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Solvability of the Gakhov Equation in S. N. Kudryashov's Example

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Abstract—The picture of the solvability of Gakhov's equation in the classical example of S.N. Kudryashov is fully explored. A complete description of the dynamics of the roots of Gakhov's equation for the family of level lines generated by the Kudryashov function is given.

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1. INTRODUCTION

One of the historically significant examples in the classics of the theory of exterior inverse boundary value problems (IBVP) for analytic functions is the example of S. N. Kudryashov. This example appeared in the work [1] as an illustration of sharpness to the now legendary Kudryashov's theorem on the uniqueness of the solution of the exterior IBVP in the class of convex functions in the exterior of the unit disk. The purpose of the example was to show that the exterior IBVP could have several univalent solutions (as an answer to the question from [2], p. 55). Interest in this aspect was renewed in due time in connection with studies on the Nuzhin hypothesis (see [3, 4]). The example itself was regularly mentioned in the IBVP monographs (see, e.g., [5, 6]).

The name "Kudryashov's example" refers to the family of functions

$$F_\alpha(z) = \alpha z(z-\alpha)(\alpha z - 1)^{-1}, \quad \alpha > 1, \tag{1}$$

that are given in the exterior of the unit circle, $E^- = \{|z| > 1\}$. Each of the functions of the family (1) is considered as the main solution of the exterior IBVP, i.e., solution with pole at infinity, $z = \infty$. This means that the function $F_\alpha(z)$ has the form

$$F_\alpha(z) = \int f'_\alpha(1/z)dz, \tag{2}$$

where $f_\alpha(\zeta)$ denotes the solution of corresponding interior IBVP; function $f_\alpha(\zeta)$ is defined in the unit disk, $E = \{|\zeta| < 1\}$, we will call it the Kudryashov function. It follows from (1) and (2) that

$$f'_\alpha(\zeta) = \alpha(\alpha\zeta^2 - 2\zeta + \alpha)(\zeta - \alpha)^{-2}. \tag{3}$$

Solutions of the exterior IBVP correspond to the roots of the Gakhov equation

$$f''(\zeta)/f'(\zeta) = 2\bar{\zeta}/(1 - |\zeta|^2), \tag{4}$$

which are exactly the critical points of the surface of the conformal radius

$$R_f(\zeta) = |f'(\zeta)|(1 - |\zeta|^2) \tag{5}$$

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for a holomorphic and locally univalent function f in the disk E (see [7]). The set of such points lying in E will be denoted by M_f. As found in [8], the elements of the set M_f can be the points of only three types: maxima, saddles, and semi-saddles of the surface (5).

Returning to the Kudryashov function, we find from (3) that for $f = f_\alpha$ equation (4) is

$$f''_\alpha(\zeta)/f'_\alpha(\zeta) \equiv 2(1-\alpha^2)\zeta(\alpha\zeta^2 - 2\zeta + \alpha)^{-1}(\zeta - \alpha)^{-1} = 2\bar{\zeta}/(1 - |\zeta|^2). \tag{6}$$

Interest in Kudryashov's example was supported by the fact that in [1] the Gakhov equation was solved only over $\mathbb{R}$—this was enough to prove the existence of more than one solution. Since then, the following question has not been solved for a long time: whether the set M_{f_α} is exhausted by the real roots of (6), or not. In the present note this question is answered positively (Section 2). A complete picture of the solvability of the equation (4) is also constructed for $f = f_{\alpha,r}$, where

$$f_{\alpha,r} = f_\alpha(r\zeta)/r, \quad 0 \le r \le 1, \tag{7}$$

is a family of the level sets of the Kudryashov function f_α, $\alpha > 1$ (Sections 3–6). Gakhov barrier $\bar{r}_{f_\alpha}$ of the family (7) is defined as the least upper bound of numbers $t \in [0,1]$ such that for every $r \in [0,t]$ the function $f_{\alpha,r}$ belongs to the regular Gakhov class, i.e. the conformal radius $R_{f_{\alpha,r}}(\zeta)$ has a unique critical point, which is its local maximum [9].

Remark 1. The problem of calculating Gakhov barriers for level lines was posed by the second author in his 1983 student course work. The stages of further development of this topic—in a wider context of parametric families of conformal radii – are noted in [10] (see also [9]).

Let $\alpha > 1$ be an arbitrary fixed number. The following assertions are true. The first is essentially part of the second; however, we single out the first in view of its historical significance.

Theorem 1. *The set M_{f_α} contains exactly three points: two maxima at $\zeta = 0$ and $\zeta = 1/\alpha$, and the saddle at $\zeta = \alpha - \sqrt{\alpha^2 - 1}$.*

Theorem 2. *Gakhov barrier $\bar{r}_{f_\alpha}$ of the family (7) is equal to the value*

$$\bar{r}(\alpha) = (9\alpha^2 + 2(4\alpha^2 - 1)A(\alpha))^{1/2}/(3\alpha\sqrt{3}), \tag{8}$$

where $A(\alpha) = 2\alpha^2 + 1 - \sqrt{(4\alpha^2 - 1)(\alpha^2 - 1)}$. When $r \in [0, \bar{r}_{f_\alpha})$, the only root of the Gakhov equation is the point $\zeta = 0$—maximum of $R_{f_{\alpha,r}}(\zeta)$, to which, for $r = \bar{r}_{f_\alpha}$, the new root is added at the point

$$\zeta_\alpha = A(\alpha)/(3\alpha\bar{r}_{f_\alpha}) \tag{9}$$

– the semi-saddle of the surface $R = R_{f_{\alpha,r}}(\zeta)$. As r increases between the Gakhov barrier $\bar{r}_{f_\alpha}$ and unity, the root ζ_α bifurcates into two roots, a saddle and a maximum. The saddle moves towards the origin, and at $r = 1$ comes to the point $\alpha - \sqrt{\alpha^2 - 1}$, the maximum grows to the unit circle, and at $r = 1$ stops at the point $1/\alpha$.

2. SOLUTION OF THE GAKHOV EQUATION FOR THE KUDRYASHOV FUNCTION

We represent the equation (6) in the form $(1-\alpha^2)\zeta(1-|\zeta|^2) = \bar{\zeta}(\alpha\zeta^3 - (2+\alpha^2)\zeta^2 + 3\alpha\zeta - \alpha^2)$. The standard change $\zeta = \rho e^{i\theta}$ ($0 \le \rho \le 1$, θ varies over a semi-segment of length 2π) allows us to immediately separate the solution $\rho = 0$. Multiplying both parts of the resulting relation by $e^{-i\theta}$, we obtain the equation $1 - \alpha^2 + (1 + 2\alpha^2)\rho^2 = \alpha\rho^3 e^{i\theta} + 3\alpha\rho e^{-i\theta} - \alpha^2 e^{-i2\theta}$. Separating the real and imaginary parts in it, we will have

$$1 - \alpha^2 + (1 + 2\alpha^2)\rho^2 = \alpha\rho^3\cos\theta + 3\alpha\rho\cos\theta - \alpha^2\cos 2\theta, \tag{10}$$

$$0 = \alpha\sin\theta(\rho^3 - 3\rho + 2\alpha\cos\theta). \tag{11}$$

Obviously, equation (11) splits into two: 1) $\cos\theta = \rho(3-\rho^2)/(2\alpha)$ or 2) $\sin\theta = 0$.

Case 1. Let's substitute the above expression for $\cos\theta$ into (10); we obtain

$$\rho^6 - 3\rho^4 + (2\alpha^2 + 1)\rho^2 - (2\alpha^2 - 1) \equiv (\rho^2 - 1)[(\rho^2 - 1)^2 + 2(\alpha^2 - 1)] = 0. \tag{12}$$

The only positive solution of equation (12) gives two roots $\zeta = \zeta_{\pm}(\alpha) \equiv 1/\alpha \pm i\sqrt{1-1/\alpha^2}$ $(\alpha > 1)$ of the equation (6). These roots lie on the unit circle and are first-order poles of the pre-Schwarzian, f''_α/f'_α, and are also the zeroes of the derivative, f'_α, of the function f_α. Since $R_{f_\alpha}(\partial E) = 0$, the points $\zeta = \zeta_{\pm}(\alpha)$ do not make any contribution to the geometry of the surface (5) for $f = f_\alpha$ (in the sense of [11]) and do not give new roots to the Gakhov equation in E.

Case 2. The equation $\sin\theta = 0$ exactly corresponds to the case of real roots, which was completely studied in [1]. The three real roots $\zeta = 0$, $\zeta = 1/\alpha$ and $\zeta = \alpha - \sqrt{\alpha^2-1}$ obtained in [1] exhaust the set M_{f_α}. The geometry of these points as critical for the conformal radius R_{f_α} is established using the classification according to the sign of curvature from [12].

Theorem 1 is proved.

3. GAKHOV EQUATION FOR THE LEVEL LINES OF KUDRYASHOV'S FUNCTION

Gakhov's equation for the function $f_{\alpha,r}$ has the form

$$f''_{\alpha,r}(\zeta)/f'_{\alpha,r}(\zeta) \equiv 2(1-\alpha^2)r^2\zeta(\alpha r^2\zeta^2 - 2r\zeta + \alpha)^{-1}(r\zeta - \alpha)^{-1} = 2\bar{\zeta}/(1-|\zeta|^2). \tag{13}$$

Recall that $\alpha > 1$ and that the roots are found in the unit disk E. Acting as in Section 2, we substitute representation $\zeta = \rho e^{i\theta}$ into (13) to separate the solution $\rho = 0$ and to obtain the following analogues of equations (10) and (11)—in order to find the remaining solutions

$$r^2[1-\alpha^2+(1+2\alpha^2)\rho^2] = \alpha r\rho(3+r^2\rho^2)\cos\theta - \alpha^2\cos 2\theta, \tag{14}$$

$$0 = \alpha\sin\theta[r^3\rho^3 - 3r\rho + 2\alpha\cos\theta]. \tag{15}$$

Equation (15) splits into two: 1) $\cos\theta = r\rho(3-r^2\rho^2)/(2\alpha)$ or 2) $\sin\theta = 0$.

Case 1. Substitution of the above expression for $\cos\theta$ in (14) leads to the equation

$$r^6\rho^6 - 3r^4\rho^4 + (2\alpha^2+1)r^2\rho^2 + r^2(1-\alpha^2) - \alpha^2 = 0, \tag{16}$$

and the replacement $r^2\rho^2 = x^2$ simplifies (16) to the equation

$$(x-1)^3 + 2(\alpha^2-1)(x-1) + (1-r^2)(\alpha^2-1) = 0. \tag{17}$$

Let us put our situation in the context of the general theory of a cubic equation (see, e.g., [13]).

Changes $y = x-1$, $p_1 = 2(\alpha^2-1)$ and $q_1 = (1-r^2)(\alpha^2-1)$ allow us to rewrite equation (17) in the reduced form $y^3 + p_1y + q_1 = 0$. In view of the conditions $p_1 > 0$ and $q_1 \geq 0$ we conclude that the discriminant of the polynomial on the left hand side of (17) is strictly greater than zero: $\Delta_1 = (q_1/2)^2 + (p_1/3)^3 > 0$. In this case, according to the "general theory", equation (17) has one real and two complex conjugate solutions. By the construction of x, we are only interested in the real solution of the equation (17). We have a number of the following simple statements:

Lemma 1. *The only real root of equation (17) is positive for any* $r \in [0,1]$, *strictly less than unity for any* $r \in [0,1)$, *and equal to unity for* $r = 1$.

The root from Lemma 1 is calculated using the Cardano formula and has the form $x = 1 + u_+ + u_-$, where $u_\pm = \sqrt[3]{-q_1/2 \pm \sqrt{\Delta_1}}$ (we omit the dependence on r). It is clear that the situation $r = 1$ corresponds to case 1) from Section 2. Now let's move on to ρ.

Lemma 2. *When* $r = 0$, *equation (16) has no roots. For any* $r \in (0,1]$ *equation (16) has a unique positive root* $\rho(r) = \sqrt{1+u_+(r)+u_-(r)}/r$.

The properties of the function $\rho = \rho(r)$ are studied in the following

Lemma 3. *Equalities* $\lim_{r\to 0-}\rho(r) = +\infty$ *and* $\rho(1) = 1$ *are satisfied. An inequality* $\rho(r) > 1$ *is valid for all* $r \in (0,1)$. *The function* $\rho = \rho(r)$ *strictly decreases on* $[0,1]$ *from* $+\infty$ *to* $r = 1$.

Lemma 3 shows that case 1) does not lead to new roots of the Gakhov equation.

Case 2. The substitution of $\sin\theta = 0$ and $\cos\theta = -1$ in (14) does not lead to new roots. On the contrary, the situation $\sin\theta = 0$, $\cos\theta = +1$ (i.e. $\theta = 0$) turns out to be generating for the roots of equation (13). In this case, equation (14) will have the form

$$\alpha r^3\rho^3 - (2\alpha^2+1)r^2\rho^2 + 3\alpha r\rho - (\alpha^2 - r^2(\alpha^2-1)) = 0. \tag{18}$$

Now this is our *main* equation.

We transform equation (18) by the replacement $x = r\rho$ and dividing all its coefficients by α:

$$x^3 - (2\alpha + 1/\alpha)x^2 + 3x - (\alpha^2 - r^2(\alpha^2 - 1))/\alpha = 0. \tag{19}$$

This will be our *auxiliary* equation. For the new coefficients, we introduce the notation $R = -(2\alpha^2 + 1)/\alpha$, $S = 3$ and $T = -(\alpha^2 - r^2(\alpha^2 - 1))/\alpha$. Coefficients of the reduced form

$$y^3 + py + q = 0 \tag{20}$$

of the equation (19), which can be written as $x^3 + Rx^2 + Sx + T = 0$, are obtained by the change of variable $x = y - R/3 = y + (2\alpha^2 + 1)/(3\alpha)$. We have

$$p = S - R^2/3 = -(\alpha^2 - 1)(4\alpha^2 - 1)/(3\alpha^2)(< 0), \tag{21}$$

$$q = 2R^3/27 - RS/3 + T = -(\alpha^2 - 1)[16\alpha^4 + 13\alpha^2 - 2 - 27\alpha^2 r^2]/(27\alpha^3)(< 0). \tag{22}$$

The discriminant is

$$\Delta = \Delta(r) = (q/2)^2 + (p/3)^3 = ((\alpha^2 - 1)/(4\alpha^2))(r^2 - r_+^2)(r^2 - r_-^2), \tag{23}$$

where $r_- = \bar{r}(\alpha)$ (see (8)) and $r_+^2 + r_-^2 = 2(16\alpha^4 + 13\alpha^2 - 2)/(27\alpha^2)$. It is clear that $0 < r_- < 1$ and $r_+ > 1$. In what follows, we will often use the notation

$$P = \sqrt{-p/3}. \tag{24}$$

In the framework of the case 2), we will stay until the end of the article. On the other hand, the stages of the further solution will be "colored in the tone" of the general theory of the cubic equation, which will naturally determine the further division of the text into sections.

4. GLOBAL PICTURE OF SOLVABILITY

1⁰ *Case* $0 < r < r_-$. In view of (23), we have $\Delta > 0$. Here the equation (20) has three different roots—two complex conjugate and one real. Since among the roots of equation (18) we are looking for values for the polar radius, then among the above three roots of equation (20) we keep only the real one. As in Lemma 1, the only real root of (20) is calculated by the Cardano formula

$$y = U_+ + U_-, \quad U_\pm = \sqrt[3]{-q/2 \pm \sqrt{\Delta}}, \tag{25}$$

where $U_+^3 + U_-^3 = -q$, and $y = -q/((U_+ - U_-)^2 - p/3) > 0$. Then the corresponding unique real root of the equation (19) has the form $x = 1 + U_+ + U_- + (\alpha - 1)(2\alpha - 1)/(3\alpha) > 1$ (we remember that $\alpha > 1$). Recalling that $x = r\rho$, $r \le 1$, from the last inequality we conclude that the only root ρ of our basic equation (18) satisfies the strict inequality $\rho > 1$ (more details: $\rho \ge r\rho = x > 1$). This means that if $0 < r < r_-$, then Gakhov's equation (13) has no nonzero roots.

2⁰ *Case* $r = r_-$. Here $\Delta = 0$. In this case, the reduced equation (20) has two "geometrically different" real roots: the root "going from" (25), $y = U_+ + U_- = 2\sqrt[3]{-q/2}$, and the double root, $\tilde{y} = -y/2 = -\sqrt[3]{-q/2}$, resulting the merge of two complex conjugate roots from the case **1⁰**.

Using (22) and (8) we set the expression for $-q/2$, and hence the explicit view of roots y and $\tilde{y}$; the corresponding roots of the equation (19) will be as follows: $x = [2\alpha^2 + 1 + 2\sqrt{(\alpha^2 - 1)(4\alpha^2 - 1)}]/(3\alpha)$ and $\tilde{x} = [2\alpha^2 + 1 - \sqrt{(\alpha^2 - 1)(4\alpha^2 - 1)}]/(3\alpha)$. The root x continues the family from subsection **1⁰** and therefore is excluded ($x > 1$). A strict inequality $\tilde{\rho}^2 = \tilde{x}^2 r_-^2 < 1$ is established by a simple but routine check. Let's summarize the preliminary output.

Proposition 1. Gakhov barrier $\bar{r}_{f_\alpha}$ of the level set family (7) is equal to $\bar{r}_{f_\alpha} = \bar{r}(\alpha)$, where the value $\bar{r}(\alpha)$ is defined in (8). When $r \in [0, \bar{r}_{f_\alpha})$, the only root of the Gakhov equation (13) is the point $\zeta = 0$—the maximum of the conformal radius (5) for $f = f_\alpha$. If $r = \bar{r}_{f_\alpha}$, the set M_{f_α} consists of exactly two points—the maximum $\zeta = 0$ and the semi-saddle $\zeta_\alpha = \tilde{x}/\bar{r}_{f_\alpha}$.

Maximum of the conformal radius $R_{f_{\alpha,r}}(\zeta)$ at $\zeta = 0$ takes place due to [12], the presence of a semi-saddle of $R_{f_{\alpha,r}}(\zeta)$ at a point ζ_α is established using the formula of M.I. Kinder (see, e.g., [11]).

3^0 *Case* $r_- < r < 1$. We are in the situation $\Delta < 0$. In the general theory of the cubic equation, this is the so-called "casus irreducibilis". Within the framework of this "casus", equation (20) has three different real roots; to represent them, we use a continuous (and even differentiable) parametrization by r. Thus, we immerse the situation in an analytical context. It follows from (21) and (22) that for $\Delta < 0$ it will be $0 < -(q/2)/P^3 < 1$, where P is defined in (24). Hence, for any $r \in (r_-, 1]$ there exists a unique $\theta_0 = \theta_0(r) \in (0, \pi/2)$ such that $\theta_0(r) \equiv \arccos \lambda(r)$, where

$$\lambda(r) = -(27\alpha^3/2)q[(\alpha^2-1)(4\alpha^2-1)]^{-3/2}, \tag{26}$$

and the roots of equation (20) can be represented as

$$y_m(r) = 2P\cos(\theta_{m-1}(r)/3), \quad \theta_{m-1}(r) = \theta_0(r) + 2\pi(m-1), \quad m = 1,2,3. \tag{27}$$

Taking into account (26) and formulas for $r_\pm^2$, we conclude that the function $\lambda(r)$ decreases on the segment $[r_-, r_+]$ from value $\lambda(r_-) = +1$ to value $\lambda(r_+) = -1$. It is clear that the function $\theta = \theta_0(r)$ increases on $[r_-, r_+]$ as a superposition of decreasing functions.

4^0 *Stitching real roots (20) over* $(0, r_-]$ *and* $[r_-, 1]$ *into the "global picture of resolvability"*. When $0 < r < r_-$ ($\Delta > 0$), there is a single real root (25), which takes the form $y = 2\sqrt[3]{-q/2}$ when passing $\Delta \downarrow 0$ to the case $r = r_-$ ($\Delta = 0$), where a double root $\tilde{y} = -\sqrt[3]{-q/2}$ also appears. If $(1 >)r > r_-$ ($\Delta < 0$), then we have three real roots (27).

Let's see what happens when $\Delta \uparrow 0$, i.e. when $r \downarrow r_-$. The passage to the limit $\lambda(r) \uparrow \lambda(r_-) = +1$ when $r \downarrow r_-$, established above, entails the passage to the limit $\theta_0(r) = \arccos \lambda(r) \downarrow \theta_0(r_-) = 0$. Let us watch for the behavior of the roots (27) when $r \downarrow r_-$.

The root $y_1(r)$. If $r \downarrow r_-$, then $y_1(r) = 2P\cos(\theta_0(r)/3) \to y_1(r_-) = 2P$. Since $\Delta = 0$, we have $P = \sqrt[3]{-q/2}$, whence $y_1(r_-) = y$, i.e., for $r \downarrow r_-$ the root $y_1(r)$ (from the case $\Delta < 0$) goes to the root $y = 2\sqrt[3]{-q/2}$ (corresponding to the case $\Delta = 0$). Thus, the last root continuously sews the roots (25) and $y_1(r)$ from (27) into one root.

The roots $y_{2,3}(r)$. Tending $r \downarrow r_-$ leads to the passages $y_2(r), y_3(r) \to \tilde{y}$. So, the root $\tilde{y}$ appearing at $r = r_-$ with further growth of r bifurcates into two continuous roots $y_{2,3}(r)$, $r \in (r_-, 1]$.

5. PASSING TO THE MAIN EQUATION

1. *The root* $\rho_1(r)$. The inequality $\rho_1(r) > 1$, which is valid for $r \in (0, r_-)$, was proved in Section 4. Let us prove it for $r = r_-$ and for $r \in (r_-, 1]$. For the root $x_1(r_-)$ of equation (19) the relation $x_1(r_-) = 1 + (\alpha-1)(2\alpha-1)/(3\alpha) + 2P > 1$ is valid.

Now let's look at the root $x_1(r)$ when $r \in (r_-, 1]$. By virtue of (27) with $m = 1$ we have the equality $x_1(r) = 1 + (\alpha-1)(2\alpha-1)/(3\alpha) + 2P\cos(\theta_0(r)/3)$, whence $x_1(r) > 1$ for $r \in (r_-, 1]$ due to the estimates $1 > \cos(\theta_0(r)/3) > 0$, following from the inequalities $0 < \theta_0(r) < \pi/2$, which are established in Section 4 for any $r \in (r_-, 1]$. As a result we have $x_1(r) > 1$ for $r \in [r_-, 1]$. So, for the solution $\rho_1(r)$ of the main equation (18) the inequalities $\rho_1(r) = x_1(r)/r > 1/r \geq 1$ are valid for $r \in [r_-, 1]$. Thus, $\rho_1(r) > 1$ also for $r \in (0, 1]$.

2. *The root* $\rho_2(r)$. We are to prove that $\rho_2(r)$ decreases with respect to r. From the second formula (27) we obtain $x_2'(r) = -2P\sin(\theta_1(r)/3)\theta_0'(r)/3$. First, we remember from Section 4 that the derivative of theta is positive. Next, for any $r \in (r_-, 1]$ there will be $2\pi/3 < \theta_1(r)/3 < 5\pi/6$. It is clear that in this case we have $\sin(\theta_1(r)/3) > 0$. Finally, $x_2'(r) < 0$, $r \in (r_-, 1]$, which means that the function $x_2(r)$, and hence $\rho_2(r) = x_2(r)/r$ decreases on $(r_-, 1]$.

3. *Function* $r\theta_0'(r)$. To study the dynamics of the third root, $\rho_3(r)$, of our main equation, we need a closer look at the behavior of the function θ_0. Recall that $\theta_0 = \theta_0(r) = \arccos \lambda(r)$, where the function $\lambda(r)$ is determined in (26). Further, due to the relation (23) and expressions for $r_\pm^2$ we have $\theta_0'(r) = 2r/\sqrt{(r_+^2 - r^2)(r^2 - r_-^2)}$ and $\theta_0'(r) > 0$ for any $r \in (r_-, r_+)$.

Let's prove that the function $r\theta_0'(r)/2$ decreases on the segment $(r_-, 1]$. To do this, we will work with a function $h(t) = t^2/((r_+^2 - t)(t - r_-^2))$ with increasing t from r_-^2 to 1. We have $h'(t) = (r_+^2 + r_-^2)t(r_+^2 - t)^{-2}(t - r_-^2)^{-2}[t - 2r_+^2 r_-^2/(r_+^2 + r_-^2)]$, where $1 < 2r_+^2 r_-^2/(r_+^2 + r_-^2) < r_+^2$. In fact, the right-hand inequality is obvious, the left-hand one follows from the formulas for $r_\pm^2$: $2r_+^2 r_-^2/(r_+^2 + r_-^2) = 1 + 2(\alpha^2-1)(8\alpha^2-1)/(16\alpha^4+13\alpha^2-2) > 1$. So, when $r_-^2 < t \le 1$, the inequality $t - 2r_+^2 r_-^2/(r_+^2 + r_-^2) < 0$ is satisfied, and hence also $h'(t) < 0$. Then the function $h(t)$ decreases on $(r_-^2, 1]$, and the function $r\theta_0'(r)/2 = \sqrt{h(r^2)}$ decreases in $r \in (r_-, 1]$. Thus,

$$r\theta_0'(r) > \theta_0'(1), \quad r \in (r_-, 1). \tag{28}$$

4. *The root* $\rho_3(r)$. The third root (27) of the reduced equation (20) generates the root of auxiliary equation (19): $x_3(r) = (2\alpha^2+1)/(3\alpha) + y_3(r) = (2\alpha^2+1)/(3\alpha) + 2P\cos(\theta_2(r)/3)$. We claim that the corresponding root of the main equation (18) is the increasing function $\rho_3(r) = x_3(r)/r$. For this it is enough to show that the expression $N(r) = x_3'(r)r - x_3(r)$ is positive. We have $N(r) = -2P\sin(\theta_2(r)/3)\cdot\theta_0'(r)\cdot r/3 - (2\alpha^2+1)/(3\alpha) - 2P\cos(\theta_2(r)/3)$.

Recall that $0 < \theta_0(r) < \pi/2$ for any $r \in (r_-, 1]$ (Section 4). Then, for every $r \in (r_-, 1]$, the inequalities $4\pi/3 < \theta_2(r)/3 < 3\pi/2$ are valid. These inequalities determine: for the cosine—the interval of increase, and for the sine—the interval of decrease. Consequently, $-\cos(\theta_2(r)/3) > 0$ and $-\sin(\theta_2(r)/3) > \sqrt{3}/2$. Therefore, $N(r) > P\cdot(\sqrt{3}/3)\cdot r\theta_0'(r) - (2\alpha^2+1)/(3\alpha)$. Due to (21), (28) and $(r_+^2-1)(1-r_-^2) = 4(\alpha^2-1)/(27\alpha^2)$ this estimate continues to $N(r) > (32\alpha^4 - 13\alpha^2 - 1)/\{3\alpha[3\alpha(4\alpha^2-1)^{1/2} + (2\alpha^2+1)]\} > 0$. Hence, $\rho_3'(r) = N(r)/r^2 > 0$, $r \in (r_-, 1]$. Thus, the root $\rho_3(r)$ of equation (18) increases in r.

6. RETURNING TO THE GAKHOV EQUATION

Let us assemble a picture of the solvability of equation (18) over the field of real numbers.

We have: one function, $\rho_1(r)$, continuous on $(0, 1]$, and two functions, $\rho_2(r)$ and $\rho_3(r)$, continuous on $[r_-, 1]$ and coinciding only for $r = r_-$. These are the functions that describe the set of solutions to equation (18) over $r \in (0, 1]$. Namely, for each $r \in (0, r_-)$, equation (18) has a single root $\rho_1(r) > 1$; for $r = r_-$ it has two roots—the root $\rho_1(r_-) > 1$ and the multiple root $\rho_2(r_-) = \rho_3(r_-) < 1$; for each $r \in (r_-, 1]$ equation (18) has three different roots—the root $\rho_1(r) > 1$ and the roots $\rho_2(r)$ and $\rho_3(r)$.

The function $\rho_1(r)$ is differentiable for $r \in (0, 1) \setminus \{r_-\}$, the functions $\rho_2(r)$ and $\rho_3(r)$ − for $r \in (r_-, 1]$. Here, if $r \in (r_-, 1]$, then $\rho_2'(r) < 0$ and $\rho_3'(r) > 0$.

Lemma 4. *The roots $\rho_2(r)$ and $\rho_3(r)$ of the equation (18) do not exceed unity for $r \in [r_-, 1]$.*

Proof. Since $\rho_2'(r) < 0$, we have $\rho_2(r) < \rho_2(r_-) < 1$ for every $r \in (r_-, 1]$.

Inequality $\rho_3(r) < \rho_1(r)$ extends from the point $r = r_-$ to the half-segment $(r_-, 1]$ in view of the presence of exactly three different roots of (18) for every $r \in (r_-, 1]$. Since $\rho_2(r_-) = \rho_3(r_-) (= \zeta_\alpha)$, then $\rho_2(r) < \rho_3(r)$ on the half-segment $(r_-, 1]$, because $\rho_2(r)$ decreases and $\rho_3(r)$ increases on it. So, for any $r \in (r_-, 1]$ there will be $\rho_2(r) < \rho_3(r) < \rho_1(r)$. In particular, $\rho_2(1) < \rho_3(1) < \rho_1(1)$. But in Section 2 we established that when $r = 1$, the equation (18) has the roots $\alpha - \sqrt{\alpha^2-1} < 1/\alpha < \alpha + \sqrt{\alpha^2-1}$. Therefore, $\rho_2(1) = \alpha - \sqrt{\alpha^2-1}$, $\rho_3(1) = 1/\alpha$, $\rho_1(1) = \alpha + \sqrt{\alpha^2-1}$.

Since $\rho_3(r)$ increases for $r \in (r_-, 1]$, then the inequality $\rho_3(r) < \rho_3(1)$ holds for $r \in (r_-, 1)$, and since $\rho_3(1) = 1/\alpha$, then, finally, $\rho_3(r) < 1$, $r \in [r_-, 1]$. □

Corollary. *The inequalities $\rho_2(1) = \alpha - \sqrt{\alpha^2-1} < \rho_2(r) < \zeta_\alpha$ and $\zeta_\alpha < \rho_3(r) < \rho_3(1) = 1/\alpha$ are valid for every $r \in (r_-, 1)$.*

Since the points $\rho_2(r)$ and $\rho_3(r)$—the real roots of the equation (18)—lie in the unit disk E, then they satisfy the Gakhov equation. Its solvability in E (without the zero root) reduced to the solvability of equation (18) on the interval $(0, 1)$.

According to [11], the roots of the Gakhov equation with decreasing moduli are saddles, and the ones with increasing moduli are maxima of the surface of conformal radius. Hence, for any $r \in (r_-, 1]$ the point $\rho_2(r)$ is the saddle, and the point $\rho_3(r)$ is the maximum of the surface $R_{f_\alpha}(\zeta)$.

The following addition to Proposition 1 completes our study.

Proposition 2. If $r \in (0, \bar{r}_{f_\alpha})$, then $M_{f_\alpha} = \{0\}$; when $r = \bar{r}_{f_\alpha}$, there will be $M_{f_\alpha} = \{0, \zeta_\alpha\}$; in the case $r \in (\bar{r}_{f_\alpha}, 1]$ we have $M_{f_\alpha} = \{0, \rho_2(r), \rho_3(r)\}$. Here, $\zeta = \zeta_\alpha$ is the semi-saddle, $\zeta = \rho_2(r)$ is the saddle, and the points $\zeta = 0$ and $\zeta = \rho_3(r)$ are maxima of the surface of the conformal radius $R_{f_\alpha}(\zeta)$.

Theorem 2 is proved.

Remark 2. According to [9], the value $\bar{r}_{f_\alpha}$ is the parameter of the $\bigcup$-exit out of the regular Gakhov class along the level lines of the Kudryashov function f_α.

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