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Projections and Traces on von Neumann Algebras

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Abstract—Let P,Q be projections on a Hilbert space. We prove the equivalence of the following conditions: (i) $PQ+QP \le 2(QPQ)^p$ for some number $0<p\le 1$; (ii) PQ is paranormal; (iii) PQ is M^*-paranormal; (iv) $PQ=QP$. This allows us to obtain the commutativity criterion for a von Neumann algebra. For a positive normal functional φ on von Neumann algebra $\mathcal{M}$ it is proved the equivalence of the following conditions: (i) φ is tracial; (ii) $\varphi(PQ+QP) \le 2\varphi((QPQ)^p)$ for all projections $P,Q\in\mathcal{M}$ and for some $p=p(P,Q)\in(0,1]$; (iii) $\varphi(PQP)\le\varphi(P)^{1/p}\varphi(Q)^{1/q}$ for all projections $P,Q\in\mathcal{M}$ and some positive numbers $p=p(P,Q)$, $q=q(P,Q)$ with $1/p+1/q=1$, $p\ne 2$. Corollary: for a positive normal functional φ on $\mathcal{M}$ the following conditions are equivalent: (i) φ is tracial; (ii) $\varphi(A+A^*)\le 2\varphi(|A^*|)$ for all $A\in\mathcal{M}$.

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1. INTRODUCTION

This work continues the research of the authors [1–4], which contain new conditions for the commutativity of projections and various characterizations of the trace among all positive normal functionals on von Neumann algebras.

Let P, Q be projections on a Hilbert space. In Theorem 3.2 we prove the equivalence of the following conditions: (i) $PQ+QP\le 2(QPQ)^p$ for some number $0<p\le 1$; (ii) PQ is paranormal; (iii) PQ is M^*-paranormal; (iv) $PQ=QP$. This allows us to obtain the commutativity criterion for a von Neumann algebra (Corollary 3.3). In Theorem 4.1 for a positive normal functional φ on von Neumann algebra $\mathcal{M}$ we prove the equivalence of the following conditions: (i) φ is tracial; (ii) $\varphi(PQ+QP)\le 2\varphi((QPQ)^p)$ for all projections $P,Q\in\mathcal{M}$ and for some $p=p(P,Q)\in(0,1]$; (iii) $\varphi(PQP)\le\varphi(P)^{1/p}\varphi(Q)^{1/q}$ for all projections $P,Q\in\mathcal{M}$ and some positive numbers $p=p(P,Q)$, $q=q(P,Q)$ with $1/p+1/q=1$, $p\ne 2$. Corollary 4.2: for a positive normal functional φ on $\mathcal{M}$ the following conditions are equivalent: (i) φ is tracial; (ii) $\varphi(A+A^*)\le 2\varphi(|A^*|)$ for all $A\in\mathcal{M}$.

2. NOTATION, DEFINITIONS AND PRELIMINARIES

Let $\mathcal{H}$ be a Hilbert space over field $\mathbb{C}$, and $\mathcal{B}(\mathcal{H})$ be the $*$-algebra of all linear bounded operators on $\mathcal{H}$. An operator $X\in\mathcal{B}(\mathcal{H})$ is said to be paranormal (respectively, M^*-paranormal for some number $M>0$), if $||X^2\xi||\ge||X\xi||^2$ (respectively, $M||X^2\xi||\ge||X^*\xi||^2$) for all $\xi\in\mathcal{H}$ with $||\xi||=1$. An operator $X\in\mathcal{B}(\mathcal{H})$ is a projection, if $X=X^2=X^*$. By the commutant of a set $\mathcal{X}\subset\mathcal{B}(\mathcal{H})$ we mean the set

$$\mathcal{X}'=\{Y\in\mathcal{B}(\mathcal{H}):\ XY=YX\ \text{ for all }\ X\in\mathcal{X}\}.$$

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A $*$-subalgebra $\mathcal{M}$ of the algebra $\mathcal{B}(\mathcal{H})$ is called a von Neumann algebra acting on the Hilbert space $\mathcal{H}$ if $\mathcal{M} = \mathcal{M}''$. If $\mathcal{X} \subset \mathcal{B}(\mathcal{H})$ then $\mathcal{X}'$ is a von Neumann algebra, and $\mathcal{X}''$ is the smallest von Neumann algebra that containes $\mathcal{X}$. For a von Neumann algebra $\mathcal{M}$ let $\mathcal{M}^{\mathrm{sa}}$, $\mathcal{M}^{+}$, $\mathcal{M}^{\mathrm{u}}$, $\mathcal{M}^{\mathrm{id}}$ and $\mathcal{M}^{\mathrm{pr}}$ denote its Hermitian, positive, unitary parts, the set of idempotents and the set of projections, respectively. For every operator $A \in \mathcal{M}$ its modulus $|A| = \sqrt{A^*A}$ lies in $\mathcal{M}^+$. Let I and $\mathcal{Z}(\mathcal{M}) = \mathcal{M} \cap \mathcal{M}'$ denote the unit and the center of the algebra $\mathcal{M}$, respectively. For $Q \in \mathcal{M}^{\mathrm{id}}$ we have $Q^{\perp} = I - Q \in \mathcal{M}^{\mathrm{id}}$. For $P, Q \in \mathcal{M}^{\mathrm{pr}}$ we write $P \sim Q$ (the *Murray–von Neumann equivalence*) if $P = U^*U$ and $Q = UU^*$ for some $U \in \mathcal{M}$. A positive functional φ on a von Neumann algebra $\mathcal{M}$ is said to be *a state*, if $\varphi(I) = 1$; *tracial*, if $\varphi(Z^*Z) = \varphi(ZZ^*)$ for all $Z \in \mathcal{M}$; *normal*, if $A_i \nearrow A$ $(A_i, A \in \mathcal{M}^+) \Rightarrow \varphi(A) = \sup\limits_i \varphi(A_i)$. By $\mathcal{M}_*^+$ we denote the set of all positive normal functionals on $\mathcal{M}$.

Lemma 2.1 [5, Theorem 2.3.3]. *Let $\mathcal{N}$ be a von Neumann algebra of type I_n (n is a cardinal number). Then $\mathcal{N}$ is *-isomorphic to the tensor product $\mathcal{Z}(\mathcal{N})\overline{\otimes}\mathcal{B}(\mathcal{K})$, where $\mathcal{K}$ is a Hilbert space with* $\dim \mathcal{K} = n$.

Lemma 2.2 [6, Corollary 3.3]. *For every $X \in \mathbb{M}_n(C(\Omega))^{sa}$ the algebra $\mathbb{M}_n(C(\Omega))$ contains a unitary operator U such that $U^*XU(\omega)$* is diagonal for each $\omega \in \Omega$.

Lemma 2.3 [7, Chap. 5, item (ii) of Theorem 1.41]. *If a von Neumann algebra $\mathcal{N}$ is generated by two projections P, $R \in \mathcal{B}(\mathcal{H})^{\mathrm{pr}}$, then there exists a unique projection $Z \in \mathcal{Z}(\mathcal{N})$ such that the algebra $\mathcal{N}_Z$ is of type I_2 and the algebra $\mathcal{N}_{Z^\perp}$ is Abelian, moreover,* $\dim_{\mathbb{C}}\mathcal{N}_{Z^\perp} \leq 4$.

Let $g(t) = \sqrt{t(1-t)}$ for $0 \leq t \leq 1$ and $\delta \in \mathbb{C}$,A $|\delta| = 1$. By $R^{(\delta,t)}$ we denote the projection

$$R^{(\delta,t)} = \begin{pmatrix} t & \delta g(t) \\ \overline{deltag}(t) & 1-t \end{pmatrix}, \tag{1}$$

which lies in $\mathbb{M}_2(\mathbb{C})^{\mathrm{pr}}$.

3. THE NEW COMMUTATIVITY CRITERIONS OF PROJECTIONS

The search of the new commutativity criterions of projections was motivated by the following definition: Two (possibly unbounded) self-adjoint operators A and B are said to be commuting, if all projections of the its corresponding projection-valued measures commute [8, Chap. VIII, § 5, Definition].

Proposition 3.1. *If $X \in \mathcal{B}(\mathcal{H})^{sa}$, $Q \in \mathcal{B}(\mathcal{H})^{id}$ and XQ is an M^*-paranormal operator then* $XQ = Q^*X$.

Proof. It is well known [9] that an operator $T \in \mathcal{B}(\mathcal{H})$ is M^*-paranormal if and only if

$$M^2T^{*2}T^2 - 2\lambda TT^* + \lambda^2 I \geq 0 \quad \text{for all} \quad \lambda \in \mathbb{R}.$$

For $T = XQ$ this inequality takes the form

$$M^2Q^*XQ^*X^2QXQ - 2\lambda XQQ^*X + \lambda^2 I \geq 0 \quad \text{for all} \quad \lambda \in \mathbb{R}.$$

We multiply both sides of this inequality from the left by the operator $Q^{*\perp} = Q^{\perp *}$ and from the right by the operator $Q^{\perp}$, and obtain

$$-2\lambda Q^{\perp *}XQQ^*XQ^{\perp} + \lambda^2 Q^{\perp *}Q^{\perp} \geq 0 \quad \text{for all} \quad \lambda \in \mathbb{R}.$$

Then division by $\lambda > 0$ yields $-2Q^{\perp *}XQQ^*XQ^{\perp} + \lambda Q^{\perp *}Q^{\perp} \geq 0$ for all $\lambda > 0$. Passing to the limit as $\lambda \to 0$, we obtain $-2Q^{\perp *}XQQ^*XQ^{\perp} = -2|Q^*XQ^{\perp}|^2 \geq 0$. Hence, $Q^*XQ^{\perp} = 0$ and $Q^*X = Q^*XQ = (Q^*XQ)^* = (Q^*X)^* = XQ$. The assertion is proved. $\square$

Theorem 3.2. *For $P, Q \in \mathcal{B}(\mathcal{H})^{pr}$ the following conditions are equivalent:*

(i) $\dfrac{PQ+QP}{2} \leq (QPQ)^p$ *for some number* $0 < p \leq 1$;

(ii) *PQ is paranormal;*

(iii) *PQ is M^*-paranormal;*

(iv) $PQ = QP$.

Proof. (iv) ⇒ (i). For $0 < p \le 1$ we have

$$\frac{PQ+QP}{2} = QPQ \le (QPQ)^p$$

by the inequality $QPQ \le QIQ = Q \le I$ and the Spectral Theorem.

(i) ⇒ (iv). With P and Q we associate the von Neumann algebra $\mathcal{N} = \{P, Q\}''$ generated by them. By Lemma 2.3 there exists a unique projection $Z \in \mathcal{Z}(\mathcal{N})$ such that the algebra $\mathcal{N}_Z$ is of type I_2 and the algebra $\mathcal{N}_{Z^\perp}$ is Abelian. Obviously, the projections $PZ^\perp$ and $QZ^\perp$ commute. Gelfand's theorem on the representation of an Abelian unital C^*-algebra (see, for example, [7, Chap. 3, Theorem 1.18]) implies that the algebra $\mathcal{Z}(\mathcal{N}_Z)$ is $*$-isomorphic to the C^*-algebra $C(\Omega)$ of all complex-valued continuous functions on the Stone space Ω of all characters of the algebra $\mathcal{Z}(\mathcal{N}_Z)$. Now from Lemma 2.1 it follows that the algebra $\mathcal{N}_Z$ is $*$-isomorphic to the matrix algebra $\mathbb{M}_2(C(\Omega))$.

We define the domains of the rank constancy (or, equivalently, the domains of the canonical trace tr) for $\tilde{R} \in \mathbb{M}_2(C(\Omega))^{\mathrm{pr}}$ as follows:

$$\Omega_j(\tilde{R}) = \{\omega \in \Omega \,:\, \tilde{r}_{11}(\omega) + \tilde{r}_{22}(\omega) = j\}, \quad j \in \{0, 1, 2\}.$$

The sets $\Omega_j(\tilde{R})$ are closed (being the preimages of the closed sets $\{j\} \subset \mathbb{C}$ under a continuous mapping) and constitute a covering of the space Ω by disjoint sets.

The projections PZ and QZ are identified with $\tilde{P}, \tilde{Q} \in \mathbb{M}_2(C(\Omega))^{\mathrm{pr}}$, respectively. Let

$$\Omega_{ij} = \Omega_i(\tilde{P}) \cap \Omega_j(\tilde{Q}), \quad i, j \in \{0, 1, 2\}.$$

All the nine sets Ω_{ij} are open-closed and constitute a covering of the space Ω by disjoint sets. If $\omega \in \Omega \setminus \Omega_{11}$, then $\tilde{P}\tilde{Q}(\omega) = \tilde{Q}\tilde{P}(\omega)$.

Lemma 2.2 implies that there exist a unitary $U \in \mathbb{M}_2(C(\Omega))^{\mathrm{u}}$ and a closed subset $\Omega_1'(\tilde{P}) \subset \Omega_1(\tilde{P})$, such that $U^*(\omega)\tilde{P}(\omega)U(\omega) = \mathrm{diag}(1, 0)$ for all $\omega \in \Omega_1'(\tilde{P})$ and $U^*(\omega)\tilde{P}(\omega)U(\omega) = \mathrm{diag}(0, 1)$ for all $\omega \in \Omega_1(\tilde{P}) \setminus \Omega_1'(\tilde{P})$. Therefore, it is sufficient to consider the case

$$P = \mathrm{diag}(1, 0), \quad Q = R^{(\delta, t)}, \quad \delta \in \mathbb{C}, \quad |\delta| = 1, \quad 0 \le t \le 1;$$

see (1). Since $QPQ = tQ$, we have $(QPQ)^t = t^p Q$. Since

$$\frac{PQ+QP}{2} = \begin{pmatrix} t & * \\ * & * \end{pmatrix}, \quad t^p Q - \frac{PQ+QP}{2} = \begin{pmatrix} t^{p+1} - t & * \\ * & * \end{pmatrix}$$

(here and elsewhere, the symbol "$*$" denotes matrix entries whose values will not be needed), we have $t^{p+1} - t \ge 0$ and $t \in \{0, 1\}$.

(ii) ⇒ (iv). It is well known [10, Problem 9.5], that an operator $T \in \mathcal{B}(\mathcal{H})$ is paranormal if and only if

$$|T|^2 \le \frac{1}{2}(\lambda^{-1}|T^2|^2 + \lambda I) \quad \text{for all} \quad \lambda > 0.$$

For $T = PQ$ this inequality takes the form

$$2QPQ \le \lambda^{-1}(QPQ)^3 + \lambda I \quad \text{for all} \quad \lambda > 0. \tag{2}$$

If $PQ \ne QP$, then an operator QPQ is not a projection and its spectrum $\sigma(QPQ)$ contains some number $x \in (0, 1)$. By (2) and the Spectral Theorem the number x satisfies the inequality

$$2x \le \lambda^{-1}x^3 + \lambda \quad \text{for all} \quad \lambda > 0.$$

However, for $\lambda = x^{3/2}$ it does not hold.

The implication (iii) ⇒ (iv) follows from Proposition 3.1. □

Corollary 3.3. *For a von Neumann algebra $\mathcal{M}$ the following conditions are equivalent:*

(i) $\frac{PQ+QP}{2} \le (QPQ)^p$ *for all $P, Q \in \mathcal{M}^{\mathrm{pr}}$ and some $p = p(P, Q) \in (0, 1]$;*

(ii) *PQ is paranormal for all $P, Q \in \mathcal{M}^{\mathrm{pr}}$;*

(iii) PQ is *M^*-paranormal for all $P, Q \in \mathcal{M}^{\mathrm{pr}}$ and some number $M = M(P,Q) > 0$;*

(iv) *$\mathcal{M}$ is Abelian.*

Proof. (i) $\Rightarrow$ (iv). By assumption, all projections from $\mathcal{M}$ commute. Then by the Spectral Theorem all the Hermitian operators from $\mathcal{M}$ also commute. Recall that every operator A from $\mathcal{M}$ can be represented as the sum $A = T + iS$ with Hermitian operators $T = (A + A^*)/2$, $S = (A - A^*)/(2i)$ from $\mathcal{M}$.

(iv) $\Rightarrow$ (i). An Abelian von Neumann algebra $\mathcal{M}$ is *-isomorphic to the algebra $L_\infty(\Omega, \mathfrak{A}, \mu)$ on a localized measure space $(\Omega, \mathfrak{A}, \mu)$. For all $P, Q \in \mathcal{M}^{\mathrm{pr}}$ there exist $A, B \in \mathfrak{A}$ such that $P = \chi_A$, $Q = \chi_B$ and inequality (i) for the indicators turns into equality. The assertion is proved. □

4. TRACE CHARACTERIZATION ON VON NEUMANN ALGEBRAS

Theorem 4.1. *For $\varphi \in \mathcal{M}_*^+$ the following conditions are equivalent:*

(i) *φ is tracial;*

(ii) $\varphi\left(\dfrac{PQ + QP}{2}\right) \leq \varphi((QPQ)^p)$ *for all $P, Q \in \mathcal{M}^{\mathrm{pr}}$ and for some number $p = p(P,Q) \in (0, 1]$;*

(iii) $\varphi(PQP) \leq \varphi(P)^{1/p}\varphi(Q)^{1/q}$ *for all $P, Q \in \mathcal{M}^{\mathrm{pr}}$ and for some positive numbers $p = p(P,Q)$, $q = q(P,Q)$ with $1/p + 1/q = 1$, $p \neq 2$.*

Proof. (i) $\Rightarrow$ (ii). For all $P, Q \in \mathcal{M}^{\mathrm{pr}}$ and $0 < p \leq 1$ by the inequality $QPQ \leq (QPQ)^p$ and monotonocity of φ on $\mathcal{M}^{\mathrm{sa}}$ we have

$$\varphi\left(\frac{PQ + QP}{2}\right) = \varphi(QPQ) \leq \varphi((QPQ)^p).$$

The implication (i) $\Rightarrow$ (iii) follows from the inequalities $PQP \leq PIP = P$, $QPQ \leq QIQ = Q$ and monotonocity of φ on $\mathcal{M}^+$.

Let us show that for an arbitrary von Neumann algebra, the proof of the inverse implications can be reduced to the case of the algebra $\mathbb{M}_2(\mathbb{C})$ just as this was done in a number of other similar cases (see [11] or [12]).

It is well known [11] that $\varphi \in \mathcal{M}_*^+$ is tracial if and only if $\varphi(P) = \varphi(Q)$ for all $P, Q \in \mathcal{M}^{\mathrm{pr}}$ with $PQ = 0$ and $P \sim Q$ (also see [12, Lemma 2]). Assume that a *-algebra $\mathcal{N}$ in the reduced algebra $(P + Q)\mathcal{M}(P + Q)$ is generated by a partial isometry $V \in \mathcal{M}$ realizing the equivalence of P and Q. Then $\mathcal{N}$ is *-isomorphic to $\mathbb{M}_2(\mathbb{C})$, while inequalities (ii) and (iii) remain valid for operators from $\mathcal{N}$ and the restriction of the functional $\varphi|_{\mathcal{N}}$. We shall show that such a restriction is a tracial functional on $\mathcal{N}$; therefore, $\varphi(P) = \varphi(Q)$.

As it is well known, every linear functional φ on $\mathbb{M}_2(\mathbb{C})$ can be represented in the form $\varphi(\cdot) = \mathrm{tr}(S_\varphi \cdot)$. The two-by-two matrix S_φ is called the density matrix of φ. It is easily seen that without loss of generality we can assume that

$$S_\varphi = \mathrm{diag}\left(\frac{1}{2} - s, \frac{1}{2} + s\right), \quad 0 \leq s \leq \frac{1}{2}.$$

Thus $\varphi(X)$ equals $(1/2 - s)x_{11} + (1/2 + s)x_{22}$ for $X = [x_{ij}]_{i,j=1}^2$ in $\mathbb{M}_2(\mathbb{C})$.

Let $0 < \varepsilon \leq 1/2$, $h(\varepsilon) = 1/4 - \varepsilon^2 (= g(1/2 - \varepsilon)^2)$. Consider two one-dimensional projections $P = R^{(1,1/2-\varepsilon)}$ and $Q = R^{(1,1/2+\varepsilon)}$, see (1). We have

$$QP = \begin{pmatrix} 2h(\varepsilon) & (1 + 2\varepsilon)\sqrt{h(\varepsilon)} \\ (1 - 2\varepsilon)\sqrt{h(\varepsilon)} & 2h(\varepsilon) \end{pmatrix}, \quad QPQ = 4h(\varepsilon)Q.$$

Hence $(QPQ)^p = (1 - 4\varepsilon^2)^p Q$ and inequality in (ii) takes the form

$$\frac{1}{2} - 2\varepsilon^2 \leq (1 - 4\varepsilon^2)^p \left(\frac{1}{2} - 2\varepsilon s\right), \quad 0 < \varepsilon \leq \frac{1}{2}. \tag{3}$$

By the Taylor's formula with the remainder in the Peano form we obtain

$$(1-4\varepsilon^2)^p = 1 - 4p\varepsilon^2 + o(\varepsilon^2) \quad \text{as} \quad \varepsilon \to 0$$

and from (3) we obtain

$$\frac{1}{2} - 2\varepsilon^2 \le \frac{1}{2} - 2\varepsilon s - 2p\varepsilon^2 + o(\varepsilon^2) \quad \text{as} \quad \varepsilon \to 0.$$

By assumption $s \ge 0$, hence inequality (3) holds for all $\varepsilon \in (0, 1/2]$ only in the case $s = 0$.

(iii) $\Rightarrow$ (i). Let, for certainty, $p > q$. In our notation we have $PQP = 4h(\varepsilon)P$, $\varphi(P) = 1/2 + 2s\varepsilon$, $\varphi(Q) = 1/2 - 2s\varepsilon$ and $\varphi(PQP) = 1/2 + 2s\varepsilon - 2\varepsilon^2 - 8s\varepsilon^3$. The inequality $\varphi(PQP) \le \varphi(P)^{1/p}\varphi(Q)^{1/q}$ takes the form

$$\frac{1}{2} + 2s\varepsilon - 2\varepsilon^2 - 8s\varepsilon^3 \le \left(\frac{1}{2} + 2s\varepsilon\right)^{1/p} \left(\frac{1}{2} - 2s\varepsilon\right)^{1/q}.$$

Multiplying both sides of this inequality by $2 = 2^{1/p}2^{1/q}$, we obtain

$$1 + 4s\varepsilon - 4\varepsilon^2 - 16s\varepsilon^3 \le (1+4s\varepsilon)^{1/p}(1-4s\varepsilon)^{1/q}. \tag{4}$$

Taylor's formula implies the asymptotic equalities

$$(1+4s\varepsilon)^{1/p} = 1 + \frac{4s}{p}\varepsilon + o(\varepsilon), \quad (1-4s\varepsilon)^{1/q} = 1 - \frac{4s}{q}\varepsilon + o(\varepsilon)$$

as $\varepsilon \to 0+$, and the right-hand side of inequality (4) is equal to

$$1 + \frac{4(q-p)s}{pq}\varepsilon + o(\varepsilon) \quad (\varepsilon \to 0+).$$

Since $s \ge 0$, inequality (4) for all $0 < \varepsilon \le 1/2$ holds only in the case of $s = 0$. □

Corollary 4.2. *For $\varphi \in \mathcal{M}_*^+$ the following conditions are equivalent:*

(i) *φ is tracial;*

(ii) $\varphi\left(\frac{A+A^*}{2}\right) \le \varphi(|A^*|)$ *for all $A \in \mathcal{M}$.*

Proof. (i) $\Rightarrow$ (ii). Recall [13], that for all $X, Y \in \mathcal{M}$ there exist partial isometries $U, V \in \mathcal{M}$ such that

$$|X+Y| \le U|X|U^* + V|Y|V^*.$$

If $Z = W|Z|$ is the polar decomposition of an operator $Z \in \mathcal{M}$ then $W \in \mathcal{M}$ and $|Z^*| = W|Z|W^*$. Also $|\varphi(T)| \le \varphi(|T|)$ for all $T \in \mathcal{M}$, see [7, p. 320]. By virtue of this and the monotonicity of the tracial functional φ for all $A \in \mathcal{M}$ we have

$$\begin{aligned}\varphi\left(\frac{A+A^*}{2}\right) &\le \left|\varphi\left(\frac{A+A^*}{2}\right)\right| \le \varphi\left(\left|\frac{A+A^*}{2}\right|\right) = \frac{1}{2}\varphi(|A+A^*|) \le \frac{1}{2}\varphi(U|A|U^* + V|A^*|V^*) \\ &= \frac{1}{2}(\varphi(U|A|U^*) + \varphi(V|A^*|V^*)) = \frac{1}{2}(\varphi(UW|A^*|W^*U^*) + \varphi(V|A^*|V^*)) \le \varphi(|A^*|).\end{aligned}$$

(ii) $\Rightarrow$ (i). For $A = PQ$ with $P, Q \in \mathcal{M}^{\mathrm{pr}}$ condition (ii) of Theorem 4.1 is met with $p = 1/2$. The assertion is proved. □

For other trace characterizations, see [14–24] and the references therein.

Corollary 4.3. *For a von Neumann algebra $\mathcal{M}$ the following conditions are equivalent:*

(i) *an algebra $\mathcal{M}$ is Abelian;*

(ii) $\varphi\left(\frac{PQ+QP}{2}\right) \le \varphi((QPQ)^p)$ *for all normal states φ on $\mathcal{M}$ and $P, Q \in \mathcal{M}^{\mathrm{pr}}$ and some number $p = p(P,Q) \in (0,1]$;*

(iii) $\varphi(PQP) \le \varphi(P)^{1/p}\varphi(Q)^{1/q}$ *for all normal states φ on $\mathcal{M}$ and $P, Q \in \mathcal{M}^{\mathrm{pr}}$ and some positive numbers $p = p(P,Q)$, $q = q(P,Q)$ with $1/p + 1/q = 1$, $p \ne 2$.*

Proof. By Theorem 4.1 every normal state on $\mathcal{M}$ is tracial, i.e., $\varphi(XY) = \varphi(YX)$ for all $X, Y \in \mathcal{M}$. Since the set $\mathcal{M}_*^+$ separates the points of the algebra $\mathcal{M}$ [25, Chap. III, Theorem 2.4.5], it follows from the last condition that $XY = YX$ $(X, Y \in \mathcal{M})$, and thus the von Neumann algebra $\mathcal{M}$ is commutative. □

Proposition 4.4. *For every number $K > 0$ there exists a positive functional φ on $\mathbb{M}_2(\mathbb{C})$ such that*

(i) $\varphi(A^2 + B^2 + K(AB + BA)) \geq 0$ *for all* $A, B \in \mathbb{M}_2^+(\mathbb{C})$;

(ii) $\varphi \neq \lambda \mathrm{tr}$ *for all numbers* $\lambda > 0$.

Proof. Let $K > 1$ and consider

$$X = \begin{pmatrix} K^2+1 & 0 \\ 0 & K^2-1 \end{pmatrix}, \quad A = \begin{pmatrix} x_1 & \delta_1 x_2 \\ \overline{\delta_1} x_2 & x_3 \end{pmatrix}, \quad B = \begin{pmatrix} y_1 & \delta_2 y_2 \\ \overline{\delta_2} y_2 & y_3 \end{pmatrix},$$

where $x_1, x_2, x_3, y_1, y_2, y_3 \geq 0$, $|\delta_i| = 1, \delta_i \in \mathbb{C}$, $x_1 x_3 \geq x_2^2$, $y_1 y_3 \geq y_2^2$. Then

$$\begin{aligned}
&\varphi(A^2 + B^2 + K(AB + BA)) \\
&= (K^2+1)(x_1^2 + x_2^2) + (K^2-1)(x_2^2 + x_3^2) + (K^2+1)(y_1^2 + y_2^2) + (K^2-1)(y_2^2 + y_3^2) \\
&\quad + 2K((K^2+1)x_1 y_1 + ((K^2+1)\mathrm{Re}\delta_1\overline{\delta_2} + (K^2-1)\mathrm{Re}\overline{\delta_1}\delta_2)x_2 y_2 + (K^2-1)x_3 y_3) \\
&= (K^2+1)(x_1^2 + y_1^2) + (K^2-1)(x_3^2 + y_3^2) + 2K^2(x_2 - y_2)^2 \\
&\quad + 2K((K^2+1)x_1 y_1 + 2(K + K^2\mathrm{Re}\delta_1\overline{\delta_2})x_2 y_2 + (K^2-1)x_3 y_3) \\
&\geq (K^2+1)(x_1^2 + y_1^2) + 2K^2(x_2 - y_2)^2 + (K^2-1)(x_3^2 + y_3^2) \\
&\quad + 2K((K^2+1)x_1 y_1 + 2K(1-K)\sqrt{x_1 y_1 x_3 y_3} + (K^2-1)x_3 y_3) \\
&= (K^2+1)(x_1^2 + y_1^2) + 2K^2(x_2 - y_2)^2 + (K^2-1)(x_3^2 + y_3^2) \\
&\quad + 2K(K+1)x_1 y_1 + 2K(K-1)x_3 y_3 + 2K^2(K-1)\left(\sqrt{x_1 y_1} - \sqrt{x_3 y_3}\right)^2 \geq 0.
\end{aligned}$$

Proposition is proved. □

Consider a matrix

$$X = \begin{pmatrix} x_1 & \delta_1 x_2 \\ \overline{\delta_1} x_2 & x_3 \end{pmatrix} \quad \text{with} \quad x_1, x_2, x_3 \geq 0, \quad x_1 x_3 \geq x_2^2, \quad \delta_1 \in \mathbb{C}, \quad |\delta_1| = 1. \tag{5}$$

Lemma 4.5. *Let X be as in* (5) *and for $\lambda \in [0,1], a > 0, \delta_2 \in \mathbb{C}, |\delta_2| = 1$ put $A = \mathrm{diag}(a, 0)$, $B = R^{(\delta_2, \lambda)}$. Then the following conditions are equivalent:*

(i) $\exists K_0 > -1\, \forall K > K_0\, \forall A, B \in \mathbb{M}_2^+(\mathbb{C})\ \mathrm{tr}(X(A^2 + B^2 + K(AB + BA))) \geq 0$;

(ii) $AX = XA$ (*i.e.*, $x_2 = 0$).

Proof. We have $A^2 = aA$, $AB = (BA)^*$ and $\varphi(A^2) = \mathrm{tr}(aXA) = a^2 x_1$,

$$\begin{aligned}
\varphi(B^2) = \mathrm{tr}(XB) &= \mathrm{tr}\begin{pmatrix} \lambda x_1 + \delta_1\overline{\delta_2}x_2\sqrt{\lambda(1-\lambda)} & * \\ * & \overline{\delta_1}\delta_2 x_2\sqrt{\lambda(1-\lambda)} + x_3(1-\lambda) \end{pmatrix} \\
&= \lambda x_1 + 2\mathrm{Re}(\delta_1\overline{\delta_2})x_2\sqrt{\lambda(1-\lambda)} + x_3(1-\lambda),
\end{aligned}$$

$$\varphi(AB) = \mathrm{tr}\begin{pmatrix} a x_1 \lambda & * \\ * & \overline{\delta_1}\delta_2 a x_2\sqrt{\lambda(1-\lambda)} \end{pmatrix} = a x_1 \lambda + \overline{\delta_1}\delta_2 a x_2 \sqrt{\lambda(1-\lambda)},$$

$$\varphi(BA) = \varphi((AB)^*) = \overline{\varphi(AB)} = a x_1 \lambda + \delta_1\overline{\delta_2} a x_2 \sqrt{\lambda(1-\lambda)}.$$

Thus,

$$\varphi(A^2 + B^2 + K(AB + BA)) = (2aK\lambda + a^2 + \lambda)x_1 + (Ka + 1)\delta_2\overline{\delta_1}\sqrt{\lambda(1-\lambda)}x_2$$

$$+ (Ka+1)\overline{\delta_2}\delta_1\sqrt{\lambda(1-\lambda)}x_2 + (1-\lambda)x_3.$$

If $\lambda = 1$, then $\varphi(A^2+B^2+K(AB+BA)) = (a^2+2aK+1)x_1 \geq 0$ for all $x_1 \geq 0$, hence $K \geq -1$. Let $\lambda \in [0,1)$, and $t = \sqrt{\lambda/(1-\lambda)} \in \mathbb{R}^+$, then

$$\frac{1}{1-\lambda}\varphi(A^2+B^2+K(AB+BA)) = (2aK+1)x_1t^2 + a^2x_1(t^2+1) + 2\mathrm{Re}\delta_1\overline{\delta_2}(Ka+1)x_2t + x_3$$
$$= (a^2+2aK+1)x_1t^2 + 2\mathrm{Re}\delta_1\overline{\delta_2}(Ka+1)x_2t + x_3 + a^2x_1.$$

If $\varphi(A^2+B^2+K(AB+BA)) \geq 0$ for all $A, B \in \mathbb{M}_2^+(\mathbb{C})$, then (for $\mathrm{Re}\delta_1\overline{\delta_2} = -1$)

$$(a^2+2aK+1)x_1t^2 - 2x_2(Ka+1)t + x_3 + a^2x_1 \geq 0 \quad \text{for any} \quad t \geq 0, a \geq 0, K > K_0. \tag{6}$$

Consider the equation $\alpha t^2 - 2\beta t + \gamma = 0$, with

$$\alpha = x_1(a^2+2aK+1) \geq 0, \quad \beta = x_2(Ka+1), \quad \gamma = x_3 + a^2x_1 \geq 0.$$

Note that $x_1x_3 \geq x_2^2$, and $\alpha, \gamma \geq 0$. Clearly, if $x_1 > 0$ and $(a-1)^2 + 2a(K+1) = x_1 = 0$, then $a = 1$, $K = -1$ and thus $\beta = x_2(Ka+1) = 0$. Inequality (6) holds only in the following three cases:

1. $\alpha = 0, \beta = 0, \gamma \geq 0$;
2. $\alpha > 0, D = \beta^2 - \alpha\gamma \leq 0$;
3. $\alpha > 0, D = \beta^2 - \alpha\gamma \geq 0, t_{1,2} = \beta \pm \sqrt{D} \leq 0$.

In the first case of $\alpha = 0 = \beta$ we have $(a-1)^2 + 2a(K+1) = 0$ and $x_2(Ka+1) = 0$, but it is not possible simultaneously for any $a \geq 0$ and $K > K_0 > -1$.

In the second case,

$$a^2x_2^2K^2 + 2(x_2^2 - x_1x_3 - a^2x_1^2)aK + (x_2^2 - (a^2+1)x_1(x_3+a^2x_1)) \leq 0.$$

Note that for any a we have $x_2^2 - x_1x_3 - a^2x_1^2 \leq 0$ and $x_2^2 - (a^2+1)x_1(x_3+a^2x_1) \leq 0$. If we fix $a > 0$ and assume that K is sufficiently large, then obviously this inequality is violated for any $x_2 \neq 0$, which leads us to the consideration of the third case. Else, if $x_2 = 0$, then the inequality

$$2(-x_1x_3 - a^2x_1^2)aK + (-(a^2+1)x_1(x_3+a^2x_1)) \leq 0$$

holds for any $a > 0$, $K > K_0$. But also, if $x_2 = 0$, then $XA = AX$, since

$$\mathrm{diag}(x_1, x_3)\mathrm{diag}(a, 0) = \mathrm{diag}(a, 0)\mathrm{diag}(x_1, x_3). \tag{7}$$

In the third case, for any $a > 0$, $K > K_0 > -1$ we have

$$a^2x_2^2K^2 + 2(x_2^2 - x_1x_3 - a^2x_1^2)aK + (x_2^2 - (a^2+1)x_1(x_3+a^2x_1)) \geq 0$$

and $\beta \leq -\sqrt{D} \leq 0$. This is true only if $\beta \leq -\sqrt{\beta^2 - \alpha\gamma}$, which in its turn holds if and only if $\beta \leq 0$, thus either $aK \leq -1$, which is impossible for any $K > K_0$ and $a \geq 0$, simultaneously, or $x_2 = 0$ for all $a > 0, K > K_0$. Again, we have (7). $\square$

Teorem 4.6. *For $\varphi \in \mathcal{M}_*^+$ the following conditions are equivalent:*

(i) *φ is tracial;*

(ii) $\forall K \geq -1\, \forall A, B \in \mathcal{M}^+\ \varphi(A^2+B^2+K(AB+BA)) \geq 0$;

(iii) $\exists K_0 > -1\, \forall K > K_0\, \forall A, B \in \mathcal{M}^+\ \varphi(A^2+B^2+K(AB+BA)) \geq 0$.

Proof. (i)$\Rightarrow$(ii). We have $\sqrt{A}B\sqrt{A} \geq 0$ and $\varphi(AB) = \varphi(BA) = \varphi(\sqrt{A}B\sqrt{A}) \geq 0$, thus $\varphi(A^2 + B^2 + K(AB+BA)) = \varphi((A-B)^2) + (K+1)\varphi(AB+BA) \geq 0$.

The implication (ii)$\Rightarrow$(iii) is obvious.

(iii) $\Rightarrow$ (i). It is sufficient to consider the case $\mathcal{M} = \mathbb{M}_2(\mathbb{C})$, see the proof of Theorem 4.1. Let P, Q be some arbitrary one-dimensional projections in $\mathbb{M}_2(\mathbb{C})^{\mathrm{pr}}$. There exists an orthonormal basis $\{\xi_1, \xi_2\}$ in $\mathcal{H} = \mathbb{C}^2$ such that $P = \mathrm{diag}(1,0)$, $Q = R^{(\delta_2,\lambda)}$ for $\lambda \in [0,1], \delta_2 \in \mathbb{C}, |\delta_2| = 1$. Also, without loss of generality we may assume that $\varphi(\cdot) = \mathrm{tr}(X\cdot)$ with X as in (5). Let $a > 0$, then the inequality $\varphi((aP)^2 + Q^2 + K(aP+Q)) \geq 0$ implies

$$\mathrm{tr}(X(A^2+B^2+K(A+B))) \geq 0$$

in terms of Lemma 4.5. Thus, $x_2 = 0$ and $PX = XP$ in some basis $\{\xi_1, \xi_2\}$, where $\xi_1 \in P\mathcal{H} = \{P\xi : \xi \in \mathcal{H}\}$ and $\xi_2 \in P^{\perp}\mathcal{H} = \{P^{\perp}\xi : \xi \in \mathcal{H}\}$. The commutativity property does not depend on the representation of an operator in matrix form. Thus, simpliy for operators X and an arbitrary P the equality $PX = XP$ holds. Hence the operator X commutes with all of the operators of $\mathcal{B}(\mathcal{H})$ and the matrix X commutes with all of the matrices in $\mathbb{M}_2(\mathbb{C})^{\mathrm{pr}}$, which is possible only if $X = \lambda I$ for some $\lambda \in \mathbb{R}^+$. $\square$

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