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Meromorphization of M. I. Kinder's Formula Via the Change of Contours

A. V. Kazantsev*

(Submitted by A. M. Elizarov)

Kazan (Volga Region) Federal University, ul. Kremlevskaya 18, Kazan, 420008 Russia

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Abstract—Parametrical families of the exterior inverse boundary value problems going back to well-known R. B. Salimov's book became a plentiful source of new statements and methods in the study of the above problems. Critical points of conformal radii acting as the free parameters of such problems show interesting interrelations between their parametrical dynamics and geometric behavior. M.I. Kinder's formula connecting the numbers of local maxima and saddles of a conformal radius is generalized here on the case when the derivative of the mapping function has zeros and poles in the unit disk and on its boundary.

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1. INTRODUCTION

Problems on the change of contours have arisen as the class of the exterior inverse boundary value problems (IBVP) with the boundary conditions depending on the additional parameters. The charm of these problems is connected with the "Sturm und Drang Periode" in the development of the exterior IBVP when the model problems didn't separate yet from the applied ones, and mathematicians and mechanics were united not only by the *Memory*, but mainly by the *memories* of the heroic cooperation.

The monograph [1] in which the additional parameter has been first introduced in the statement of IBVP became one of the tops crowned the above "Periode" in the development of the exterior problems.

As well as in any historical milestone, it is possible to see the mystery elements in the treatise [1]: in spite of the fact that at each stage of the solution of exterior IBVP on change of contours the linearization with respect to the parameter is carried out, the condition of construction of the *approximate* solution of the above IBVP ([1], p. 64) turns out to be the condition of existence and uniqueness of its *exact* solution [2].

Thus, the book [1] stimulated a further study of the parametrical families of exterior problems and their bifurcations (e.g., [2, 3], etc.), including the research undertaken in the present note.

Recall that here we deal with the exterior inverse boundary value problem with respect to the parameter s in F. D. Gakhov's posing [4], and that first of all due to the papers [5–7] the following discourse has been formed to articulate the traditional description of the picture of solvability of such a problem. Namely, for the fixed boundary data the set of the solutions of the exterior IBVP is exhausted by the collection of the integral representations of the form

$$F(\zeta) = \int_b^\zeta f'(t) \left(\frac{1-\bar{a}t}{t-a}\right)^2 dt. \qquad (1)$$

*E-mail: avkazantsev63@gmail.com

Each of (1)'s is defined in the unit disk $\mathbb{D} = \{\zeta \in \mathbb{C} : |\zeta| < 1\}$ modulo translations and rotations in the F-plane where a run over the set $\mathrm{M}_f \subset \mathbb{D}$ of the roots of the Gakhov equation

$$\Phi(\zeta, \bar{\zeta}) := f''(\zeta)/f'(\zeta) - 2\bar{\zeta}/(1 - |\zeta|^2) = 0 \tag{2}$$

for a holomorphic and locally univalent function f in $\mathbb{D}$. The latter function represents the solution of the interior IBVP with respect to the same boundary data; $b \in \mathbb{D} \setminus \mathrm{M}_f$. The set M_f is exactly the basin of the critical points of the hyperbolic derivative (conformal radius)

$$h_f(\zeta) = (1 - |\zeta|^2)|f'(\zeta)| \tag{3}$$

of the function f (see [7, 8]).

In the framework of the discourse just pronounced the above mentioned R. B. Salimov's condition in [1] (p. 64) may be written as the inequality

$$|\{f, a\}| \neq 2/(1 - |a|^2)^2, \tag{4}$$

where $\{f, \zeta\} = (f''/f')'(\zeta) - (f''/f')^2(\zeta)/2$ is the Schwarzian derivative of the function f. The inequality (4) means that the Jacobian of the Gakhov equation (2) is non-vanishing at a point $a \in \mathrm{M}_f$, and that $\zeta = a$ is a maximum or a saddle of the surface $h = h_f(\zeta)$ given by a function (3) over the unit disk. So, the geometric character of elements $a \in \mathrm{M}_f$ is defined by the Gaussian curvature which is proportional to the Jacobian of (2). For a study of the geometry of this surface in the neighborhoods of the *isolated* elements $a \in \mathrm{M}_f$ more precise characterization is provided by the index

$$\gamma_f(a) = -\frac{1}{2\pi i} \int\limits_{|\zeta - a| = \rho} d\ln\{(\ln h_f)_\zeta\} \tag{5}$$

of the point $\zeta = a$ as a singular point of the vector field $\nabla \ln h_f(\zeta) \cong \bar{\Phi} = 2(\ln h_f)_{\bar{\zeta}}$ where $\Phi = \Phi(\zeta, \bar{\zeta})$ is defined in (2). Radius of the integration ρ in (5) is chosen such that $\mathrm{M}_f \cap \{|\zeta - a| \leq \rho\} = \{a\}$. The crucial role that the index plays in the geometric analysis for the critical points of the conformal radii (and their multiply connected analogies) in the context of IBVP has been discovered by M.I. Kinder; see, for example, [9]. The questions concerning the equivalence of the fields $\nabla \ln h_f$ and ∇h_f is not considered here.

After the gradual passage (mainly due to [6, 7, 10]) from the study of surjections $f \mapsto F$ in (1) to the treatment of corresponding elements $a \in \mathrm{M}_f$ the notion of the change of contours began to associate more and more with the parametrical families of conformal radii of the form (3). The particular cases of the change of contours often appear as an instrument to obtain the new facts about the sets M_f for the functions f; see, e.g., [3, 11, 12]. The present note acts in the framework of this tradition and deduces the generalization of M. I. Kinder's formula

$$M - S = 1 \tag{6}$$

to the simplest meromorphic case. In the holomorphic counterpart [9] by M and S the numbers of maxima ($\gamma_f = +1$) and saddles ($\gamma_f = -1$) are denoted; the semi-saddles have the indices $\gamma_f = 0$, and there are no other opportunities for a's in M_f than just mentioned ones.

We remark that in the case of the meromorphic function f an exterior IBVP leading to the solutions (1) can be called the exterior IBVP in the Gakhov–Nuzhin mixed posing where Gakhov's "non-fixed poles" $a \in \mathrm{M}_f$—poles of (1)'s—are added to Nuzhin's "fixed poles"—poles of the function f. See [13–15] as the versions of such a problem.

An exit out of the holomorphy expands a set of the singular points of the vector field $\nabla \ln h_f$. Throughout the work we shall assume that f is meromorphic function in the disk $\mathbb{D}_R = \{\zeta \in \mathbb{C} : |\zeta| < R\}$ of radius $R > 1$. It follows that f has finite number of poles in $\overline{\mathbb{D}}$, let M^∞ be their number within $\mathbb{D}$, and M^∂—on $\partial\mathbb{D}$. Besides this, f' has finite number of zeros in $\overline{\mathbb{D}}$; let n be the number of such zeros in the open unit disk. Thus, the condition of a local univalence is broken only in finite number of points. The set of all poles of the function f lying on $\partial\mathbb{D}$ is denoted by P_f; recall that the number of the elements of P_f is equal to M^∂. We don't provide the designations for the set of f's zeros on $\partial\mathbb{D}$ and for their number since, as it will be shown below in Lemma 3, such zeros won't participate in the derivation of our analogue of the formula (6).

We denote by S_f the whole set of the poles of the function f and of the zeros of its derivative f' in $\mathbb{D}$. Let q_f be the number of elements of S_f, so $q_f = M^\infty + n < \infty$. Let us denote by Λ_f the set of all singularities of the field $\nabla \ln h_f$ in $\mathbb{D}$. We set $\mathrm{M}_f := \Lambda_f \setminus \mathrm{S}_f$. The following statement shows that the choice of the latter notation is correct: in fact, for the meromorphic f the set M_f contains only those types of singularities which took place in the holomorphic situation too.

Proposition 1. *Let a function f be meromorphic in the disk $\mathbb{D}_R$ with $R > 1$. Then $\mathbb{D} \setminus \mathrm{S}_f$ is the smoothness domain for the surface $h = h_f(\zeta)$, and the set of the singularities of the field $\nabla \ln h_f$ in this domain, i.e. the set M_f, is exhausted by the finite maxima, and also saddles and semi-saddles.*

This statement ascertains that if the domain of holomorphy and local univalence of the function f is shrunken up to the subdomain of $\mathbb{D}$, then this subdomain contains neither flattening points, nor (zero or non-zero) minima. Instead of the superharmonicity of the function $\ln h_f$ over its domain of smoothness in the case in question we must lean on the fact that the principal normal curvatures $k_\pm(a)$ of the surface $h = \ln h_f(\zeta)$ at a point $a \in \mathrm{M}_f$ give the sum $k_+(a) + k_-(a) = -4/(1-|a|^2)^2$. Hence, they can't vanish simultaneously.

Let k_f be the number of elements of M_f. Our main result is the following

Theorem. *Let f be a meromorphic function in $\mathbb{D}_R$ where $R > 1$, and let the set M_f is free of continua. Then $k_f < \infty$, and*

$$M + M^\infty + M^\partial - S = 1 - n. \tag{7}$$

In the second paragraph the indices of elements of the set S_f will be calculated, the third section will be intended to study the dynamics of the boundary zeros and poles of the function f when it immerses into the level lines family

$$f_r(\zeta) = f(r\zeta)/r, \quad 0 \le r \le 1, \tag{8}$$

which is used to establish the relation (7) as the limit of the formulae of the form (6) when $r \to 1-$ in the fourth section.

2. INDEX ON S_f

We denote $K_\varepsilon(a) = \{\zeta \in \mathbb{C} : |\zeta - a| < \varepsilon\}$ and $\check{K}_\varepsilon(a) = K_\varepsilon(a) \setminus \{a\}$ for $a \in \mathbb{C}$ and $\varepsilon > 0$. Recall that we consider the meromorphic f in $\mathbb{D}_R$ of the radius $R > 1$. For the sake of accuracy we formulate the following

Assumption A. *A point $\zeta = a$ is the zero of the function $f(\zeta) - f(a)$ of order $n \ge 2$ or the pole of the function $f(\zeta)$ of order $n \ge 1$ and lies in the domain $\mathbb{D}_R \setminus \partial\mathbb{D}$.*

Now we provide the base for the calculation of the index $\gamma_f(a)$ of a singular point $\zeta = a$ of the vector field $\nabla \ln h_f$ satisfying the Assumption A:

Lemma 1. *Let the function f be meromorphic in $\mathbb{D}_R$ with $R > 1$, and let the Assumption A is fulfilled. There exists a real number $\varepsilon > 0$ such that*

$$f''(\zeta)/f'(\zeta) = \alpha_n/(\zeta - a) + \phi(\zeta), \quad \zeta \in \check{K}_\varepsilon(a), \tag{9}$$

where $\alpha_n = -(n+1)$ (pole) or $\alpha_n = n - 1$ (zero), a function ϕ is holomorphic in $K_\varepsilon(a)$, and

$$|\phi(\zeta) - 2\bar{\zeta}/(1 - |\zeta|^2)| < |\alpha_n|/\rho, \quad \zeta = a + \rho e^{i\theta} \in \check{K}_\varepsilon(a). \tag{10}$$

Proof of the expansion (9) does not exit out of the traditional standard in the local theory of the analytic functions. Estimate (10) is obtained by the shrinking of ε proceeding from the boundedness of the left hand side of (10) in $K_\varepsilon(a)$ due to the holomorphy ϕ in $K_\varepsilon(a)$ and the choice $\varepsilon < |1 - |a||$ leading to the relation $\overline{K_\varepsilon(a)} \cap \partial\mathbb{D} = \varnothing$.

Corollary 1. *In the assumptions of Lemma 1 the disk $K_\varepsilon(a)$ doesn't contain any singularity of the vector field $\nabla \ln h_f$, except $\zeta = a$.*

Proof. Singular points of the field $\nabla \ln h_f(\zeta) \cong 2(\ln h_f)_{\bar{\zeta}}$ are its zeros, and also the zeros and poles of the derivative f', that are the points satisfying the Assumption A. An absence of the latter points in the ring $\check{K}_\varepsilon(a)$ is caused by the holomorphy of the function f''/f' in it. The inequality $2(\ln h_f)_{\bar\zeta} \neq 0$, $\zeta \in \check{K}_\varepsilon(a)$, follows from the property $sgn\ \mathrm{Re}\{2e^{i\theta}(\ln h_f)_\zeta\} = sgn\ \alpha_n$, $\zeta = a + \rho e^{i\theta} \in \check{K}_\varepsilon(a)$, deducing from the relations (9), (10) and

$$2(\ln h_f)_\zeta = f''(\zeta)/f'(\zeta) - 2\bar\zeta/(1 - |\zeta|^2) \tag{11}$$

on the base of the fact that the fields $2e^{i\theta}(\ln h_f)_\zeta$ and $2(\ln h_f)_{\bar\zeta}$ have the same null-sets. □

The following statement calculates the indices of points under the Assumption A.

Proposition 2. *Let the function f be meromorphic in $\mathbb{D}_R$, $R > 1$, and the Assumption* A *is fulfilled for the point $\zeta = a$. The index of this point as the singular for the field $\nabla \ln h_f$ is equal to $\gamma_f(a) = +1$.*

Proof. We have

$$\gamma_f(a) = -\frac{1}{2\pi}\int\limits_\Gamma d\arg\{(\ln h_f)_\zeta\} = 1 + \tilde\gamma_f(a),$$

where Γ is the circle $\zeta = a + \eta e^{i\theta}$ with arbitrarily fixed $\eta \in (0, \varepsilon)$, and

$$\tilde\gamma_f(a) = -\frac{1}{2\pi}\int\limits_\Gamma d\arg\{e^{i\theta}(\ln h_f)_\zeta\}$$

is the index of the singular point $\zeta = a$ of the vector field $\Psi = 2e^{-i\theta}(\ln h_f)_{\bar\zeta}$.

Let us prove that $\tilde\gamma_f(a) = 0$. By virtue of (9) and (11) the estimate (10) is equivalent to the inequality $|\Psi_0 - \Psi| < |\Psi_0|$, $\zeta \in \check{K}_\varepsilon(a)$, hence the field $\Psi_0 = \alpha_n e^{-i\theta}/(\bar\zeta - \bar a)$ is the principal part of the field Ψ in some neighborhood of $\zeta = a$. According to Corollary 1 the point $\zeta = a$ is an isolated singular point of the fields Ψ and Ψ_0, therefore their indices at this point are the same, that is $\tilde\gamma_f(a) = 0$, as it was desired (see [16], p. 61–62). □

Now the following is evident.

Corollary 2. *Let the function f be meromorphic in $\mathbb{D}_R$, $R > 1$. If $a \in \mathrm{S}_f$, then $\gamma_f(a) = +1$.*

3. DYNAMICS OF THE BOUNDARY ZEROS AND POLES

In this case we use the Gakhov equation for the family (8) in two forms—as in (2) with f_r instead of f, i.e. in the form of

$$\Phi_r(\zeta, \bar\zeta) := f_r''(\zeta)/f_r'(\zeta) - 2\bar\zeta/(1 - |\zeta|^2) = 0, \tag{12}$$

and also in the form suitable for $\partial\mathbb{D}$,

$$F_r(\zeta, \bar\zeta) := f_r'(\zeta)/f_r''(\zeta) - (1 - |\zeta|^2)/(2\bar\zeta) = 0, \tag{13}$$

with the Jacobian $J_{F_r}(\zeta)$. The dynamics is as follows.

Lemma 2. *Suppose the function f is holomorphic in some punctured neighborhood U of the point $b \in \partial\mathbb{D}$, and the function $\zeta = \zeta(r)$ with values in U, $\zeta(1) = b$, is continuously differentiable with respect to $r \in [r_0, 1]$ (from the left in $r = 1$) and satisfies the following conditions when $r \in [r_0, 1)$: 1) $|\zeta(r)| \neq 1$; 2) $F_r(\zeta(r), \overline{\zeta(r)}) \equiv 0$ and 3) $J_{F_r}(\zeta(r)) \neq 0$. If $J_{F_1}(b) < 0$, then $\zeta(r) \in U \cap \mathbb{D}$, $r \in [r_0, 1)$, and if $J_{F_1}(b) > 0$, then $\zeta(r) \in U \setminus \overline{\mathbb{D}}$, $r \in [r_0, 1)$.*

Proof. Differentiating the identity $\Phi_r(\zeta(r), \overline{\zeta(r)}) \equiv 0$ as a result of the substitution of the function $\zeta = \zeta(r)$, $r \in [r_0, 1]$, in the equation (12), after the series of transformations we obtain the identity

$$(1 - |\zeta|^2)^2|\{f_r, \zeta\}| \equiv 2|(r\zeta'/\zeta - 1)/(r\zeta'/\zeta + 1)|, \quad \zeta = \zeta(r), \quad r \in [r_0, 1). \tag{14}$$

Let $J_{F_1}(b) < 0$. Third lemma's condition may be rewrite in the form of $J_{F_r}(\zeta(r)) < 0$, $r \in [r_0, 1)$. Since the Jacobians of the mappings in (12) and (13) at a point $\zeta(r)$ have the same sign, and since the Jacobian in (12) is equal to $J_{\Phi_r}(\zeta(r)) = |\{f_r, \zeta(r)\}|^2 - 4/(1 - |\zeta(r)|^2)^4$, $r \in [r_0, 1)$, we come to the inequality

$$(1 - |\zeta|^2)^2|\{f_r, \zeta(r)\}| < 2, \quad r \in [r_0, 1),$$

meaning that for every $r \in [r_0, 1)$ the point $\zeta(r)$ is the local maximum of the hyperbolic derivative h_{f_r} (see, e.g., [11]). Hence it follows from (14) that for $\zeta = \zeta(r)$ and $r \in [r_0, 1)$ the ratio $(r\zeta'/\zeta - 1)/(r\zeta'/\zeta + 1)$ takes its values in the unit disk, consequently, the inequality $\mathrm{Re}\ \zeta'(r)/\zeta(r) > 0$, $r \in [r_0, 1)$, is fulfilled. It is obvious that the latter is equivalent to the estimate $d|\zeta(r)|/dr > 0$, $r \in [r_0, 1)$, whence $|\zeta(r)| < |\zeta(1)| = 1$. This finishes the proof of Lemma 2 (the second implication is checked similarly, but now the point $\zeta(r)$ will be a saddle of h_{f_r}). □

By the use of the lemma just proved the following statement is obtained.

Lemma 3. *Let V be some neighborhood of a point $b \in \partial\mathbb{D}$, and let the function f be holomorphic in $\check{V} = V \setminus \{b\}$. There exist a real number $r_0 \in (0, 1)$ and a neighborhood $U \subset V$ of $\zeta = b$ such that 1) if b is a pole of n-th order of the function f $(n \geq 1)$, then for every $r \in [r_0, 1)$ the function h_{f_r} has the unique critical point of the local maximum $\zeta(r)$ in $U \cap \mathbb{D}$ $(\zeta(1) = b)$;* 2) if *b is a zero of $(n-1)$-th order of the derivative f' $(n \geq 2)$, then for any $r \in [r_0, 1)$ the domain $U \cap \mathbb{D}$ is free of the critical points of h_{f_r}.*

Proof. Without loss of generality we assume that f is holomorphic and locally univalent in $\check{V}$, and one of the expansions $f(\zeta) = (\zeta - b)^{\pm n}\varphi_\pm(\zeta)$, $\zeta \in V$, takes place where the functions $\varphi_\pm$ are holomorphic in V with $\varphi_\pm(b) \neq 0$. It is evident from these expansions that in both cases $\zeta = b$ is a root of the equation (13) when $r = 1$, which permits us to use the implicit function theory. Jacobian J_{F_1} at $\zeta = b$ is equal to

$$J_{F_1}(b) = \begin{cases} [(n-1)^2/(n+1)^2 - 1]/4 < 0, \\ [(n+1)^2/(n-1)^2 - 1]/4 > 0; \end{cases} \tag{15}$$

upper line in (15) corresponds to the pole, lower line—to the zero of the function f' at the point $\zeta = b$.

In view of (15) by the implicit function theorem there exist the numbers $r_0 \in (0, 1)$ and $r_1 > 1$, a neighborhood $U \subset V$ of the point $\zeta = b$, and a continuously differentiable function $\zeta = \zeta(r)$ with $\zeta(1) = b$ such that for any $r \in [r_0, r_1]$ the point $\zeta(r)$ is the unique root of the equation (13) in U, $J_{F_r}(\zeta(r)) \neq 0$, and for any $r \in [r_0, 1)$ the following inclusion takes place,

$$\{r\zeta : \zeta \in \overline{U} \cap \partial\mathbb{D}\} \subset V \cap \mathbb{D}. \tag{16}$$

Further, for any $r \in [r_0, 1)$ the inequality $|\zeta(r)| \neq 1$ is fulfilled. In fact, assuming the contrary, we should obtain that for some $r \in [r_0, 1)$ the following identities

$$f'(r\zeta(r))/f''(r\zeta(r)) \equiv r f_r'(\zeta(r))/f_r''(\zeta(r)) \equiv 0 \tag{17}$$

hold with $r\zeta(r) \in V \cap \mathbb{D}$ (see (13) and (16)). This is not the case in view of the holomorphy and the local univalence of the function f in $\check{V}$, whence we have also the equivalence of the equations (12) and (13) in $\check{U}$. So, we have verified the conditions of Lemma 2 which completes the proof. □

4. PROOF OF THE THEOREM

According to the results of [8] the assumption of the infinity of the set M_f leads to the presence of the continua in M_f which is excluded by the conditions of Theorem. Hence $k_f < \infty$. Since also $q_f < \infty$, the set Λ_f will be finite. Moreover, the set P_f of the poles of the function f on $\partial\mathbb{D}$ will be finite too.

Two latter facts allows us to establish the existence of the numbers $\varepsilon > 0$ and $r_0 \in (0, 1)$ such that for any $a \in \Lambda_f \cup \mathrm{P}_f$ there are $k = k(a)$ functions $c_1, c_2, \ldots, c_k \in C^1[r_0, 1]$ having the following properties when $r \in [r_0, 1)$:

1) $\Lambda_{f_r} \cap K_\varepsilon(a) = \{c_1(r), \ldots, c_k(r)\}$;

2) $\Lambda_{f_r} \subset \bigcup_{a' \in \Lambda_f \cup \mathrm{P}_f} K_\varepsilon(a')$;

3) $\sum_{j=1}^{k} \gamma_{f_r}(c_j(r)) = \gamma_f(a)$ when $a \in \Lambda_f$, and $k = 1$ with $\gamma_{f_r}(c_1(r)) = +1$ when $a \in \mathrm{P}_f$.

The base for such a conclusion is made by the results of [3]—for the points $a \in \mathrm{M}_f$, by the direct verification using (8)—for the elements $a \in \mathrm{S}_f$, and by Lemma 3—for the elements of the set P_f. By Lemma 3 we also conclude that the boundary zeros of f don't branching into the unit disk for the values $r < 1$ near 1. When $r \in [r_0, 1)$, all of the points in Λ_{f_r} will be the isolated singularities of the field $\nabla \ln h_f$, and the relation $\lim_{\zeta\to\partial\mathbb{D}} h_{f_r}(\zeta) = 0$ is valid. The proof of the formula (6) transferred on this case word by word from [9], therefore for every $r \in [r_0, 1)$ we have

$$1 = \sum_{a\in\Lambda_f\cup\mathrm{P}_f} \sum_{j=1}^{k(a)} \gamma_{f_r}(c_j(r)) = \sum_{a\in\Lambda_f} \gamma_f(a) + M^{\partial},$$

and it remains only to calculate the indices (for the elements of the set S_f—by Corollary 2). The proof is complete.

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