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Chapter 1
Mathematical Analysis and Numerical Modelling of the Guided Modes of the Step-Index Optical Fibers

Evgenii Karchevskii *

Abstract

The original physical problem of the guided modes of the step-index optical fibers is reduced to a nonlinear spectral problem for Fredholm holomorphic (on some Reiman surface) operator-valued function. It is demonstrated that the spectrum may consist only of isolated points. Galerkin's method is proposed for the calculating of approximate complex eigenvalues. The convergence of Galerkin's method is studied.

1 Formulation of the Problem

An optical fiber is a cylindrical structure which consists of a core of a dielectric material, surrounded by a cladding of another dielectric material. By n_1 denote the refractive index of the core. By n_2 denote the refractive index of the cladding. Assume that $n_1 > n_2$. Suppose the fiber extend infinitely along axis z and the fiber is perfectly cylindrical. By R^2 denote a plain $\{z = \text{const}\}$. Let $S_1 \subset R^2$ be the core region, bounded curve Γ; let $S_2 = R^2 \setminus \bar{S}_1$ be the cladding region; let M be a point of R^2.

We look for particular solution of the Maxwell equations

$$\mathrm{rot}\mathcal{H} = \varepsilon_0 n^2 \frac{\partial \mathcal{E}}{\partial t}, \quad \mathrm{rot}\mathcal{E} = -\mu_0 \frac{\partial \mathcal{H}}{\partial t},$$

which can be written as

$$(1)\ \mathcal{E}(M,z,t) = \mathrm{Re}\left(E(M)\exp\left(i\beta z - i\omega t\right)\right), \ \mathcal{H}(M,z,t) = \mathrm{Re}\left(H(M)\exp\left(i\beta z - i\omega t\right)\right).$$

Here $\mathcal{E}$ is the electric field, $\mathcal{H}$ is the magnetic field; E and H are complex amplitudes of $\mathcal{E}$ and $\mathcal{H}$; $\omega > 0$ is the frequency of oscillations; β is the complex propagation constant; ε_0 is the dielectric constant; μ_0 is the magnetic constant.

The problem of the guided modes of form (1) can be formulated as an eigenvalue problem, in terms of the complex-value parameter β, for which there exist non-trivial solutions (u, v) of the following boundary-value problem:

$$(2) \qquad \Delta u + \chi_j^2(\beta)u = 0, \ \Delta v + \chi_j^2(\beta)v = 0, \quad M \in S_j, \ j = 1,2,$$

$$(3) \quad u^+ = u^-, \quad \chi_1^{-2}(\beta)\left(\beta\frac{\partial v}{\partial \tau} + \varepsilon_1\omega\frac{\partial u^-}{\partial \nu}\right) = \chi_2^{-2}(\beta)\left(\beta\frac{\partial v}{\partial \tau} + \varepsilon_2\omega\frac{\partial u^+}{\partial \nu}\right), \quad M \in \Gamma,$$

$$(4) \quad v^+ = v^-, \quad \chi_1^{-2}(\beta)\left(\beta\frac{\partial u}{\partial \tau} - \mu_0\omega\frac{\partial v^-}{\partial \nu}\right) = \chi_2^{-2}(\beta)\left(\beta\frac{\partial u}{\partial \tau} - \mu_0\omega\frac{\partial v^+}{\partial \nu}\right), \quad M \in \Gamma.$$

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Here $u = E_z$, $v = H_z$, $\partial u/\partial \nu$ ($\partial u/\partial \tau$) is the normal (tangential) derivative on Γ, u^- (u^+) is the limiting value of the function u from inside (outside) of Γ, $\chi_j(\beta) = \sqrt{k_0^2 n_j^2 - \beta^2}$, $k_0^2 = \omega^2 \varepsilon_0 \mu_0$. Let the curve Γ is twice continuously differentiable. We shall suppose that the functions u and v for the enough large r can be presented as

$$(5) \qquad u = \sum_{n=-\infty}^{\infty} \alpha_n H_n^{(1)}(\chi_2(\beta) r) \exp(int), \quad v = \sum_{n=-\infty}^{\infty} \gamma_n H_n^{(1)}(\chi_2(\beta) r) \exp(int),$$

where (r,t) are the polar coordinates of the point M; $H_n^{(1)}$ is the Hankel function of the first kind and index n.

Denote by Λ_j the Reiman surface of the function $\ln \chi_j(\beta)$; $\Lambda = \Lambda_1 \bigcup \Lambda_2$. Denote by Λ_0 the principal ("physical") sheet of Λ, which is specified by the conditions $-\pi < \arg \chi_j(\beta) < \pi, \mathrm{Im}\chi_j(\beta) \geq 0, j = 1,2$. By U denote the space of continuous and continuously differentiable in $\bar{S}_1$ and $\bar{S}_2$, twice continuously differentiable in S_1 and S_2 functions.

DEFINITION 1.1. An $(u,v) \in (U \setminus \{0\})^2$ is called an eigenvector of problem (2) – (5) and a $\beta \in \Lambda$ is called an eigenvalue of problem (2) – (5) if the (u,v) and β satisfy conditions (2) – (5).

THEOREM 1.1. *The eigenvalues of problem (2) – (5) can not belong to imaginary and real axes of the sheet* Λ_0*, except for the set* $G = \{\beta \in \Lambda_0 : \mathrm{Im}\beta = 0,\ k_0 n_2 < |\beta| < k_0 n_1\}$.

This theorem was proved by A.I.Nosich in [1].

2 Regularization of the Problem

Denote by $C^{0,\alpha}$ the space of the Hölder continuous functions. Denote by $C^{1,\alpha}$ the space of the Hölder continuously differentiable functions. We use the representation of the eigenvector (u,v) of problem (2) – (5) in the form of the single-layer potentials:

$$(6) \qquad \begin{bmatrix} u(M) \\ v(M) \end{bmatrix} = \int_\Gamma \Phi_j(\beta; M, M_0) \begin{bmatrix} \varphi_j(M_0) \\ \psi_j(M_0) \end{bmatrix} dl_{M_0}, \quad \Phi_j = \frac{i}{4} H_0^{(1)}(\chi_j(\beta) |M - M_0|),$$

Here $M \in S_j$, $j = 1,2$; the unknown densities φ_j, $\psi_j \in C^{0,\alpha}$. Suppose the contour Γ is given in the parametric form as $r = r(t)$, $t \in [0, 2\pi]$. Using (3), (4), (6), and properties of single-layer potentials, we get nonlinear spectral problem:

$$(7) \qquad A(\beta) w \equiv (C(\beta) + R(\beta)) w = 0,\ \beta \in \Lambda, \quad A(\beta) : H \to H,\ H = \left(C^{0,\alpha}\right)^4,$$

The vector $w = \left(w^{(j)}\right)_{j=1}^4$ has elements $w^{(1)} = (\varphi_1 - \varphi_2)|r'|$, $w^{(2)} = (\psi_1 - \psi_2)|r'|$, $w^{(3)} = \varphi_1 |r'|$, and $w^{(4)} = \psi_1 |r'|$. The operator $R(\beta) = (R_{i,j}(\beta))_{i,j=1}^4$ is defind by his representation as

$$R_{1,1} = L^{-1} R_2^{(1)},\ R_{1,2} = 0,\ R_{1,3} = L^{-1}(R_1^{(1)} - R_2^{(1)}),\ R_{1,4} = 0,$$

$$R_{2,1} = 0,\ R_{2,2} = L^{-1} R_2^{(1)},\ R_{2,3} = 0,\ R_{2,4} = L^{-1}(R_1^{(1)} - R_2^{(1)}),$$

$$R_{3,1} = \frac{-\omega \varepsilon_2}{\chi_2^2} R_2^{(2)},\ R_{3,2} = \frac{\beta}{\chi_2^2} R_2^{(3)},\ R_{3,3} = \frac{-\omega \varepsilon_1}{\chi_1^2} R_1^{(2)} - \frac{\omega \varepsilon_2}{\chi_2^2} R_2^{(2)},\ R_{3,4} = \frac{\beta}{\chi_1^2} R_1^{(3)} - \frac{\beta}{\chi_2^2} R_2^{(3)},$$

$$R_{4,1} = \frac{\beta}{\chi_1^2} R_2^{(3)},\ R_{4,2} = \frac{\omega \mu_0}{\chi_2^2} R_2^{(2)},\ R_{4,3} = \frac{\beta}{\chi_1^2} R_1^{(3)} - \frac{\beta}{\chi_2^2} R_2^{(3)},\ R_{4,4} = \frac{\omega \mu_0}{\chi_1^2} R_1^{(2)} + \frac{\omega \mu_0}{\chi_2^2} R_2^{(2)},$$

where

$$R_j^{(k)}(\beta)x = \frac{1}{2\pi}\int_0^{2\pi} h_j^{(k)}(\beta;t,t_0)x(t_0)dt_0,\ h_j^{(1)}(\beta;t,t_0) = 2\pi\Phi_j(\beta;M,M_0) + \ln|\sin\frac{t-t_0}{2}|,$$

$$h_j^{(2)}(\beta;t,t_0) = 4\pi|r'(t)|\frac{\partial}{\partial\nu_M}\Phi_j(\beta;M,M_0),\ \ h_j^{(3)}(\beta;t,t_0) = 2|r'(t)|\frac{\partial}{\partial\tau_M}h_j^{(1)}(\beta;t,t_0) - i.$$

The operator $C(\beta) = (C_{i,j}(\beta))_{i,j=1}^4$ is defined by his representation as

$$C_{1,1} = I,\ C_{1,2} = 0,\ C_{1,3} = 0,\ C_{1,4} = 0,\ C_{2,1} = 0,\ C_{2,2} = I,\ C_{2,3} = 0,\ C_{2,4} = 0,$$

$$C_{3,1} = \frac{\omega\varepsilon_2}{\chi_2^2}I,\ C_{3,2} = \frac{\beta}{\chi_2^2}S,\ C_{3,3} = \left(\frac{\omega\varepsilon_1}{\chi_1^2} + \frac{\omega\varepsilon_2}{\chi_2^2}\right)I,\ C_{3,4} = \left(\frac{\beta}{\chi_1^2} - \frac{\beta}{\chi_2^2}\right)S,$$

$$C_{4,1} = \frac{\beta}{\chi_2^2}S,\ C_{4,2} = \frac{-\omega\mu_0}{\chi_2^2}I,\ C_{4,3} = \left(\frac{\beta}{\chi_1^2} - \frac{\beta}{\chi_2^2}\right)S,\ C_{4,4} = \left(\frac{\omega\mu_0}{\chi_1^2} - \frac{\omega\mu_0}{\chi_2^2}\right)I.$$

Here I is the identity operator; the operators

$$Sx = \frac{1}{2\pi}\int_0^{2\pi}\mathrm{ctg}\frac{t_0-t}{2}x(t_0)dt_0 + \frac{i}{2\pi}\int_0^{2\pi}x(t_0)dt_0,\quad S: C^{0,\alpha}\to C^{0,\alpha},$$

$$Lx = -\frac{1}{2\pi}\int_0^{2\pi}\ln|\sin\frac{t-t_0}{2}|x(t_0)dt_0,\quad L: C^{0,\alpha}\to C^{1,\alpha},$$

are continuously invertible.

THEOREM 2.1. *The operator-valued function $A(\beta)$, $\beta \in \Lambda$, is holomorphic one. The operator $A(\beta): H \to H$ is the Fredholm one for any $\beta \in \Lambda$.*

Proof. For any $\beta \in \Lambda$, the operators $R_j^{(1)}(\beta): C^{0,\alpha} \to C^{1,\alpha}$, $R_j^{(k)}(\beta): C^{0,\alpha} \to C^{0,\alpha}$, $k = 2,3$, $j = 1,2$, are completely continuous, hence it appears that the operator $R(\beta): H \to H$ is also completely continuous. The operator $C(\beta): H \to H$ is continuously invertible for any $\beta \in \Lambda$. Therefore, the operator $A(\beta): H \to H$ is the Fredholm one for any $\beta \in \Lambda$. The functions $h_j^{(k)}(\beta;t,t_0)$, $k = 1,2,3$, $j = 1,2$, are analytic functions of $\beta \in \Lambda$ for any point $(t,t_0) \in [0,2\pi]\times[0,2\pi]$. Hence, as follows from [2], the operator-valued function $A(\beta)$, $\beta \in \Lambda$, is holomorphic one. □

DEFINITION 2.1. A $w \in H$ is called an eigenvector of operator-valued function $A(\beta)$ and a $\beta \in \Lambda$ is called an eigenvalue of operator-valued function $A(\beta)$ if the w and β satisfy equation (7).

By Ω denote the set $\{\beta \in \Lambda_0 : \mathrm{Re}\beta = 0\} \bigcup \{\beta \in \Lambda_0 : \mathrm{Im}\beta = 0,\ -k_0n_2 < |\beta| < k_0n_2\}$.

THEOREM 2.2. *If a $\beta \in \Lambda_0 \setminus \Omega$ is the eigenvalue of operator-valued function $A(\beta)$; then the same β is the eigenvalue of problem (2) – (5). If a $\beta \in \Lambda_0$ is the eigenvalue of problem (2) – (5); then the same β is the eigenvalue of operator-valued function $A(\beta)$.*

Proof. By using the methods of potential theory, one can prove that for all $\beta \in \Lambda_0$ any eigenvector (u,v) of problem (2) - (5) can be presented in the form of the single-layer potentials (6). Hence, if a $\beta \in \Lambda_0$ is the eigenvalue of problem (2) – (5); then the same β is the eigenvalue of operator-valued function $A(\beta)$. Also, if for a certain $\beta \in \Lambda_0 \setminus \Omega$ the single-layer potential equals zero in S_j, then its density is an identical zero on Γ. From this it follows that if a $\beta \in \Lambda_0 \setminus \Omega$ is the eigenvalue of operator-valued function $A(\beta)$; then the same β is the eigenvalue of problem (2) – (5). □

Denote by $\sigma(A) \subset \Lambda$ the spectrum of operator-valued function $A(\beta)$, that is the set of all eigenvalues of Fredholm operator-valued function $A(\beta)$.

THEOREM 2.3. *The set $\sigma(A)$ can be only a set of isolated points.*

Proof. From theorems 1.1, 2.1, and 2.2 it follows that for all $\beta \in \Lambda_0$ satisfying the conditions $\mathrm{Im}\beta = 0$, $|\beta| > k_0 n_1$ the operator $A(\beta)$ is invertible. Hence, based on the results of [3], the spectrum of operator-valued function $A(\beta)$ can be only a set of isolated points. □

3 Galerkin's Method

We use the representation of the approximate eigenvector of operator-valued function $A(\beta)$ in the form:

$$w_n = \left(w_n^{(j)}\right)_{j=1}^4, \quad w_n^{(j)}(t) = \sum_{k=-n}^{n} \alpha_k^{(j)} e_k(t), \quad e_k(t) = \exp(ikt), \quad j = 1,2,3,4.$$

We look for unknown coefficients $\alpha_k^{(j)}$ by Galerkin's Method:

$$\int_0^{2\pi} (Aw_n)^{(j)}(t) e_{-k}(t)dt = 0, \quad k = -n,\dots,n, \quad j = 1,2,3,4.$$

The trigonometric functions $e_k(t)$ are the orthogonal eigenfunctions of the singular operators $L^{-1} : C^{1,\alpha} \to C^{0,\alpha}$ and $S : C^{0,\alpha} \to C^{0,\alpha}$, corresponding to the following eigenvalues: $\lambda_m = \{1/\ln 2 \text{ if } m = 0,\ 2|m| \text{ if } m \neq 0\}$ for the operator L^{-1} and $\lambda_m = \{i \text{ if } m = 0,\ i\,\mathrm{sign}(m) \text{ if } m \neq 0\}$ for the operator S. Hence, the action of the main (singular) parts of the integral operators in (7) on the basis functions is expressed in explicit form.

Denote by H_n^T the set of all trigonometric polynomials of the orders up to n. Denote by $H_n \subset H$ the space of the elements $w_n = \left(w_n^{(j)}\right)_{j=1}^4$, where $w_n^{(j)} \in H_n^{(T)}$, $j = 1,2,3,4$. Let $p_n : H \to H_n$ be the corresponding projection operator.

Using the Galerkin's method for solving problem (7), we get finite-dimensional nonlinear spectral problem:

(8) $A_n(\beta)w_n \equiv p_n A(\beta)w_n \equiv (I + p_n B(\beta))w_n \equiv (I + B_n(\beta))w_n = 0,\ A_n(\beta) : H_n \to H_n,$

where $B(\beta) = C^{-1}(\beta)R(\beta)$, $\beta \in \Lambda$. The operator-valued function $A_n(\beta)$, $\beta \in \Lambda$ is holomorphic one. The operator $A_n(\beta) : H_n \to H_n$ is the Fredgholm one for any $\beta \in \Lambda$.

DEFINITION 3.1. A $w_n \in H$ is called an eigenvector of operator-valued function $A_n(\beta)$ and a $\beta_n \in \Lambda$ is called an eigenvalue of operator-valued function $A_n(\beta)$ if the w_n and β_n satisfy equation (8). Denote by $\sigma(A_n) \subset \Lambda$ the spectrum of operator-valued function $A_n(\beta)$, that is the set of all eigenvalues of Fredholm operator-valued function $A_n(\beta)$.

We seek the eigenvalues β_n of operator-valued function $A_n(\beta)$ as the zeros of the determinant of the matrix eqation equivalent to (8):

$$\det\left(A_n(\beta)\right) = 0, \quad \beta \in \Lambda.$$

Denote by N', N'', N''' the infinite subsets of the set of integers N. Denote by $w_n \to w$, $n \in N'$ the convergence $w_n \to w$ for $n \to \infty$, $n \in N'$.

THEOREM 3.1. *The set $\sigma(A_n)$ can be only a set of isolated points. If $\beta_0 \in \sigma(A)$; then there exists some sequence $\{\beta_n\}_{n\in N}$, with $\beta_n \in \sigma(A_n)$, that $\beta_n \to \beta_0$, $n \in N$. If $\beta_n \in \sigma(A_n)$, $\beta_n \to \beta_0 \in \Lambda$, $n \in N' \subseteq N$; then $\beta_0 \in \sigma(A)$.*

Proof. It is reduced to the verification of fulfillment of all the conditions of theorem 1 from [4]. The operator p_n is linear one, $||p_n|| = 1$, and $||p_n y||_H \to ||y||_H$, $n \in N$ for any $y \in H$. Using the definitions of the operators $A(\beta)$ and $A_n(\beta)$, we get $||A_n(\beta)|| \leq ||A(\beta)|| \leq c(\beta)$, $n \in N$, $\beta \in \Lambda$, where $c(\beta)$ is a continuous function in Λ. The function $c(\beta)$ is determined by the norms of holomorphic operators $R_j^{(k)}(\beta)$, $k = 1, 2, 3$, and by the analytic functions $\chi_j(\beta)$, $j = 1, 2$. Therefore, the norms $||A_n(\beta)||$ are uniformly bounded with respect to n and β in any compact domain $D \subset \Lambda$.

For all $\beta \in \Lambda$ the sequence of operators $\{A_n(\beta)\}_{n\in N}$ properly converges to the operator $A(\beta)$, i.e., for all $\beta \in \lambda$ the following conditions are satisfied:

1. If the sequence $\{y_n\}_{n\in N}$, $y_n \in H_n$, P-converges to $y \in H$; then the sequence $\{A_n y_n\}_{n\in N}$ P-converges to Ay.
2. From the uniform boundedness of $\{y_n\}_{n\in N}$, $||y_n|| \leq const$, $n \in N$, and P-compactness of the sequence $\{A_n y_n\}_{n\in N}$, it follows that the sequence $\{y_n\}_{n\in N}$ is a P-compact one.

P-convergence $\{y_n\}_{n\in N}$ to $y \in H$, means that $||y_n - p_n y|| \to 0$, $n \in N$, and hence the validity of the first condition follows from the inequality $||A_n y_n - p_n A y|| \leq ||A_n|| ||y_n - p_n y|| + ||p_n|| ||A|| ||p_n y - y||$, $n \in N$ and the obvious limiting relationship $||p_n y - y|| \to 0$, $n \in N$.

Now let us verify the second condition. P-compactness of the sequence $\{A_n y_n\}_{n\in N}$ means that for any $N' \subseteq N$ there exists such a $N'' \subseteq N'$ that $\{A_n y_n = y_n + B_n y_n\}_{n\in N''}$ P-converges to $w \in H$. If $||y_n|| \leq const$, $n \in N''$, then there exists a weakly convergent sub-sequence $\{y_n\}_{n\in N'''}$, $N''' \subset N''$. Completely continuous operator B transforms it to a strongly convergent one: $||By_n - u|| \to 0$, $n \in N'''$, $u \in H$. From here, by inquality $||B_n y_n - p_n u|| \leq ||p_n|| ||By_n - u||$, we conclude that the sequence $\{B_n y_n\}_{n\in N'''}$ P-converges to $u \in H$. Hence, $\{y_n\}_{n\in N'''}$ P-converges to $y = w - u \in H$, and the second condition is satisfied as well. Hence, all the conditions of theorem 1 from [4] are fulfilled in the considered case. This proves the theorem. □

A practical efficiency of this method was shown in [5], [6] by the comparing of solutions of some problems of the theory of electromagnetic waves with experimental data and results obtained by other methods.

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