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Ideal spaces of measurable operators affiliated to a semifinite von Neumann algebra. II

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Abstract

Let τ be a faithful semifinite normal trace on a von Neumann algebra $\mathcal{M}$, let $S(\mathcal{M}, \tau)$ be the *-algebra of all τ-measurable operators. Let $\mu(t; X)$ be the generalized singular value function of the operator $X \in S(\mathcal{M}, \tau)$. If $\mathcal{E}$ is a normed ideal space (NIS) on $(\mathcal{M}, \tau)$, then

$$\|A\|_{\mathcal{E}} \leq \|A + \mathrm{i}B\|_{\mathcal{E}} \qquad (*)$$

for all self-adjoint operators $A, B \in \mathcal{E}$. In particular, if $A, B \in (L_1 + L_\infty)(\mathcal{M}, \tau)$ are self-adjoint, then we have the (Hardy–Littlewood–Pólya) weak submajorization, $A \preceq_w A + \mathrm{i}B$. Inequality $(*)$ cannot be extended to the Shatten–von Neumann ideals $\mathfrak{S}_p, 0 < p < 1$. Hence, the well-known inequality $\mu(t; A) \leq \mu(t; A+\mathrm{i}B)$ for all $t > 0$, positive $A \in S(\mathcal{M}, \tau)$ and self-adjoint $B \in S(\mathcal{M}, \tau)$ cannot be extended to all self-adjoint operators $A, B \in S(\mathcal{M}, \tau)$. Consider self-adjoint operators $X, Y \in S(\mathcal{M}, \tau)$, let $K(X)$ be the Cayley transform of X. Then, $\mu(t; K(X) - K(Y)) \leq 2\mu(t; X - Y)$ for all $t > 0$. If $\mathcal{E}$ is an F-NIS on $(\mathcal{M}, \tau)$ and $X - Y \in \mathcal{E}$, then $K(X) - K(Y) \in \mathcal{E}$ and $\|K(X) - K(Y)\|_{\mathcal{E}} \leq 2\|X - Y\|_{\mathcal{E}}$.

Keywords Hilbert space · Von Neumann algebra · Normal trace · Measurable operator · Normed ideal space · Weak submajorization

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1 Introduction

The section of functional analysis, called noncommutative integration theory, is an important part of the theory of operator algebras. This article is devoted to noncommutative analogs of the classical function spaces. Development of the corresponding aspect of noncommutative integration theory started with the works of Segal and Dixmier, who in the early 1950s created a theory of integration with respect to a trace on a semifinite von Neumann algebra, see [23]. The results of these investigations found spectacular applications in the duality theory for unimodular groups and stimulated the progress of "noncommutative probability theory". The theory of algebras of measurable and locally measurable operators is rapidly developing and has interesting applications in various areas of functional analysis, mathematical physics, statistical mechanics, and quantum field theory. In [31–33], Muratov introduced and investigated ideal spaces of measurable operators on a finite von Neumann algebra. They were also studied by Chilin in [19]. In the above-mentioned works, the ideal spaces serve primarily as the object of investigation. Recently, there have appeared publications in which they act as a tool see, for instance, [6]. In [7, 15] new methods were proposed for constructing ideal spaces on semifinite von Neumann algebras and the geometric and topological properties of the obtained spaces were studied.

Our article continues the research of works [11] and [12] of the first author. Let a von Neumann algebra $\mathcal{M}$ of operators act on a Hilbert space $\mathcal{H}$, τ be a faithful semifinite normal trace on $\mathcal{M}$. Let $S(\mathcal{M},\tau)$ be the *-algebra of all τ-measurable operators, let $\mu(t;X)$ be the generalized singular value function of the operator $X\in S(\mathcal{M},\tau)$. Our main results are obtained in the context of semifinite von Neumann algebras; some results are new even in the case of the algebra $\mathcal{M}=\mathcal{B}(\mathcal{H})$, equipped with $\tau=\mathrm{tr}$. If $\mathcal{E}$ is a normed ideal space (NIS) on $(\mathcal{M},\tau)$, then $\|A\|_{\mathcal{E}}\le\|A+\mathrm{i}B\|_{\mathcal{E}}$ for all self-adjoint operators $A,B\in\mathcal{E}$ (Theorem 3.3). In particular, if $A,B\in(L_1+L_\infty)(\mathcal{M},\tau)$ are self-adjoint, then we have the (Hardy–Littlewood–Pólya) weak submajorization, $A\preceq_w A+\mathrm{i}B$ (Corollary 3.4). Theorem 3.3 cannot be extended to the Shatten–von Neumann ideals $\mathfrak{S}_p$, $0<p<1$ (Theorem 3.5). Hence, the well known inequality $\mu(t;A)\le\mu(t;A+\mathrm{i}B)$ for all $t>0$, $A\in S(\mathcal{M},\tau)^+$ and self-adjoint $B\in S(\mathcal{M},\tau)$ cannot be extended to all self-adjoint operators $A,B\in S(\mathcal{M},\tau)$. If $\langle\mathcal{E},\|\cdot\|_{\mathcal{E}}\rangle$ is a NIS on $(\mathcal{M},\tau)$, then $\|A-B\|_{\mathcal{E}}\le\|A+B\|_{\mathcal{E}}$ for all $A,B\in\mathcal{E}^+$ (Theorem 3.7).

Consider self-adjoint operators $X,Y\in S(\mathcal{M},\tau)$, let $K(X)$ be the Cayley transform of X. Let $\mathcal{E}$ be an F-NIS on $(\mathcal{M},\tau)$. We have $\mu(t;K(X)-K(Y))\le2\mu(t;X-Y)$ for all $t>0$ (Theorem 3.10). If $X-Y\in\mathcal{E}$, then $K(X)-K(Y)\in\mathcal{E}$ and $\|K(X)-K(Y)\|_{\mathcal{E}}\le2\|X-Y\|_{\mathcal{E}}$ (Theorem 3.11).

2 Definitions and notation

Let a von Neumann operator algebra $\mathcal{M}$ act on a Hilbert space $\mathcal{H}$, I be the unit of $\mathcal{M}$, $\mathcal{M}_1=\{X\in\mathcal{M}:\ \|X\|\le1\}$. Let $\mathcal{M}^{\mathrm{pr}}$ be the lattice of projections ($P=P^2=P^*$) in $\mathcal{M}$ and $P^\perp=I-P$ for $P\in\mathcal{M}^{\mathrm{pr}}$, let $\mathcal{M}^+$ be the cone of all positive operators in $\mathcal{M}$. An operator $U\in\mathcal{M}$ is called a *partial isometry*, if U^*U is a projection; *unitary*, if $U^*U=UU^*=I$. Two projections $P,Q\in\mathcal{M}^{\mathrm{pr}}$ are said to be *Murray–von*

Neumann equivalent if there exists a partial isometry $U \in \mathcal{M}$ such that $U^*U = P$ and $UU^* = Q$. In this case, we write $P \sim Q$. If $P \leq Q$, then we say that P is a *subprojection* of Q. If P is equivalent to a subprojection of Q, we write $P \prec Q$. We say that $P \in \mathcal{M}^{\text{pr}}$ is a *finite projection* if $P \sim Q \leq P$ implies $P = Q$.

A mapping $\varphi : \mathcal{M}^+ \to [0, +\infty]$ is called *a trace*, if $\varphi(X + Y) = \varphi(X) + \varphi(Y)$, $\varphi(\lambda X) = \lambda\varphi(X)$ for all $X, Y \in \mathcal{M}^+$, $\lambda \geq 0$ (moreover, $0 \cdot (+\infty) \equiv 0$); $\varphi(Z^*Z) = \varphi(ZZ^*)$ for all $Z \in \mathcal{M}$. A trace φ is called (see [37, Chap. V, §2])

- *faithful*, if $\varphi(X) > 0$ for all $X \in \mathcal{M}^+$, $X \neq 0$;
- *normal*, if $X_i \uparrow X$ $(X_i, X \in \mathcal{M}^+) \Rightarrow \varphi(X) = \sup \varphi(X_i)$;
- *semifinite*, if $\varphi(X) = \sup\{\varphi(Y) : \ Y \in \mathcal{M}^+, \ Y \leq X, \ \varphi(Y) < +\infty\}$ for every $X \in \mathcal{M}^+$.

An operator on $\mathcal{H}$ (not necessarily bounded or densely defined) is said to be *affiliated to the von Neumann algebra* $\mathcal{M}$ if it commutes with any unitary operator from the commutant $\mathcal{M}'$ of the algebra $\mathcal{M}$. Let τ be a faithful normal semifinite trace on $\mathcal{M}$. A closed operator X, affiliated to $\mathcal{M}$ and possessing a domain $\mathfrak{D}(X)$ everywhere dense in $\mathcal{H}$ is said to be τ*-measurable* if, for any $\varepsilon > 0$, there exists a projection $P \in \mathcal{M}^{\text{pr}}$ such that $P\mathcal{H} \subset \mathfrak{D}(X)$ and $\tau(P^\perp) < \varepsilon$. The set $S(\mathcal{M}, \tau)$ of all τ-measurable operators is a *-algebra under passage to the adjoint operator, multiplication by a scalar, and operations of strong addition and multiplication resulting from the closure of the ordinary operations [38, Chap. IX].

Let $\mathcal{L}^+$ and $\mathcal{L}^{\text{h}}$ denote the positive and Hermitian parts of a family $\mathcal{L} \subset S(\mathcal{M}, \tau)$, respectively. We denote by $\leq$ the partial order in $S(\mathcal{M}, \tau)^{\text{h}}$ generated by its proper cone $S(\mathcal{M}, \tau)^+$. If $X \in S(\mathcal{M}, \tau)$ and $X = U|X|$ is the polar decomposition of X, then $U \in \mathcal{M}$ and $|X| = \sqrt{X^*X} \in S(\mathcal{M}, \tau)^+$.

An operator $A \in S(\mathcal{M}, \tau)$ is called *hyponormal*, if $A^*A \geq AA^*$; *cohyponormal*, if the operator A^* is hyponormal. Denote by $[A, B] = AB - BA$ the commutator of operators $A, B \in S(\mathcal{M}, \tau)$. The generalized singular value function $\mu(\cdot; X) : t \to \mu(t; X)$ of the operator X is defined by setting

$$\mu(t; X) = \inf\{\|XP\| : \ \ P \in \mathcal{M}^{\text{pr}} \ \text{ and } \ \tau(P^\perp) \leq t\}, \quad t > 0.$$

It is a non-increasing right-continuous function.

Lemma 2.1 ([27]) *Let* $X, Y \in S(\mathcal{M}, \tau)$*,* $A, B \in \mathcal{M}$ *and* $U, V \in \mathcal{M}$ *be unitary. Then*

(i) $\mu(t; X) = \mu(t; |X|) = \mu(t; X^*) = \mu(t; UXV)$ *for all* $t > 0$*;*
(ii) *If* $|X| \leq |Y|$*, then* $\mu(t; X) \leq \mu(t; Y)$ *for all* $t > 0$*;*
(iii) $\mu(t; AXB) \leq \|A\|\,\|B\|\,\mu(t; X)$ *for all* $t > 0$*;*
(iv) $\mu(s + t; X + Y) \leq \mu(s; X) + \mu(t; Y)$ *for all* $s, t > 0$*;*
(v) $\mu(t; f(|X|)) = f(\mu(t; X))$ *for all continuous functions* $f : \mathbb{R}^+ \to \mathbb{R}^+$ *with* $f(0) = 0$ *and* $t > 0$.

Let m be the linear Lebesgue measure on $\mathbb{R}$. Noncommutative Lebesgue L_p-space $(0 < p < \infty)$, associated with $(\mathcal{M}, \tau)$, may be defined as

$$L_p(\mathcal{M}, \tau) = \{X \in S(\mathcal{M}, \tau) : \ \mu(\cdot; X) \in L_p(\mathbb{R}^+, m)\}$$

with the F-norm (norm for $1 \le p < \infty$) $\|X\|_p = \|\mu(\cdot; X)\|_p$, $X \in L_p(\mathcal{M}, \tau)$. The extension of τ to the unique linear functional on the whole space $L_1(\mathcal{M}, \tau)$ we denote by the same letter τ. A linear subspace $\mathcal{E} \subset S(\mathcal{M}, \tau)$ is called an *ideal space* on $(\mathcal{M}, \tau)$, if

1. $X \in \mathcal{E} \Rightarrow X^* \in \mathcal{E}$;
2. $X \in \mathcal{E}$, $Y \in S(\mathcal{M}, \tau)$ and $|Y| \le |X| \Rightarrow Y \in \mathcal{E}$.

Such are, for example, the algebra $\mathcal{M}$, the collection of all elementary operators $\mathcal{F}(\mathcal{M}, \tau)$ and $L_p(\mathcal{M}, \tau)$ for $0 < p < \infty$. For every ideal space $\mathcal{E}$ on $(\mathcal{M}, \tau)$ we have $\mathcal{M}\mathcal{E}\mathcal{M} \subset \mathcal{E}$ [11, Lemma 5]. An ideal space $\mathcal{E}$ on $(\mathcal{M}, \tau)$, equipped with an F-norm $\|\cdot\|_{\mathcal{E}}$, is called an *F-normed ideal space* (F-NIS) on $(\mathcal{M}, \tau)$, if

1. $\|X\|_{\mathcal{E}} = \|X^*\|_{\mathcal{E}}$ for all $X \in \mathcal{E}$;
2. $X, Y \in \mathcal{E}$ and $|Y| \le |X| \Rightarrow \|Y\|_{\mathcal{E}} \le \|X\|_{\mathcal{E}}$ (see [7–13]).

If $\tau(I) < +\infty$ then every F-NIS on $(\mathcal{M}, \tau)$ is continuously embedded into $S(\mathcal{M}, \tau)$ in the measure topology t_τ, see [6, 14].

A linear subspace $\mathcal{E}$ in $S(\mathcal{M}, \tau)$, endowed with an F-norm $\|\cdot\|_{\mathcal{E}}$ is said to be an *F-normed symmetric space* (F-NSS) on $(\mathcal{M}, \tau)$ if $X \in \mathcal{E}$, $Y \in S(\mathcal{M}, \tau)$ and $\mu(t; Y) \le \mu(t; X)$ for all $t > 0$ imply that $Y \in \mathcal{E}$ and $\|Y\|_{\mathcal{E}} \le \|X\|_{\mathcal{E}}$.

Such are, for example, the algebra $\mathcal{M}$, the collection of all elementary operators $\mathcal{F}(\mathcal{M}, \tau)$, the ideal of all τ-compact operators

$$S_0(\mathcal{M}, \tau) = \left\{X \in S(\mathcal{M}, \tau) : \lim_{t \to +\infty} \mu(t; X) = 0\right\},$$

the Banach space $(L_1 + L_\infty)(\mathcal{M}, \tau)$ and the Lebesgue spaces $L_p(\mathcal{M}, \tau)$ for $0 < p < \infty$. A wide class of Orlicz F-NSS on $(\mathcal{M}, \tau)$ was investigated in [15]; for other examples see also [22] and [23]. For $A, B \in (L_1 + L_\infty)(\mathcal{M}, \tau)$ we write $A \preceq_w B$, the (Hardy–Littlewood–Pólya) weak submajorization, if

$$\int_0^t \mu(s; A)\, \mathrm{d}t \le \int_0^t \mu(s; B)\, \mathrm{d}t \quad \text{for all} \quad t > 0.$$

Interesting examples of such submajorizations were obtained in [4, 5, 16, 24, 25, 29, 35, 36] and others.

If $\mathcal{M} = \mathcal{B}(\mathcal{H})$, the *-algebra of all bounded linear operators on $\mathcal{H}$, and $\tau = \mathrm{tr}$ is the canonical trace, then $S(\mathcal{M}, \tau)$ coincides with $\mathcal{B}(\mathcal{H})$, $S_0(\mathcal{M}, \tau)$ coincides with the ideal $\mathfrak{S}_\infty$ of compact operators on $\mathcal{H}$, the space $L_p(\mathcal{M}, \tau)$ coincides with the Shatten–von Neumann *-ideal $\mathfrak{S}_p(\mathcal{H})$ of compact operators in $\mathcal{B}(\mathcal{H})$ and

$$\mu(t; X) = \sum_{n=1}^{\infty} s_n(X) \chi_{[n-1,n)}(t), \quad t > 0,$$

where $\{s_n(X)\}_{n=1}^{\infty}$ is the sequence of s-numbers of the operator X; χ_A is the indicator function of the set $A \subset \mathbb{R}$ [28, Chap. II].

If $\mathcal{M}$ is Abelian (i.e., commutative), then $\mathcal{M} \simeq L^{\infty}(\Omega, \Sigma, \nu)$ and $\tau(f) = \int_{\Omega} f \, \mathrm{d}\,\nu$, where (Ω, Σ, ν) is a localizable measure space, the *-algebra $S(\mathcal{M}, \tau)$ coincides with the algebra of all complex measurable functions f on (Ω, Σ, ν), bounded everywhere but for a set of finite measure. The function $\mu(t; f)$ coincides with the nonincreasing rearrangement of the function $|f|$; see properties of such rearrangements in [30]. The algebra $\mathcal{M}$ contains no compact operators if and only if the measure ν has no atoms [3, Theorem 8.4].

3 The main results

The following assertion was established in [26, Lemma 9] and, independently, in [34, Proposition 3].

Theorem 3.1 *We have* $\mu(t; A) \le \mu(t; A + \mathrm{i}B)$ *for all* $t > 0$, $A \in S(\mathcal{M}, \tau)^{+}$ *and* $B \in S(\mathcal{M}, \tau)^{\mathrm{h}}$.

Corollary 3.2 *If* $\mathcal{E}$ *is an* F*-NSS on* $(\mathcal{M}, \tau)$*, then* $\|A\|_{\mathcal{E}} \le \|A + \mathrm{i}B\|_{\mathcal{E}}$ *for all* $A \in \mathcal{E}^{+}$ *and* $B \in \mathcal{E}^{\mathrm{h}}$.

In the case of NIS $\mathcal{E}$, we generalize our Corollary 3.2.

Theorem 3.3 (cf. with Proposition 3.7 of [2]) *If* $\mathcal{E}$ *is a NIS on* $(\mathcal{M}, \tau)$*, then* $\|A\|_{\mathcal{E}} \le \|A + \mathrm{i}B\|_{\mathcal{E}}$ *for all* $A, B \in \mathcal{E}^{\mathrm{h}}$.

Proof Recall that $\|Z\|_{\mathcal{E}} = \| \, |Z| \, \|_{\mathcal{E}} = \|Z^*\|_{\mathcal{E}}$ for all $Z \in \mathcal{E}$. If $X \in \mathcal{E}$ and $Y \in \mathcal{E}^{\mathrm{h}}$, then from the representation

$$X - \mathrm{Re}X = \frac{X - Y}{2} - \frac{X^* - Y}{2} = \frac{X - Y}{2} - \frac{(X - Y)^*}{2}$$

and from the triangle inequality for the norm $\| \cdot \|_{\mathcal{E}}$ we infer that $\|X - \mathrm{Re}X\|_{\mathcal{E}} \le \|X - Y\|_{\mathcal{E}}$. Put $X = \mathrm{i}A$ for $A \in \mathcal{E}^{\mathrm{h}}$. Then $|X| = |A|$, $\mathrm{Re}X = 0$ and for all $B \in \mathcal{E}^{\mathrm{h}}$ we have

$$\begin{aligned} \|A\|_{\mathcal{E}} &= \| \, |A| \, \|_{\mathcal{E}} = \| \, |X| \, \|_{\mathcal{E}} = \|X\|_{\mathcal{E}} = \|X - \mathrm{Re}X\|_{\mathcal{E}} \le \|X - B\|_{\mathcal{E}} \\ &= \|\mathrm{i}A - B\|_{\mathcal{E}} = \|A + \mathrm{i}B\|_{\mathcal{E}}, \end{aligned}$$

since $|\mathrm{i}A - B| = |\mathrm{i}(A + \mathrm{i}B)| = |A + \mathrm{i}B|$. The theorem is proved. □

Corollary 3.4 *If* $X, Y \in (L_1 + L_{\infty})(\mathcal{M}, \tau)^{\mathrm{h}}$*, then* $X \preceq_w X + \mathrm{i}Y$.

Proof For every fixed number $t > 0$ the mapping

$$X \mapsto \int_0^t \mu(s; X) \, \mathrm{d}t$$

is a norm on NSS $(L_1 + L_{\infty})(\mathcal{M}, \tau)$. □

Theorem 3.3 cannot be extended to the class of all F-NSS on $(\mathcal{M}, \tau)$.

Theorem 3.5 *Theorem 3.3 cannot be extended to the Shatten–von Neumann *-ideals $\mathfrak{S}_p$, $0 < p < 1$.*

Proof For $\mathcal{M} = \mathbb{M}_2(\mathbb{C})$, $\tau = \mathrm{tr}$ and $x \in \mathbb{R}$ put

$$A = \begin{pmatrix} 1 & -x \\ -x & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & \mathrm{i}x \\ -\mathrm{i}x & 0 \end{pmatrix}. \tag{3.1}$$

Then, $|A|^2 = A^2 = (1+x^2)I$, hence $|A|^p = (|A|^2)^{p/2} = (1+x^2)^{p/2}I$ and $\mathrm{tr}(|A|^p) = 2(1+x^2)^{p/2}$ for all $0 < p < +\infty$. The eigenvalues of a matrix

$$|A + \mathrm{i}B|^2 = \begin{pmatrix} 1 & -2x \\ -2x & 1+4x^2 \end{pmatrix}$$

are $\lambda_1 = 1 + 2x^2 + 2x\sqrt{1+x^2}$, $\lambda_2 = 1 + 2x^2 - 2x\sqrt{1+x^2}$. By the Taylor formula with Peano remainder:

$$(1+x^2)^{p/2} = 1 + \frac{p}{2}x^2 + o(x^2) \quad (x \to 0),$$

therefore,

$$\begin{aligned}
\mathrm{tr}(|A|^p) &= 2 + px^2 + o(x^2) \quad (x \to 0),\\
\mathrm{tr}(|A + \mathrm{i}B|^p) &= \lambda_1^{p/2} + \lambda_2^{p/2}\\
&= \left(1 + 2x^2 + 2x\left(1 + \frac{x^2}{2} + o(x^2)\right)\right)^{p/2}\\
&\quad + \left(1 + 2x^2 - 2x\left(1 + \frac{x^2}{2} + o(x^2)\right)\right)^{p/2}\\
&= 1 + px + px^2 + \frac{p}{2}\left(\frac{p}{2} - 1\right)\frac{(2x)^2}{2} + o(x^2)\\
&\quad + 1 - px + px^2 + \frac{p}{2}\left(\frac{p}{2} - 1\right)\frac{(-2x)^2}{2} + o(x^2)\\
&= 2 + p^2x^2 + o(x^2) \quad (x \to 0).
\end{aligned}$$

Thus, the inequality $\|A\|_p^p = \mathrm{tr}(|A|^p) \le \mathrm{tr}(|A + \mathrm{i}B|^p) = \|A + \mathrm{i}B\|_p^p$ for every $0 < p < 1$ does not hold. The theorem is proved. □

Corollary 3.6 *Theorem 3.1 cannot be extended to all operators $A, B \in S(\mathcal{M}, \tau)^{\mathrm{h}}$.*

Proof There exist $A, B \in \mathbb{M}_2(\mathbb{C})^{\mathrm{sa}}$ such that for $1 < t \le 2$ the inequality $\mu(t; A) \le \mu(t; A + \mathrm{i}B)$ is not true, see (3.1). □

Theorem 3.7 *If $\langle \mathcal{E}, \|\cdot\|_{\mathcal{E}} \rangle$ is a NIS on $(\mathcal{M}, \tau)$, then $\|A - B\|_{\mathcal{E}} \le \|A + B\|_{\mathcal{E}}$ for all $A, B \in \mathcal{E}^+$.*

Proof If $X \in \mathcal{E}^+$, $Y \in S(\mathcal{M}, \tau)^{\mathrm{h}}$ and $-X \leq Y \leq X$, then $Y \in \mathcal{E}^{\mathrm{h}}$ and $\|Y\|_{\mathcal{E}} \leq \|X\|_{\mathcal{E}}$ [8]. (This implies the assertion of [20, Proposition 1.2].) Put $X = A+B$ and $Y = A-B$.

Corollary 3.8 *If $X, Y \in (L_1 + L_\infty)(\mathcal{M}, \tau)^+$, then $X - Y \preceq_w X + Y$.*

In the particular case, when $\tau(I) = 1$, the assertion of Corollary 3.8 was obtained by another method in [21, Lemma 2.1]. For $\mathcal{M} = \mathbb{M}_2(\mathbb{C})$ and $\tau = \mathrm{tr}$ put

$$X = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}, \quad Y = \begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix}.$$

Then, $s_1(X+Y) = 9$, $s_2(X+Y) = 1$ and $s_1(X-Y) = s_2(X-Y) = 3$. Hence for $1 < t \leq 2$ the inequality $\mu(t; X-Y) \leq \mu(t; X+Y)$ does not hold.

Lemma 3.9 *If $\langle \mathcal{E}, \|\cdot\|_{\mathcal{E}} \rangle$ is an F-NIS on $(\mathcal{M}, \tau)$, $Z \in \mathcal{E}$ and $A, B \in \mathcal{M}_1$, then $AZB \in \mathcal{E}$ and $\|AZB\|_{\mathcal{E}} \leq \|Z\|_{\mathcal{E}}$.*

Proof We have $AZB \in \mathcal{E}$ by [11, Lemma 5]. Via the operator monotonocity of the function $\lambda \mapsto \sqrt{\lambda}$ ($\lambda \geq 0$), inequalities $B^*Z^*A^*AZB \leq B^*Z^*ZB$, $ZBB^*Z^* \leq ZZ^*$ and properties of the F-norm $\|\cdot\|_{\mathcal{E}}$ we obtain

$$\begin{aligned} \|AZB\|_{\mathcal{E}} &= \||AZB|\|_{\mathcal{E}} = \|\sqrt{B^*Z^*A^*AZB}\|_{\mathcal{E}} \leq \|\sqrt{B^*Z^*ZB}\|_{\mathcal{E}} \\ &= \||ZB|\|_{\mathcal{E}} = \|ZB\|_{\mathcal{E}} = \|(ZB)^*\|_{\mathcal{E}} = \|B^*Z^*\|_{\mathcal{E}} \\ &= \||B^*Z^*|\|_{\mathcal{E}} = \|\sqrt{ZBB^*Z^*}\|_{\mathcal{E}} \leq \|\sqrt{ZZ^*}\|_{\mathcal{E}} = \|Z^*\|_{\mathcal{E}} = \|Z\|_{\mathcal{E}}. \end{aligned}$$

The lemma is proved. □

By the Spectral Theorem in the multiplicator form, the Cayley transform

$$K(X) = \frac{X + \mathrm{i}I}{X - \mathrm{i}I} = (X - \mathrm{i}I)^{-1}(X + \mathrm{i}I) = (X + \mathrm{i}I)(X - \mathrm{i}I)^{-1}$$

of an operator $X \in S(\mathcal{M}, \tau)^{\mathrm{h}}$ is a unitary operator in $\mathcal{M}$. Since $\lambda + \mathrm{i} = (1+\lambda^2)/(\lambda - \mathrm{i})$ for every $\lambda \in \mathbb{R}$, we have $K(X) = (X - \mathrm{i}I)^{-2}(I + X^2)$ for all $X \in S(\mathcal{M}, \tau)^{\mathrm{h}}$.

Theorem 3.10 *Let $X, Y \in S(\mathcal{M}, \tau)^{\mathrm{h}}$ and Y be invertible in $S(\mathcal{M}, \tau)$. Then*

(i) *$K(X)^* = K(-X)$;*
(ii) *$K(Y^{-1}) = -K(-Y) = -K(Y)^*$;*
(iii) *If $X = \sum_{k=1}^n \lambda_k P_k$ with $\lambda_k \in \mathbb{R}$, $P_k \in \mathcal{M}^{\mathrm{pr}}$ and $P_k P_m = 0$ for $k \neq m$, $k, m = 1, \ldots, n$, then*

$$K(X) = \sum_{k=1}^n \frac{\lambda_k + \mathrm{i}}{\lambda_k - \mathrm{i}} P_k - Q^\perp, \quad \textit{where} \quad Q = \sum_{k=1}^n P_k = \bigvee_{k=1}^n P_k;$$

(iv) *$K(X) = \mathrm{i}X \Leftrightarrow X^2 = I$.*

Proof (i). Since the operator $K(X)$ is unitary, we have

$$K(X)^* = K(X)^{-1} = (X - \mathrm{i}I)(X + \mathrm{i}I)^{-1} = -(-X + \mathrm{i}I) \cdot -(-X - \mathrm{i}I)^{-1}$$
$$= (-X + \mathrm{i}I)(-X - \mathrm{i}I)^{-1} = K(-X).$$

(ii). We have

$$K(Y) = (Y + \mathrm{i}I)(Y - \mathrm{i}I)^{-1} = (Y + \mathrm{i}YY^{-1})(Y - \mathrm{i}Y^{-1}Y)^{-1}$$
$$= Y(I + \mathrm{i}Y^{-1})[(I - \mathrm{i}Y^{-1})Y]^{-1} = Y(I + \mathrm{i}Y^{-1})Y^{-1}(I - \mathrm{i}Y^{-1})^{-1}$$
$$= YY^{-1}(I + \mathrm{i}Y^{-1})(I - \mathrm{i}Y^{-1})^{-1} = \mathrm{i}(I + \mathrm{i}Y^{-1}) \cdot \mathrm{i}^{-1}(I - \mathrm{i}Y^{-1})^{-1}$$
$$= (-Y^{-1} + \mathrm{i}I)(\mathrm{i}I + Y^{-1})^{-1} = ((-Y)^{-1} + \mathrm{i}I)(\mathrm{i}I - (-Y)^{-1})^{-1}$$
$$= ((-Y)^{-1} + \mathrm{i}I)((-Y)^{-1} - \mathrm{i}I)^{-1} = -K((-Y)^{-1}).$$

Replace the operator Y with $-Y$ and conclude that $K(Y^{-1}) = -K(-Y)$. From (i) it follows that $-K(-Y) = -K(Y)^*$.

(iii). By the Spectral Theorem we have

$$K(X) = \left(\sum_{k=1}^{n} \lambda_k P_k - \sum_{k=1}^{n} \mathrm{i}P_k - \mathrm{i}Q^\perp\right)^{-1} \left(\sum_{k=1}^{n} \lambda_k P_k + \sum_{k=1}^{n} \mathrm{i}P_k + \mathrm{i}Q^\perp\right)$$
$$= \left(\sum_{k=1}^{n} (\lambda_k - \mathrm{i}) P_k - \mathrm{i}Q^\perp\right)^{-1} \left(\sum_{k=1}^{n} (\lambda_k + \mathrm{i}) P_k + \mathrm{i}Q^\perp\right)$$
$$= \left(\sum_{k=1}^{n} \frac{1}{\lambda_k - \mathrm{i}} P_k - \frac{1}{\mathrm{i}} Q^\perp\right)^{-1} \left(\sum_{k=1}^{n} (\lambda_k + \mathrm{i}) P_k + \mathrm{i}Q^\perp\right)$$
$$= \sum_{k=1}^{n} \frac{\lambda_k + \mathrm{i}}{\lambda_k - \mathrm{i}} P_k - Q^\perp.$$

In particular, $K(P) = \mathrm{i}P - P^\perp$ for all $P \in \mathcal{M}^{\mathrm{pr}}$.

(iv), "$\Rightarrow$". Since $K(X) = \mathrm{i}X$ is unitary, we have $I = (\mathrm{i}X)^* \cdot \mathrm{i}X = -\mathrm{i}X \cdot \mathrm{i}X = X^2$.

(iv), "$\Leftarrow$". Putting $n = 2$, $\lambda_1 = 1 = -\lambda_2$, $P_1 = (X + I)/2$, $P_2 = P_1^\perp$ in (iii) (then $Q = I$), we obtain $K(X) = \mathrm{i}X$. The theorem is proved. □

Theorem 3.11 *If $\mathcal{E}$ is an F-NIS on $(\mathcal{M}, \tau)$, $X, Y \in S(\mathcal{M}, \tau)^{\mathrm{h}}$ and $X - Y \in \mathcal{E}$, then $K(X) - K(Y) \in \mathcal{E}$ and $\|K(X) - K(Y)\|_{\mathcal{E}} \le 2\|X - Y\|_{\mathcal{E}}$.*

Proof Let $X, Y \in S(\mathcal{M}, \tau)^{\mathrm{h}}$. For a function $f(\lambda) = 1/(\lambda - \mathrm{i})$, $\lambda \in \mathbb{R}$, the inequality $|f(\lambda)| \le 1$, $\lambda \in \mathbb{R}$, holds. Hence

$$f(X) = (X - \mathrm{i}I)^{-1},\ f(Y) = (Y - \mathrm{i}I)^{-1} \in \mathcal{M}_1.$$

If $X - Y \in \mathcal{E}$, then the operator

$$\begin{aligned} K(X) - K(Y) &= (X - iI)^{-1}(X + iI) - (Y + iI)(Y - iI)^{-1} \\ &= (X - iI)^{-1}[(X + iI)(Y - iI) - (X - iI)(Y + iI)](Y - iI)^{-1} \\ &= -2i(X - iI)^{-1}(X - Y)(Y - iI)^{-1} \end{aligned} \tag{3.2}$$

lies in $\mathcal{E}$ by [11, Lemma 5]. Now by Lemma 3.9 and the triangle inequality for the F-norm $\|\cdot\|_{\mathcal{E}}$ we obtain

$$\begin{aligned} \|K(X) - K(Y)\|_{\mathcal{E}} &= \||-2i(X - iI)^{-1}(X - Y)(Y - iI)^{-1}|\|_{\mathcal{E}} \\ &= 2\|(X - iI)^{-1}(X - Y)(Y - iI)^{-1}\|_{\mathcal{E}} \\ &\leq 2\|X - Y\|_{\mathcal{E}}. \end{aligned}$$

The theorem is proved. □

Note that in the proofs of Lemma 3.9 and Theorem 3.11 we follow the scheme of the proof of Theorem 4 from [17].

Theorem 3.12 *If $X, Y \in S(\mathcal{M}, \tau)^{\mathrm{h}}$, then $\mu(t; K(X) - K(Y)) \leq 2\mu(t; X - Y)$ for all $t > 0$.*

Proof Use representation (3.2) and items (i), (ii) and (iii) of Lemma 2.1. □

Remark 3.13 (i) If $\langle \mathcal{E}, \|\cdot\|_{\mathcal{E}}\rangle$ is an F-NIS on $(\mathcal{M}, \tau)$ and $I \in \mathcal{E}$, then there is no constant $C > 0$ such that $\|K(X)\|_{\mathcal{E}} \leq C\|X\|_{\mathcal{E}}$ for all $X \in \mathcal{E}^{\mathrm{h}}$. Suppose that a such constant $C > 0$ exists. Then for every $X \in \mathcal{E}^{\mathrm{h}}$ and $x \in \mathbb{R}$ we have $|K(xX)| = I$ and

$$\|I\|_{\mathcal{E}} = \||K(xX)|\|_{\mathcal{E}} = \|K(xX)\|_{\mathcal{E}} \leq C\|xX\|_{\mathcal{E}} \to 0 \text{ as } x \to 0;$$

a contradiction.

(ii) There is no constant $C > 0$ such that $\mu(t; K(X)) \leq C\mu(t; X)$ $(t > 0)$ for all $X \in S(\mathcal{M}, \tau)^{\mathrm{h}}$. Let $P \in \mathcal{M}^{\mathrm{pr}}$ and $\tau(P) < +\infty$. Consider an operator uP, $u > 0$. Suppose that such a constant $C > 0$ exists. Then, for every $t > 0$ we have

$$\begin{aligned} \chi_{(0,\tau(I))}(t) &= \mu(t; I) = \mu(t; |K(uP)|) = \mu(t; K(uP)) \\ &\leq C\mu(t; uP) = Cu\mu(t; P) = Cu\chi_{(0,\tau(P))}(t) \to 0 \quad \text{as} \quad u \to 0+; \end{aligned}$$

a contradiction.

Lemma 3.14 ([18, Theorem 17]) *If $X, Y \in S(\mathcal{M}, \tau)$ and $XY, YX \in L_1(\mathcal{M}, \tau)$, then $\tau(XY) = \tau(YX)$.*

Theorem 3.15 *Let $\mathcal{E}$ be an ideal space on $(\mathcal{M}, \tau)$, $A, B \in S(\mathcal{M}, \tau)$, A be hyponormal, B cohyponormal and $AB \in \mathcal{E}$. Then*

(i) *$BA \in \mathcal{E}$ and if $\mathcal{E}$ is F-NIS on $(\mathcal{M}, \tau)$, then $\|BA\|_{\mathcal{E}} \leq \|AB\|_{\mathcal{E}}$;*

(ii) *If* $\mathcal{E} = L_1(\mathcal{M}, \tau)$, *then* $\tau([A, B]) = 0$.

Proof (i). The operator A is hyponormal, so

$$|AB|^2 = B^*A^*AB \geq B^*AA^*B = |A^*B|^2;$$

then by the operator monotonocity of the function $t \mapsto \sqrt{t}$ $(t \geq 0)$ we infer that $|A^*B| \leq |AB|$. Hence $A^*B \in \mathcal{E}$ and $B^*A = (A^*B)^* \in \mathcal{E}$.
Since the operator B is cohyponormal, we have

$$|B^*A|^2 = A^*BB^*A \geq A^*B^*BA = |BA|^2,$$

and by the operator monotonocity of the function $t \mapsto \sqrt{t}$ $(t \geq 0)$ we obtain $|BA| \leq |B^*A|$. Hence $BA \in \mathcal{E}$.
For an F-NIS $\mathcal{E}$ we have $\|BA\|_{\mathcal{E}} \leq \|B^*A\|_{\mathcal{E}} = \|A^*B\|_{\mathcal{E}} \leq \|AB\|_{\mathcal{E}}$.

(ii). The assertion follows from (i) and Lemma 3.14.

□

Theorem 3.16 *If* $\mathcal{E}$ *is an ideal space on* $(\mathcal{M}, \tau)$, *operators* $A, B \in S(\mathcal{M}, \tau)$ *and* $A^*A, B^*B \in \mathcal{E}$, *then* $AXB \in \mathcal{E}$ *for all* $X \in \mathcal{M}$.

Proof Since $(XB)^*XB = B^*X^*XB \leq \|X\|^2 B^*B \in \mathcal{E}$, it suffices to show that $AB \in \mathcal{E}$. The inequality $(A \pm B)^*(A \pm B) \geq 0$ implies that

$$0 \leq A^*A + B^*B + A^*B + B^*A \leq 2A^*A + 2B^*B \in \mathcal{E}^+.$$

Therefore

$$A^*B + B^*A \in \mathcal{E}. \tag{3.3}$$

The inequality $(A \pm \mathrm{i}B)^*(A \pm \mathrm{i}B) \geq 0$ shows us that $0 \leq A^*A + B^*B - \mathrm{i}A^*B + \mathrm{i}B^*A \leq 2A^*A + 2B^*B \in \mathcal{E}^+$. Hence,

$$\mathrm{i}A^*B - \mathrm{i}B^*A \in \mathcal{E}. \tag{3.4}$$

From (3.3) and (3.4) it follows that $A^*B \in \mathcal{E}$. Note that $A^*A \in \mathcal{E}^+ \Leftrightarrow AA^* \in \mathcal{E}^+$ (if $A = U|A|$ is the polar decomposition of A, then $AA^* = UA^*AU^*$) and $A \in \mathcal{E} \Leftrightarrow A^* \in \mathcal{E}$. Replace A with A^* and obtain $AB \in \mathcal{E}$. □

Put $A = B$ and $X = I$ in Theorem 3.16 and infer

Corollary 3.17 *If* $\mathcal{E}$ *is an ideal space on* $(\mathcal{M}, \tau)$, $A \in S(\mathcal{M}, \tau)$ *and* $A^*A \in \mathcal{E}$, *then* $A^2 \in \mathcal{E}$.

Theorem 3.18 *Let* $A, B \in S(\mathcal{M}, \tau)$ *and* $|A| \geq \lambda I \geq |B|$ *for some number* $\lambda > 0$. *Then*

(i) $\mu(t; |A| - \lambda I) \leq \mu(t; A - B)$ *for all* $t > 0$;

(ii) *If* $\mathcal{E}$ *is an* F*-NIS on* $(\mathcal{M}, \tau)$ *and* $A - B \in \mathcal{E}$*, then* $|A| - \lambda I \in \mathcal{E}^+$ *and* $\||A| - \lambda I\|_{\mathcal{E}} \leq \|A - B\|_{\mathcal{E}}$.

Proof *Step 1*. By the operator triangle inequality [1, 19] there exist partial isometries $U, V \in \mathcal{M}$ such that

$$|A| = |A - B + B| \leq U|A - B|U^* + V|B|V^* \leq U|A - B|U^* + \lambda I.$$

Step 2. Now, by items (i), (ii) and (iii) of Lemma 1 for all $t > 0$, we have

$$\mu(t; |A| - \lambda I) \leq \mu(t; U|A - B|U^*) \leq \|U\|\|U^*\|\mu(t; |A - B|) = \mu(t; A - B)$$

and inequality (i) is true.

(ii) Via Step 1 and Lemma 3.9, we have

$$\||A| - \lambda I\|_{\mathcal{E}} \leq \|U|A - B|U^*\|_{\mathcal{E}} \leq \||A - B|\|_{\mathcal{E}} = \|A - B\|_{\mathcal{E}}.$$

The theorem is proved. □

Corollary 3.19 *Let* $A = V|A|$ *be the polar decomposition of* $A \in S(\mathcal{M}, \tau)$ *and* $|A| \geq \lambda I$ *for some number* $\lambda > 0$*. Then*

(i) $\mu(t; A - \lambda V) = \inf\{\mu(t; A - \lambda W)|\ W \in \mathcal{M}_1\}$ *for all* $t > 0$*;*
(ii) *If* $\mathcal{E}$ *is an* F*-NIS on* $(\mathcal{M}, \tau)$*, then* $\|A - \lambda V\|_{\mathcal{E}} = \inf\{\|A - \lambda W\|_{\mathcal{E}}|\ W \in \mathcal{M}_1\}$.

Proof For any $W \in \mathcal{M}_1$ we have $|A| \geq \lambda I \geq |\lambda W|$, hence Theorem 3.18 with $B = \lambda W$ works.

(i) For all $t > 0$ by items (ii) and (iii) of Lemma 2.1, we conclude that

$$\mu(t; A - \lambda V) = \mu(t; V(|A| - \lambda I)) \leq \|V\|\mu(t; |A| - \lambda I) = \mu(t; |A| - \lambda I).$$

(ii) By Lemma 3.9 we obtain $\|A - \lambda V\|_{\mathcal{E}} = \|V(|A| - \lambda I)\|_{\mathcal{E}} \leq \||A| - \lambda I\|_{\mathcal{E}}$.

□

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