbf{B}_h} \quad \forall \mathbf{u}_1, \mathbf{u}_2, \mathbf{v} \in \mathbf{V}.$$

Consider the following iterative method for solving system (40):

$$\tau^{-1}\mathbf{B}_h(\mathbf{u}^{k+1} - \mathbf{u}^k) + \mathbf{A}_h(\mathbf{u}^k) = \mathbf{F}_h, \quad k = 0, 1, \ldots, \tag{43}$$

where $\mathbf{u}^0 \in \mathbf{V}$ is a given vector and τ is an iteration parameter.

Theorem 8. *Let conditions* $(H_{1,2})$ *hold. Then, for any* $\mathbf{u}^0 \in \mathbf{V}$ *and* $\tau \in (0, 2C_0/c_0^2)$, *iterative process* (43) *converges to the solution* $\mathbf{u}$ *of system* (40); *moreover,* $\|\mathbf{u}^{k+1} - \mathbf{u}\|_{\mathbf{B}_h} \le \rho(\tau)\|\mathbf{u}^k - \mathbf{u}\|_{\mathbf{B}_h}$, *where* $\rho(\tau)$ *is independent of h and* $\rho(\tau) \in (0, 1)$.

The proof is well known (see, e.g., [14, p. 106] and also [15, Chapter XIII, Section 1]).

Let us describe an algorithm for solving the system of linear algebraic equations $\mathbf{B}_h\mathbf{u} = \mathbf{f}$, where $\mathbf{u} = (u, \lambda)$ and $\mathbf{f} = (f, g)$. By analogy with (41) and (42), we derive the equations

$$(\nabla^0_K u_h + \nabla_{\partial K}\lambda_h, \nabla^0_K v_h)_K = f_K, \quad \sum_{K \in \mathcal{T}_h} (\nabla^0_K u_h + \nabla_{\partial K}\lambda_h, \nabla_{\partial K}\mu_h)_K = g\cdot\mu$$

or, in matrix notation,

$$[\nabla^0_K]^{\mathrm{T}}[\nabla^0_K]u_K + [\nabla^0_K]^{\mathrm{T}}[\nabla_{\partial K}]\lambda_K = f_K, \quad K \in \mathcal{T}_h, \tag{44}$$

$$\sum_{K \in \mathcal{T}_h} ([\nabla_{\partial K}]^{\mathrm{T}}[\nabla^0_K]u_K + [\nabla_{\partial K}]^{\mathrm{T}}[\nabla_{\partial K}]\lambda_K) = g, \tag{45}$$

where $[A]$ is the matrix of the operator A and $[A]^{\mathrm{T}}$ is its transpose. The variable u is elementwise eliminated from system (44), (45) to obtain a system for determining $\lambda \in \Lambda$ with a symmetric positive definite matrix L_h:

$$L_h\lambda = d_h, \quad L_h\lambda = \sum_{K \in \mathcal{T}_h} ([\nabla_{\partial K}]^{\mathrm{T}} L_K[\nabla_{\partial K}])\lambda_K,$$
$$L_K = E_{W_K} - [\nabla^0_K]\Delta_K^{-1}[\nabla^0_K]^{\mathrm{T}}, \quad \Delta_K = [\nabla^0_K]^{\mathrm{T}}[\nabla^0_K], \tag{46}$$

where E_{W_K} is the identity matrix of dimension $\dim W_K$ and $d_h = g - \sum_{K \in \mathcal{T}_h}[\nabla_{\partial K}]^{\mathrm{T}}[\nabla^0_K]\Delta_K^{-1}f_K$.

In the matrix assembly algorithm used in FEM, the matrix L_h is calculated prior to the iteration with the use of the above formulas. Thus, the following algorithm is obtained for solving the system $\mathbf{B}_h\mathbf{u} = \mathbf{f}$ at every iteration step: (a) compute the vector d_h, (b) solve system (46), and (c) determine u elementwise by applying the formulas $u_K = f_K - \Delta_K^{-1}[\nabla_K^0]^{\mathrm{T}}[\nabla_{\partial K}]\lambda^K$.

Note that the matrix L_h has a characteristic sparse structure: the equations for determining $\lambda_h|_e$, $e \in \overline{\mathscr{E}}_h$, involve the degrees of freedom λ_h associated with the faces of at most two elements adjacent to e, irrespective of the triangulation method.

The above-described procedure for the elementwise elimination of unknowns is similar to hybridizable mixed FEM schemes for linear problems, in which the unknowns q_h and u_h are locally eliminated (see, e.g., [4; 7, pp. 178–181]).

Theorem 9. *Let conditions* $(H_{1,2})$ *hold. Then the condition number of the matrix* L_h *satisfies the estimate*

$$\mathrm{cond}_2(L_h) = \lambda_{\max}(L_h)/\lambda_{\min}(L_h) \le ch^{-2}.$$

Proof. According to the definition of $\mathbf{B}_h$ and Lemma 1, we have

$$\mathbf{B}_h\mathbf{u}\cdot\mathbf{u} = \|\nabla_h\mathbf{u}_h\|_{0,\Omega}^2 \sim |[\mathbf{u}_h]|_{1,h}^2 = \sum_{K\in\mathscr{T}_h}(\|\nabla u_h\|_{0,K}^2 + h^{-1}\|\lambda_h - \langle u_h\rangle_K\|_{0,\partial K}^2).$$

The matrix L_h is the corresponding Schur complement of the block matrix $\mathbf{B}_h$. Direct computations show that $L_h\lambda\cdot\lambda = \inf_{u\in V,\,\mathbf{u}=(u,\lambda)}\mathbf{B}_h\mathbf{u}\cdot\mathbf{u}$. It follows that $L_h\lambda\cdot\lambda \sim \inf_{u_h\in V_h}|[\mathbf{u}_h]|_{1,h}^2 = I(\lambda)$. Here, $I(\lambda)$ is easily calculated if we take into account that (a) the restrictions of $u_h \in V_h$ to different elements are not interrelated and (b) both the first and second term in the definition of the norm $|[\mathbf{u}_h]|_{1,h}$, $\langle\lambda\rangle_{\partial K} = |\partial K|^{-1}(\lambda, 1)_{\partial K}$ reach their minimum at $u_h|_K = \mathrm{const} = \langle\lambda_h\rangle_{\partial K}$. Thus,

$$L_h\lambda\cdot\lambda \sim I(\lambda) = h^{-1}\sum_{K\in\mathscr{T}_h}\|\lambda_h - \langle\lambda_h\rangle_{\partial K}\|_{0,\partial K}^2.$$

Since

$$\|\lambda_h - \langle\lambda_h\rangle_{\partial K}\|_{0,\partial K}^2 = \|\lambda_h\|_{0,\partial K}^2 - \|\langle\lambda_h\rangle_{\partial K}\|_{0,\partial K}^2,$$

we have

$$I(\lambda) \le h^{-1}\sum_{K\in\mathscr{T}_h}\|\lambda_h\|_{0,\partial K}^2 \le 2h^{-1}\|\lambda_h\|_{0,\mathscr{E}_h}^2.$$

To estimate $I(\lambda)$ from below, we define a partial extension of λ_h to Ω. Let $\langle\lambda_h\rangle$ be a piecewise constant function on $\overline{\mathscr{E}}_h$ and $\langle\lambda_h\rangle|_e = |e|^{-1}(\lambda_h, 1)_e$. It is easy to see that

$$I(\lambda) = h^{-1}\sum_{K\in\mathscr{T}_h}\|\lambda_h - \langle\lambda_h\rangle\|_{0,\partial K}^2 + h^{-1}\sum_{K\in\mathscr{T}_h}\|\langle\lambda_h\rangle - \langle\lambda_h\rangle_{\partial K}\|_{0,\partial K}^2 = I_1(\lambda) + I_2(\lambda).$$

Consider $I_K(\langle\lambda_h\rangle) = \|\langle\lambda_h\rangle - \langle\lambda_h\rangle_{\partial K}\|_{0,\partial K}^2$. Since $\langle\lambda_h\rangle_{\partial K}$ is a linear combination of $d+1$ numbers $\langle\lambda_h\rangle|_e$, $e \in \partial K$, the functional $I_K(\langle\lambda_h\rangle)$ can be regarded as a function of these $d+1$ variables. Let Y_K be a set of constant functions on the faces of K with the common element $\langle\lambda_h\rangle = (\langle\lambda_h\rangle|_{e_1}, \ldots, \langle\lambda_h\rangle|_{e_{d+1}})$. Clearly, I_K defines a seminorm on Y_K, and its kernel consists of the functions $\{c(1, \ldots, 1), c \in R\}$. Using an incompatible first-order finite element, we define a linear function $\tilde{\lambda}_h$ on K whose mean value on a face $e \in \partial K$ is equal to $\langle \lambda_h \rangle|_e$. Standard rescaling yields

$$I_K(\langle \lambda_h \rangle) \sim \|\partial \tilde{\lambda}_h\|^2_{0,K}, \quad \|\langle \lambda_h \rangle\|^2_{0,\partial K} \sim \|\tilde{\lambda}_h\|^2_{0,\partial K} \le ch^{-1}\|\tilde{\lambda}_h\|^2_{0,K}. \tag{47}$$

Since $\langle [\tilde{\lambda}_h] \rangle = 0$ on $\bar{\mathscr{E}}_h$, it follows from inequality (18) that $\|\tilde{\lambda}_h\|^2_{0,\Omega} \le c\|\nabla\tilde{\lambda}_h\|^2_{0,\mathscr{T}_h}$. According to (47), this implies the estimate $I_2(\lambda) \ge ch\|\langle \lambda_h \rangle\|^2_{0,\partial\mathscr{T}_h}$. Therefore,

$$I(\lambda) \ge h^{-1}\|\lambda_h - \langle \lambda_h \rangle\|^2_{0,\partial\mathscr{T}_h} + ch\|\langle \lambda_h \rangle\|^2_{0,\partial\mathscr{T}_h} \ge ch(\|\lambda_h - \langle \lambda_h \rangle\|^2_{0,\mathscr{E}_h} + \|\langle \lambda_h \rangle\|^2_{0,\mathscr{E}_h}) = ch\|\lambda_h\|^2_{0,\mathscr{E}_h},$$

since $\|\lambda_h - \langle \lambda_h \rangle\|^2_{0,\mathscr{E}_h} = \|\lambda_h\|^2_{0,\mathscr{E}_h} - \|\langle \lambda_h \rangle\|^2_{0,\mathscr{E}_h}$. Thus, $ch\|\lambda_h\|^2_{0,\mathscr{E}_h} \le L_h\lambda \cdot \lambda \le ch^{-1}\|\lambda_h\|^2_{0,\mathscr{E}_h}$. This implies the assertion of the theorem, since

$$\|\lambda_h\|^2_{0,\mathscr{E}_h} = \sum_{e \in \mathscr{E}_h} \|\lambda_h\|^2_{0,e} \sim \sum_{e \in \mathscr{E}_h} |e|\lambda_e \cdot \lambda_e \sim h^{d-1}\lambda \cdot \lambda.$$

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