ISSN 1064-5624, Doklady Mathematics, 2018, Vol. 98, No. 3, pp. 545–548. © Pleiades Publishing, Ltd., 2018.
Original Russian Text © A.M. Bikchentaev, 2018, published in Doklady Akademii Nauk, 2018, Vol. 483, No. 1.

MATHEMATICS

On Sets of Measurable Operators Convex and Closed in Topology of Convergence in Measure

A. M. Bikchentaev

Presented by Academician of the RAS I.A. Ibragimov June 13, 2018

Received July 9, 2018

Abstract—For a von Neumann algebra with a faithful normal semifinite trace, the properties of operator "intervals" of three types for operators measurable with respect to the trace are investigated. The first two operator intervals are convex and closed in the topology of convergence in measure, while the third operator interval is convex for all nonnegative operators if and only if the von Neumann algebra is Abelian. A sufficient condition for the operator intervals of the second and third types not to be compact in the topology of convergence in measure is found. For the algebra of all linear bounded operators in a Hilbert space, the operator intervals of the second and third types cannot be compact in the norm topology. A nonnegative operator is compact if and only if its operator interval of the first type is compact in the norm topology. New operator inequalities are proved. Applications to Schatten–von Neumann ideals are obtained. Two examples are considered.

DOI: 10.1134/S1064562418070037

Let τ be a faithful normal semifinite trace on a von Neumann algebra $\mathcal{M}$. In the theory of noncommutative integration, an important role is played by the operator "intervals" $I_B = \{A: -B \leq A \leq B\}$, where the operators $A = A^*$ and $B \geq 0$ belong to in the *-algebra $\tilde{\mathcal{M}}$ of τ-measurable operators affiliated with $\mathcal{M}$ [1, 2]. Such operator intervals were investigated in [3–8]. In [5, 9, 10] the subsets $K_B = \{A \in \tilde{\mathcal{M}}: A^*A \leq B\}$ were considered in the context of the topology t_τ of convergence in measure.

In Section 2, we prove new operator inequalities. It is shown that I_B and K_B are convex and t_τ-closed for every $B \in \tilde{\mathcal{M}}^+$. The set $M_B = \{A \in \tilde{\mathcal{M}}: |A| \leq B\}$ is convex for all nonnegative operators B if and only if the algebra $\mathcal{M}$ is Abelian. We have $I_B \supset M_B \cap \tilde{\mathcal{M}}^{\text{sa}}$ for all $B \in \tilde{\mathcal{M}}^+$.

The following results are obtained in Section 3. Suppose that an algebra $\mathcal{M}$ contains a sequence $\{P_n\}_{n=1}^{\infty}$ of pairwise orthogonal projections. If $B \in \tilde{\mathcal{M}}^+$ and $bP_1 \leq B$ for some number $0 < b < 1$, then the sets K_B and M_B cannot be t_τ-compact. Specifically, if $\mathcal{M} = \mathcal{B}(\mathcal{H})$, $\dim\mathcal{H} = \infty$, and $B \in \mathcal{M}^+\setminus\{0\}$, then the sets K_B and M_B cannot be $\|\cdot\|$-compact. An operator $B \in \mathcal{B}(\mathcal{H})^+$ is compact if and only if the set I_B is $\|\cdot\|$-compact. Let $\mathfrak{S}_p(\mathcal{H})$ be a Schatten–von Neumann ideal and $0 < p < +\infty$. If $B \in \mathfrak{S}_p(\mathcal{H})^+$, then I_B is a $\|\cdot\|_p$-compact subset of $\mathfrak{S}_p(\mathcal{H})^{\text{sa}}$.

1. NOTATION AND DEFINITIONS

Let $\mathcal{M}$ be a von Neumann algebra of operators on a Hilbert space $\mathcal{H}$, $\mathcal{M}^{\text{pr}}$ be the lattice of projections in $\mathcal{M}$, I be the unity of $\mathcal{M}$, $P^\perp = I - P$ for $P \in \mathcal{M}^{\text{pr}}$, and $\mathcal{M}^+$ be the cone of positive elements in $\mathcal{M}$. A mapping $\varphi\colon \mathcal{M}^+ \to [0, +\infty]$ is called a trace if $\varphi(X + Y) = \varphi(X) + \varphi(Y)$ and $\varphi(\lambda X) = \lambda\varphi(X)$ for all $X, Y \in \mathcal{M}^+$ and $\lambda \geq 0$ (it is assumed that $0 \cdot (+\infty) \equiv 0$) and $\varphi(Z^*Z) = \varphi(ZZ^*)$ for all $Z \in \mathcal{M}$. A trace φ is called faithful if $\varphi(X) > 0$ for all $X \in \mathcal{M}^+$, $X \neq 0$; it is normal if $X_i \nearrow X$ $(X_i, X \in \mathcal{M}^+) \Rightarrow \varphi(X) = \sup\varphi(X_i)$; and it is semifinite if $\varphi(X) = \operatorname{supp}\{\varphi(Y)\colon Y \in \mathcal{M}^+,\ Y \leq X,\ \varphi(Y) < +\infty\}$ for every $X \in \mathcal{M}^+$.

An operator on $\mathcal{H}$ (not necessarily bounded or densely defined) is said to be affiliated with a von

Lobachevskii Institute of Mathematics and Mechanics, Kazan (Volga Region) Federal University, Kazan, 420008 Tatarstan, Russia
e-mail: airat.bikchentaev@kpfu.ru

Neumann algebra $\mathcal{M}$ if it commutes with any unitary operator in the commutator subalgebra $\mathcal{M}'$ of $\mathcal{M}$. A self-adjoint operator is affiliated with $\mathcal{M}$ if and only if all projections in its spectral decomposition of unity belong to $\mathcal{M}$.

Let τ be a faithful normal semifinite trace on $\mathcal{M}$. A closed operator X affiliated with $\mathcal{M}$ whose domain $\mathfrak{D}(X)$ is dense in $\mathcal{H}$ is called τ-measurable if, for any $\varepsilon > 0$, there exists $P \in \mathcal{M}^{\text{pr}}$ such that $P\mathcal{H} \subset \mathfrak{D}(X)$ and $\tau(P^{\perp}) < \varepsilon$. The set $\tilde{\mathcal{M}}$ of all τ-measurable operators is a *-algebra under passage to the adjoint operator, multiplication by a scalar, and strong addition and multiplication operations defined as closures of the usual operations [11, 12]. For a family $\mathcal{L} \subset \tilde{\mathcal{M}}$, let $\mathcal{L}^+$ and $\mathcal{L}^{\text{sa}}$ denote its positive and Hermite parts, respectively. The partial order on $\tilde{\mathcal{M}}^{\text{sa}}$ generated by its proper cone $\tilde{\mathcal{M}}^+$ is denoted by $\le$. If $X \in \tilde{\mathcal{M}}$, then $|X| = \sqrt{X^*X} \in \tilde{\mathcal{M}}^+$.

Let $\mu_t(X)$ denote the rearrangement of an operator $X \in \tilde{\mathcal{M}}$, i.e., the nonincreasing right continuous function $\mu(X)\colon (0,\infty) \to [0,\infty)$ defined by the formula

$$\mu_t(X) = \inf\{\|XP\|: P \in \mathcal{M}^{\text{pr}}, \tau(P^{\perp}) \le t\}, \quad t > 0.$$

The set of τ-compact operators $\tilde{\mathcal{M}}_0 = \{X \in \tilde{\mathcal{M}}: \lim_{t\to\infty}\mu_t(X) = 0\}$ is an ideal in $\tilde{\mathcal{M}}$. The set of elementary operators $\mathcal{F}(\mathcal{M}) = \{X \in \mathcal{M}: \mu_t(X) = 0$ for some $t > 0\}$ is an ideal in $\mathcal{M}$.

The *-algebra $\tilde{\mathcal{M}}$ is equipped with the topology t_τ of convergence in measure [12] whose fundamental system of neighborhoods around zero is given by the sets

$$U(\varepsilon,\delta) = \{X \in \tilde{\mathcal{M}}: \exists P \in \mathcal{M}^{\text{pr}}(\|XP\| \le \varepsilon \text{ and } \tau(P^{\perp}) \le \delta)\}, \quad \varepsilon > 0, \quad \delta > 0.$$

It is well known that $\langle \tilde{\mathcal{M}}, t_\tau\rangle$ is a complete metrizable topological *-algebra; moreover, $\mathcal{M}$ is dense in $\langle \tilde{\mathcal{M}}, t_\tau\rangle$.

Let m be a linear Lebesgue measure on $\mathbb{R}$. The noncommutative Lebesgue space L_p $(0 < p < +\infty)$ associated with $(\mathcal{M},\tau)$ can be defined as $L_p(\mathcal{M},\tau) = \{X \in \tilde{\mathcal{M}}: \mu(X) \in L_p(\mathbb{R}^+, m)\}$ with F-norm (norm for $1 \le p < \infty$) $\|X\|_p = \|\mu(X)\|_p$, $X \in L_p(\mathcal{M},\tau)$. We have $\mathcal{F}(\mathcal{M}) \subset L_p(\mathcal{M},\tau) \subset \tilde{\mathcal{M}}_0$ for all $0 < p < +\infty$.

If $\mathcal{M} = \mathcal{B}(\mathcal{H})$ is the *-algebra of all bounded linear operators on $\mathcal{H}$ and $\tau = \text{tr}$ is the canonical trace, then $\tilde{\mathcal{M}}$ coincides with $\mathcal{B}(\mathcal{H})$, while $\tilde{\mathcal{M}}_0$ and $\mathcal{F}(\mathcal{M})$ coincide with the ideals of compact and finite-dimensional operators on $\mathcal{H}$, respectively. We have

$$\mu_t(X) = \sum_{n=1}^{\infty} s_n(X)\chi_{[n-1,n)}(t), \quad t > 0,$$

where $\{s_n(X)\}_{n=1}^{\infty}$ is the sequence of s-numbers of the operator X [13] and χ_A is the indicator function of a set $A \subset \mathbb{R}$. Then the space $L_p(\mathcal{M},\tau)$ is the Schatten–von Neumann ideal $\mathfrak{S}_p(\mathcal{H})$, $0 < p < +\infty$.

2. CONVEX SETS OF τ-MEASURABLE OPERATORS

Definition 1. Let τ be a faithful normal semifinite trace on a von Neumann algebra $\mathcal{M}$. For every $B \in \tilde{\mathcal{M}}^+$, let $K_B = \{A \in \tilde{\mathcal{M}}: A^*A \le B\}$, $M_B = \{A \in \tilde{\mathcal{M}}: |A| \le B\}$, and $I_B = \{A \in \tilde{\mathcal{M}}^{\text{sa}}: -B \le A \le B\}$.

Lemma 1. *For each operator* $B \in \mathcal{B}(\mathcal{H})^+$, $I_B \supset M_B \cap \mathcal{B}(\mathcal{H})^{\text{sa}}$.

Corollary 1. *Let* $X, Y \in \mathcal{B}(\mathcal{H})^+$ *and* $a = \max\{\|X\|, \|Y\|\}$. *Then* $X - Y \in I_{X+Y} \cap I_{2aI-X-Y}$.

Theorem 1. *Let* τ *be a faithful normal semifinite trace on a von Neumann algebra* $\mathcal{M}$ *and* $B \in \tilde{\mathcal{M}}^+$. *Then the following assertions hold*:

(i) $K_B \subseteq M_{\sqrt{B}}$ *with equality when* $\mathcal{M}$ *is Abelian*.

(ii) $I_B \supset M_B \cap \tilde{\mathcal{M}}^{\text{sa}}$.

(iii) *If* $B \in \mathcal{M}^{\text{pr}}$, *then* $I_B \subset M_B = K_B$.

(iv) *If* $A \in I_B$, *then* $A + B \in \tilde{\mathcal{M}}^+ \cap I_{2B} \cap M_{2B}$.

Corollary 2. *Let* $B \in \tilde{\mathcal{M}}^+$ *and* $A \in I_B$. *If the operator A is invertible, then B is invertible as well.*

Corollary 3. *Let* $X, Y \in \tilde{\mathcal{M}}$. *Then* $X^*Y + Y^*X \in I_{\{X^*X+Y^*Y\}} + Y^*Y$ *and* $XY + Y^*X^* \in I_{X|Y^*|^2X^*+I}$.

Corollary 4. *Let* $X \in \tilde{\mathcal{M}}^+$ *and* $Y \in \tilde{\mathcal{M}}$. *Then* $XY^* + YX \in I_{X+YXY^*}$.

Corollary 5. *Let* $Y \in \tilde{\mathcal{M}}^{\text{sa}}$ *and* $X \in \tilde{\mathcal{M}}^+$. *Then* $Y \in I_{|Y-X|+X}$.

Proposition 1. *For all operators* $A, B \in \tilde{\mathcal{M}}^{\text{sa}}$ *and numbers* $t \in [0, 1]$ *it is true that* $(\sqrt{t}A + \sqrt{1-t}B)^2 \le A^2 + B^2$ *with equality holding if and only if* $\sqrt{1-t}A = \sqrt{t}B$. *Therefore,* $\sqrt{t}A + \sqrt{1-t}B \in I_{\sqrt{A^2+B^2}}$.

The following result is derived from item (ii) of Theorem 1 combined with well-known properties of rearrangements of τ-measurable operators.

Proposition 2. *If* $\mathcal{E} \in \{\tilde{\mathcal{M}}_0, \mathcal{F}(\mathcal{M})\}$ *and* $B \in \mathcal{E}^+$, *then* K_B, M_B, *and* I_B *are subsets of* $\mathcal{E}$.

A similar statement holds for the sets M_B, I_B, and ideal spaces of τ-measurable operators (for ideal spaces, see [14]).

Proposition 3. *Let* $A \in \tilde{\mathcal{M}}$ *and* $U \in \mathcal{M}$ *with* $\|U\| \le 1$.

(i) *If* $A \in K_B$, *then* $UA \in K_B$.

(ii) *If* $AU \in M_B$, *then* $U^*|A|U \in M_B$.

Theorem 2. *For each operator* $B \in \tilde{\mathcal{M}}^+$, *the sets* I_B *and* K_B *are convex and* t_τ*-closed in* $\tilde{\mathcal{M}}$. *If there are* $\varepsilon > 0$ *and* $P \in \mathcal{M}^{\mathrm{pr}}$ *such that* $A = \sqrt{2B_\varepsilon}P\sqrt{2B_\varepsilon} - B_\varepsilon$ *with* $B_\varepsilon = B + \varepsilon I$, *then* $A \in \operatorname{ext} I_B$.

Proof. For every $\varepsilon > 0$, define $T_\varepsilon = (2B_\varepsilon)^{-1/2}$. Since $0 \le A + B_\varepsilon \le 2B_\varepsilon$, we have $0 \le T_\varepsilon(A + B_\varepsilon)T_\varepsilon \le I$, i.e., $0 \le T_\varepsilon A T_\varepsilon + 2^{-1}I \le I$. Since $\mathcal{M}^{\mathrm{pr}} = \operatorname{ext}\{X \in \mathcal{M}^+ : \|X\| \le 1\}$ is the set of all extreme points of the positive part of the unit ball of the algebra $\mathcal{M}$, we obtain $T_\varepsilon A T_\varepsilon + 2^{-1}I = P \in \mathcal{M}^{\mathrm{pr}}$. Thus,

$$A = T_\varepsilon^{-1}PT_\varepsilon^{-1} - 2^{-1}T_\varepsilon^{-2} = \sqrt{2B_\varepsilon}P\sqrt{2B_\varepsilon} - B_\varepsilon$$

for some $\varepsilon > 0$ and $P \in \mathcal{M}^{\mathrm{pr}}$.

Corollary 6. *If* $B \in \mathcal{M}^{\mathrm{pr}}$, *then the set* M_B *is convex.*

Proof. Apply Theorem 2 and item (iii) of Theorem 1.

Proposition 4. *For every operator* $B \in \tilde{\mathcal{M}}^+$, *the set* M_B *is* t_τ*-closed in* $\tilde{\mathcal{M}}$.

Theorem 3. *The set* M_B *is convex for each operator* $B \in \tilde{\mathcal{M}}^+$ *if and only if* $\mathcal{M}$ *is Abelian.*

3. τ-COMPACT CONVEX SETS OF τ-MEASURABLE OPERATORS

Proposition 5. *If* $\tau(I) = +\infty$ *and* $B \in \tilde{\mathcal{M}}^+ \setminus \tilde{\mathcal{M}}_0$, *then the sets* K_B, M_B, *and* I_B *cannot be* t_τ*-compact.*

Proof. For such B, we have $b = \lim_{t\to\infty}\mu_t(B) > 0$. Since the trace τ is semifinite, there exists a sequence $\{P_n\}_{n=1}^\infty$ of pairwise orthogonal projections in $\mathcal{M}$ such that $\tau(P_n) \ge a > 0$ and $bP_n \le B$ for all $n \in \mathbb{N}$. Obviously, any subsequence $\{bP_{n_k}\}_{k=1}^\infty$ of $\{P_n\}_{n=1}^\infty$ is not t_τ-convergent.

Theorem 4. *Suppose that an algebra* $\mathcal{M}$ *contains a sequence* $\{P_n\}_{n=1}^\infty$ *of pairwise orthogonal nonzero equivalent projections. If* $B \in \tilde{\mathcal{M}}^+$ *and* $bP_1 \le B$ *for some number* $0 < b < 1$, *then the sets* K_B *and* M_B *cannot be* t_τ*-compact.*

Corollary 7. *If* $\mathcal{M} = \mathcal{B}(\mathcal{H})$, $\dim\mathcal{H} = \infty$, *and* $B \in \mathcal{M}^+ \setminus \{0\}$, *then the sets* K_B *and* M_B *cannot be* $\|\cdot\|$*-compact.*

Proof. Let $\langle\cdot,\cdot\rangle$ be the inner product in $\mathcal{H}$. Suppose that $\{\xi_n\}_{n=1}^\infty \subset \mathcal{H}$ is an orthonormal system such that $b\langle\cdot,\xi_1\rangle\xi_1 \le B$ for some number $0 < b < 1$. Define the sequence of partial isometries $A_n = \langle\cdot,\xi_1\rangle\xi_n$. Then $A_n^* = \langle\cdot,\xi_n\rangle\xi_1$, $A_nA_n^* = \langle\cdot,\xi_n\rangle\xi_n = P_n \in \mathcal{M}^{\mathrm{pr}}$, and $A_n^*A_n = P_1$ for all $n \in \mathbb{N}$.

Theorem 5. *An operator* $B \in \mathcal{B}(\mathcal{H})^+$ *is compact if and only if the set* I_B *is* $\|\cdot\|$*-compact.*

Now the Krein–Milman theorem implies the following result.

Corollary 8. *If an operator* $B \in \mathcal{B}(\mathcal{H})^+$ *is compact, then the set* I_B *is the* $\|\cdot\|$*-closure of the convex hull of the set of its extreme points.*

Corollary 9. *If* $0 < p < +\infty$ *and an operator* $B \in \mathfrak{S}_p(\mathcal{H})^+$, *then* I_B *is a* $\|\cdot\|_p$*-compact subset of* $\mathfrak{S}_p(\mathcal{H})^{\mathrm{sa}}$.

By the Krein–Milman theorem, we have the following result.

Corollary 10. *If* $1 \le p < +\infty$ *and an operator* $B \in \mathfrak{S}_p(\mathcal{H})^+$, *then the set* I_B *is the* $\|\cdot\|_p$*-closure of the convex hull of the set of its extreme points.*

Example 1. Let $\mathcal{M} = \ell_\infty$ and $\tau(X) = \sum_{k=1}^\infty x_k$ for $X = \{x_k\}_{k=1}^\infty \in \mathcal{M}^+$. Then $\tilde{\mathcal{M}}_0 = c_0$ is the set of all complex sequences converging to zero. It is well known that a closed set $\mathcal{A} \subset c_0$ is $\|\cdot\|$-compact if and only if there exists $B \in c_0^+$ such that $|A| \le B$ for all $A \in \mathcal{A}$.

Example 2. On the von Neumann algebra $\mathcal{M} = L_\infty([0,1],\nu)$, where ν is the Lebesgue measure on $[0, 1]$, consider the trace $\tau(f) = \int_{[0,1]} f d\nu$ and the sequence of Rademacher functions $r_n(t) = \operatorname{sign}\sin 2^n\pi t$ with $0 \le t \le 1$. The sequence $\{r_n\}_{n=1}^\infty$ does not contain t_τ-convergent subsequences:

$$\nu\{t \in [0,1]: |r_n(t) - r_k(t)| \ge 1\}$$
$$= \nu\{t \in [0,1]: r_n(t) \ne r_k(t)\} = \frac{1}{2}, \quad n \ne k.$$

Therefore, for the τ-compact operator $B = \chi_{[0,1]}$, the subsets I_B and $K_B = M_B$ are not t_τ-compact.

ACKNOWLEDGMENTS

This work was supported by grants awarded to Kazan Federal University for the state assignment in the sphere of research activities, grant nos. 1.9773.2017/8.9, 1.1515.2017/4.6.

REFERENCES

1. A. A. Novikov and O. E. Tikhonov, Lobachevskii J. Math. **37** (4), 497–499 (2016).
2. A. Novikov, Positivity **21** (1), 359–375 (2017).
3. A. M. Bikchentaev, Math. Notes **87** (5), 768–773 (2010).
4. A. M. Bikchentaev, Russ. Math. **56** (2), 75–79 (2012).
5. A. M. Bikchentaev and A. A. Sabirova, Sib. Math. J. **53** (2), 207–216 (2012).
6. A. M. Bikchentaev, Lobachevskii J. Math. **33** (3), 216–222 (2012).
7. A. M. Bikchentaev, Lobachevskii J. Math. **34** (3), 227–233 (2013).
8. V. I. Chilin and F. A. Sukochev, J. Operator Theory **31** (1), 35–65 (1994).
9. L. J. Ciach, Colloq. Math. **55** (1), 109–121 (1988).
10. W. Stainspring, Trans. Am. Math. Soc. **90** (1), 15–56 (1959).
11. I. E. Segal, Ann. Math. **57** (3), 401–457 (1953).
12. E. Nelson, J. Funct. Anal. **15** (2), 103–116 (1974).
13. I. Ts. Gokhberg, M. G. Krein, *Introduction to the Theory of Linear Nonselfadjoint Operators in Hilbert Space* (Moscow, Nauka, 1965; Am. Math. Soc., Providence, R.I., 1969).
14. A. M. Bikchentaev, Sib. Math. J. **59** (2), 243–251 (2018).

Translated by I. Ruzanova