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Existence and Properties of Solutions to the Spectral Problem of the Dielectric Waveguide Theory

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Abstract—The existence of a spectrum is proved for the eigenmodes of weakly guiding dielectric waveguides, and the properties of the dispersion curves are analyzed. The method of analysis is based on a reduction of the problem on the unbounded domain to a parametric eigenvalue problem posed on a circle with a nonlocal boundary condition nonlinear in the spectral parameter. The analysis of the reduced problem is based on the spectral theory of compact self-adjoint operators.

INTRODUCTION

Analysis of spectral problems in the theory of dielectric waveguides and development of numerical methods for their solution are the subjects of extensive current research (see, for example, [1–5] and references cited therein).

The basic problem in the theory of dielectric waveguides is the analysis of their modes of the form $\Phi(x)\exp[i(\omega t - \beta z)]$, where x is the vector of transverse coordinates, z is the longitudinal coordinate, β is the longitudinal propagation constant, ω is the frequency of oscillation in time t. Surface waves are of particular interest. Their amplitude $\Phi(x)$ decays exponentially as $|x| \longrightarrow \infty$, and the corresponding propagation constant β is real.

For cylindrical waveguides with permittivity weakly varying over the cross section, the "approximation of a weakly guiding waveguide" is widely used [2]. It leads to a parametric spectral problem for the Helmholtz equation on a plane in which the desired parameters are $\lambda = \omega^2$ and $\mu = \beta^2$. The theoretical analysis is focused on the behavior of dispersion curves, which describe the relation between the parameters λ and μ. For waveguides of canonical cross-sectional geometry (circle, ellipse, rectangle) characterized by certain permittivity distributions, the analysis can be carried out by means of separation of variables (see, for example, [2]). However, waveguides with diverse properties are used in practice (homogeneous waveguides of arbitrary geometry, fibers with permittivity varying over the cross section, waveguides consisting of two and more parallel fibers, etc.). The corresponding spectral problems cannot be solved explicitly, and numerical methods are used to solve them (see, for example, the reviews in [3, 4]). In [6–8], a method for calculating homogeneous waveguides of arbitrary cross section was suggested and studied, based on a reduction of the problem to a nonlinear spectral problem for a Fredholm integral holomorphic operator-valued function.

In this paper, we present a theoretical analysis of the dispersion curves in the general case, i.e., for inhomogeneous waveguides of arbitrary cross section. The approach suggested below further develops the method described in [9], where the existence of a spectrum was proved. It is based on a reduction of the problem under consideration to a problem on a bounded domain with a nonlocal boundary condition. The resulting parametric eigenvalue problem is nonlinear in the spectral parameter. Our analysis of this problem is based on the spectral theory of bounded self-adjoint operators.

We point out that the existence of solutions to spectral problems on unbounded domains is frequently examined by methods based on the spectral theory of unbounded operators. A number of interesting results have been obtained in this way (see, for example, [10–12]). In particular, the issues addressed in this paper were considered in [10] for a vector problem.

The method proposed here can be considered as an alternative approach to the analysis and, eventually, approximate solution to problems of this kind. It appears to be more constructive, because it leads to equations that are easier to solve numerically. Moreover, a number of new results can be obtained along these lines, in particular, the equations that determine the number of solutions (the so-called cut-off equations) can be considerably simplified.

In Section 1, we formulate the problem and the restrictions on the parameters λ and μ required to ensure the existence of solutions to the problem.

In Section 2, two equivalent definitions of the generalized solution are given: the conventional one (on a plane) and a new one (on a circle). The method of reduction of problems posed on unbounded domains to problems on bounded domains, which was applied to boundary value problems in [13, 14] and to spectral problems in [15] is employed. The new definition of the generalized solution leads to a parametric eigenvalue problem of the form $A_\mu(\lambda)u = \lambda Bu$, where $A_\mu(\lambda)$ and B are linear self-adjoint operators; in addition, $A_\mu(\lambda)$ is bounded and nonnegative, and B is compact and positive.

At a fixed μ, we have the problem $A(\lambda)u = \lambda Bu$. In Section 3, we examine the existence of solutions to an abstract problem of this form by analyzing the dependence of the eigenvalues of the conventional symmetric spectral problem $A(\lambda)u = \gamma Bu$ on λ. In this analysis, a key role is played by the monotonicity and continuity of the operator-valued function $A(\lambda)$. A finite-dimensional problem of this type was analyzed in [16], and an analogous abstract problem was considered in [17, 18].

In Section 4 the operator $A_\mu(\lambda)$ is defined at μ, λ belonging to the boundary of the domain of their admissible values. This is required to formulate the cut-off equation, which is done in Section 5.

The main results of the paper are formulated and proved in Section 6. The eigenvalues $\lambda(\mu)$ are studied as functions of the parameter μ. We construct simple equations based on the Courant–Weyl principle, which are satisfied by $\lambda(\mu)$. The dispersion curves $\lambda = \lambda(\mu)$ are described. In particular, their smoothness, monotonicity, and large μ asymptotics are studied. We show that the problem has at least one solution at any $\mu > 0$ and the number of solutions increases with μ, approaching infinity as $\mu \longrightarrow \infty$. There exist only a finite number of solutions for any finite μ. This number depends on special values of μ, which are called the cut-off points and are the solutions to the cut-off equations. Finally, a theorem on the orthogonality of eigenfunctions is proved for the original problem on a plane.

The equations obtained in the paper are transparent as concerns the construction of their numerical solutions methods. These equations can be differenced in a natural manner by a variational difference method. Algorithms that solve the resulting discrete spectral problems, which are nonlinear in the spectral parameter, were discussed, for example, in [18]. The issues arising in the numerical analysis of the problems considered in this paper will be addressed in separate publications.

1. STATEMENT OF THE PROBLEM

We seek the surface eigenmodes of weakly guiding dielectric waveguides. Under certain assumptions (e.g., see [2, p. 525]), the task reduces to finding such positive values of the parameters λ and μ that there exist nontrivial, exponentially decaying toward infinity solutions u to the following problem understood in the classical sense [19]:

$$-\Delta u + \mu u = \lambda \varepsilon u, \quad x \in \Omega_i \cup \Omega_e, \tag{1}$$

$$[u]_\gamma = 0, \quad [u_\nu]_\gamma = 0. \tag{2}$$

Here, $\varepsilon(x)$ denotes the permittivity of a waveguide at $x \in \Omega_i$ and that of the ambient medium at $x \in \Omega_e$, $\varepsilon(x) = \varepsilon_\infty = \text{const}$ in Ω_e, u_ν is the derivative along the outward normal to the boundary γ of a bounded (not necessarily connected) domain Ω_i in $\mathbb{R}^2$, $\Omega_e = \mathbb{R}^2 \backslash \overline{\Omega}_i$, and $[u]_\gamma$ is the jump of u across the contour γ. We assume that the origin lies in Ω_i, $\varepsilon \in C(\overline{\Omega}_i)$, each connected component of the boundary γ is a Lipschitz curve, and

$$\min_{x \in \overline{\Omega}_i} \varepsilon(x) \geq \varepsilon_\infty, \quad \varepsilon_+ = \max_{x \in \overline{\Omega}_i} \varepsilon(x) > \varepsilon_\infty > 0.$$

The exact solution to problem (1), (2) for a uniform waveguide ($\varepsilon(x) = \text{const}$ in Ω_i) of circular cross section is well known [2, p. 267]. The values of λ and μ for which there exist nontrivial solutions to the problem make up a countable set of dispersion curves $\lambda = \lambda_k(\mu)$ in the plane (μ, λ). Figure 1 shows the dispersion curves for $\varepsilon_\infty = 1$ and $\varepsilon_+ = 2$. These curves are smooth and belong to the domain Λ bounded by the lines $\lambda = \mu/\varepsilon_\infty$, and $\lambda = \mu/\varepsilon_+$. In what follows, we show that this behavior is also characteristic of the case under consideration.

Let us describe the set of admissible values of μ and λ. If u is a classical solution to problem (1), (2) at some (μ, λ), then the equation

$$-\Delta u + (\mu - \lambda\varepsilon_\infty)u = 0, \quad x \in \Omega_e \tag{3}$$

holds in the domain Ω_e. It has solutions exponentially decaying toward infinity only if $\lambda < \mu/\varepsilon_\infty$ (e.g., see

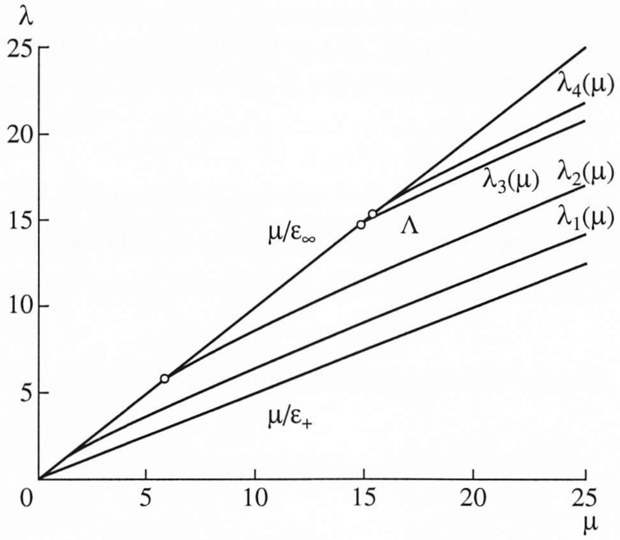

Fig. 1.

[20]). Furthermore, we multiply (1) by u and integrate the result over $x \in \mathbb{R}^2$ to obtain

$$\int_{\mathbb{R}^2} (|\nabla u|^2 + \mu u^2)dx = \lambda \int_{\mathbb{R}^2} \varepsilon u^2 dx < \lambda \varepsilon_+ \int_{\mathbb{R}^2} u^2 dx,$$

which implies that $\lambda > \mu/\varepsilon_+$. Therefore,

$$(\mu, \lambda) \in \Lambda, \quad \Lambda = \{(\mu, \lambda) : \mu/\varepsilon_+ < \lambda < \mu/\varepsilon_\infty, \mu > 0\}$$

is a necessary solvability condition for problem (1), (2). We assume that it is satisfied.

2. DEFINITIONS OF THE GENERALIZED SOLUTION

Let us examine the existence of the generalized solutions to problem (1), (2). We formulate two definitions of the generalized solution. One of them is conventional and is used to establish the properties of eigenfunctions. The other definition is a new one; it can be used to analyze the solvability of the problem by means of the theory of self-adjoint compact operators in the Hilbert space.

2.1. Generalized Solution in an Unbounded Domain

Consider the Sobolev space $H^1 = W_2^1(\mathbb{R}^2)$ with the norm

$$\|u\|^2_{1, \mathbb{R}^2} = \int_{\mathbb{R}^2} (|\nabla u|^2 + u^2)dx.$$

Consider the problem: to determine all pairs $(\mu, \lambda) \in \Lambda$ for which there exist functions $u \in H^1 \backslash \{0\}$ satisfying the identity

$$\int_{\mathbb{R}^2} (\nabla u \cdot \nabla v + \mu u v)dx = \lambda \int_{\mathbb{R}^2} \varepsilon u v dx \qquad \text{(P1)}$$

with any $v \in H^1$.

If (μ, λ, u) is a solution to problem (1), (2), then (P1) is satisfied. This is readily checked by multiplying (1) by an arbitrary $v \in H^1$ and integrating by parts. The converse is also true. Choosing a function v in (P1) with support in Ω_i (Ω_e), we find that the equation (1) is satisfied on Ω_i (Ω_e) in the sense of distributions and, since solutions to the boundary value problem are regular in the classical sense as well. Choosing an arbitrary v defined on γ, we obtain the second equality in (2). The equality $[u]_\gamma = 0$ is valid for any function in H^1. Furthermore, because u satisfies equation (3), it exponentially decays toward infinity, since $(\mu, \lambda) \in \Lambda$ (e.g., see [20]).

2.2 Generalized Solution in a Bounded Domain

Denote by Ω an open circle of radius R centered at the origin, such that $\Omega_i \subset \Omega$. Let Γ be the boundary of Ω and $\Omega_\infty = \mathbb{R}^2 \backslash \bar{\Omega}$. Define the spaces $V = W_2^1(\Omega)$, $V_\infty = W_2^1(\Omega_\infty)$, and $V_\infty^0 = \{v \in V_\infty : v|_\Gamma = 0\}$. Denote by $(\cdot, \cdot)$ the scalar product in V.

Let $u \in W_2^{1/2}(\Gamma)$. We call $u_\sigma \in V_\infty$ a metaharmonic continuation of u into Ω_∞ if $u_\sigma|_\Gamma = u$ and

$$\int_{\Omega_\infty} (\nabla u_\sigma \cdot \nabla v + \sigma^2 u_\sigma v)dx = 0 \quad \forall v \in V_\infty^0.$$

It is clear that such a continuation exists and is unique for $\sigma > 0$.

Define the operator $S_\Gamma(\sigma) : V \longrightarrow V$ as

$$(S_\Gamma(\sigma)u, v) = \int_{\Omega_\infty} (\nabla u_\sigma \cdot \nabla v_\sigma + \sigma^2 u_\sigma v_\sigma)dx, \quad \sigma > 0, \tag{4}$$

where u and v are arbitrary functions in V and u_σ and v_σ are, respectively, the metaharmonic continuations of the traces of u and v on Γ into the domain Ω_∞.

Rewrite (P1) as

$$\int_{\Omega} (\nabla u \cdot \nabla v + \mu u v)dx + \int_{\Omega_\infty} (\nabla u \cdot \nabla v + \sigma^2 u v)dx = \lambda \int_{\Omega} \varepsilon u v dx \quad \forall v \in H^1, \tag{5}$$

where $\sigma = \sqrt{\mu - \lambda \varepsilon_\infty}$. Note here that if $v = 0$ on Ω in equality (5), then

$$\int_{\Omega_\infty} (\nabla u \cdot \nabla v + \sigma^2 u v)dx = 0 \quad \forall v \in V_\infty^0;$$

i.e., $u = u_\sigma$ on Ω_∞. Restricting (5) to such $v \in H^1$ that $v = v_\sigma$ in Ω_∞ and taking into account the fact that any function in V admits a metaharmonic continuation of its trace, we find that the solution to problem (P1), when it exists, satisfies the identity

$$\int_{\Omega} (\nabla u \cdot \nabla v + \mu u v)dx + (S_\Gamma(\sigma)u, v) = \lambda \int_{\Omega} \varepsilon u v dx \quad \forall v \in V. \tag{6}$$

Obviously, the converse is also true. Indeed, suppose that (μ, λ, u) satisfies the equality (6). Consider the metaharmonic continuation of u from the domain Ω into the domain Ω_∞. The resulting function defined on $\mathbb{R}^2$ is also denoted by u. It is clear that $u \in H^1$. Using the definition of the form $(S_\Gamma(\sigma)u, v)$, and noting that the definition of metaharmonic continuation implies that

$$\int_{\Omega_\infty} (\nabla u_\sigma \cdot \nabla v_\sigma + \sigma^2 u_\sigma v_\sigma)dx = \int_{\Omega_\infty} (\nabla u_\sigma \cdot \nabla v + \sigma^2 u_\sigma v)dx \quad \forall v \in H^1,$$

we deduce (5) from (6). The equivalence of problems (P1) and (6) is construed here in this particular sense.

2.3. Explicit Form of the Operator $S_\Gamma(\sigma)$

We chose a circular domain Ω to ensure that an explicit representation for $S_\Gamma(\sigma)$ can be obtained. The operator $S_\Gamma(\sigma)$ is symmetric, is defined on the entire space V, and therefore is continuous (e.g., see [21, p. 29]). Since the set of infinitely differentiable functions is dense in V, it suffices to demonstrate the explicit form of $S_\Gamma(\sigma)$ for smooth functions. Integrating (4) by parts, we see that

$$-\Delta u_\sigma + \sigma^2 u_\sigma = 0, \quad x \in \Omega_\infty, \tag{7}$$

$$u_\sigma = u(R, \varphi), \quad x \in \Gamma, \tag{8}$$

and

$$(S(\sigma)u, v) = -\int_\Gamma \frac{\partial u_\sigma}{\partial r} v dx. \tag{9}$$

Here, (r, φ) are the polar coordinates of a point x. The solution to problem (7), (8) is readily determined by separation of variables. It has the form

$$u_\sigma(r, \varphi) = \sum_{n=0}^{\infty}{}' \frac{K_n(\sigma r)}{K_n(\sigma R)}[a_n(u)\cos(n\varphi) + b_n(u)\sin(n\varphi)],$$

where $n = 0, 1, \ldots$;

$$a_n(u) = \frac{1}{\pi}\int_0^{2\pi} u(R, \varphi)\cos(n\varphi)d\varphi, \quad b_n(u) = \frac{1}{\pi}\int_0^{2\pi} u(R, \varphi)\sin(n\varphi)d\varphi,$$

$K_n(z)$ is the modified Bessel function of nth order (e.g., see [22]), the prime over the sum means that the zeroth term is multiplied by 1/2. Consequently,

$$\left.\frac{\partial u_\sigma}{\partial r}\right|_{r=R} = -\frac{1}{R}\sum_{n=0}^{\infty}{}' H_n(R\sigma)[a_n(u)\cos(n\varphi) + b_n(u)\sin(n\varphi)],$$
$$H_n(z) = -zK_n'(z)/K_n(z). \tag{10}$$

It is well known that the modified Bessel functions $K_n(z)$ are positive at real $z > 0$ and $n \geq 0$ and

$$K_n'(z) = -K_{n-1}(z) - \frac{n}{z}K_n(z), \quad n \geq 1, \quad K_0'(z) = -K_1(z).$$

Therefore, at $z > 0$,

$$H_n(z) = n + z\frac{K_{n-1}(z)}{K_n(z)} > 0, \quad n \geq 1, \quad H_0(z) = z\frac{K_1(z)}{K_0(z)} > 0.$$

Substituting (10) into (9) we obtain the desired formula:

$$(S_\Gamma(\sigma)u, v) = \sum_{n=0}^{\infty}{}' H_n(R\sigma)[a_n(u)a_n(v) + b_n(u)b_n(v)].$$

2.4. Formulation of the Problem in Terms of Operators

Define operators A_0, $S(\sigma)$, B_0, and $B : V \longrightarrow V$ for any $(\mu, \lambda) \in \Lambda$ by the equations

$$(A_0 u, v) = \int_\Omega \nabla u \cdot \nabla v dx + \sum_{n=1}^{\infty}{}' n[a_n(u)a_n(v) + b_n(u)b_n(v)],$$

$$(S(\sigma)u, v) = 0.5R\sigma\frac{K_1(R\sigma)}{K_0(R\sigma)}[a_0(u)a_0(v) + b_0(u)b_0(v)] + \sum_{n=1}^{\infty} R\sigma\frac{K_{n-1}(R\sigma)}{K_n(R\sigma)}[a_n(u)a_n(v) + b_n(u)b_n(v)],$$

$$(B_0u, v) = \int_\Omega uv\,dx, \quad (Bu, v) = \int_\Omega \varepsilon uv\,dx,$$

where u and v are arbitrary functions in V and $\sigma = \sqrt{\mu - \lambda\varepsilon_\infty}$. Let $A_\mu(\lambda) = A_0 + \mu B_0 + S(\sigma)$.

Problem (6) can now be formulated as follows: find all $(\mu, \lambda) \in \Lambda$ such that there exist nontrivial solutions to the problem

$$u \in V : A_\mu(\lambda)u = \lambda Bu. \tag{P2}$$

At a fixed μ, problem (P2) is a symmetric eigenvalue problem of the form $A(\lambda)u = \lambda Bu$, which is nonlinear in the spectral parameter λ. It is convenient to consider first an abstract problem of this kind.

3. EXISTENCE OF SOLUTIONS TO THE PROBLEM $A(\lambda)u = \lambda Bu$

Let H be a Hilbert space with a scalar product $(\cdot, \cdot)$ and a norm $\|\cdot\|$, $H_0 = H\backslash\{0\}$, and I be the identity operator in H. The operator inequalities $A \ge B$ and $A > B$ are interpreted here as usually done, as $(Au, u) \ge (Bu, u)$ and $(Au, u) > (Bu, u)$ for any $u \in H_0$, respectively.

Let $\Delta = [\lambda_-, \lambda_+]$, where $\lambda_- \ge 0$. Consider the problem of finding such $(\lambda, u) \in \Delta \times H_0$ that

$$A(\lambda)u = \lambda Bu, \tag{11}$$

where $A(\lambda)$ and B are linear self-adjoint operators in H. Consider the Rayleigh ratio

$$R(\lambda, u) = \frac{(A(\lambda)u, u)}{(Bu, u)}, \quad \lambda \in \Delta, \quad u \in H_0.$$

Assume the following:

1. B is a compact operator, $B > 0$, and $0 \le A(\lambda) \le mI$, where $m > 0$ and is independent of λ;
2. The operator-valued function $A(\lambda)$ is continuous on Δ;
3. The functional $R(\lambda, \cdot)$ is a decreasing function of λ;
4. The eigenvalue problem

$$(\gamma, u) \in \Delta \times H_0 : \ A(\lambda_+)u = \gamma Bu \tag{12}$$

has n eigenvalues γ_i^+ (counted with their multiplicities) in the interval $(0, \lambda_+]$:

$$0 < \gamma_1^+ \le \ldots \le \gamma_n^+ \le \lambda_+;$$

5. $R(\lambda_-, u) > \lambda_-$ for any $u \in H_0$.

Theorem 1. *If conditions* 1–5 *are fulfilled, then there exist exactly n spectral pairs* (λ_i, u_i) *of problem* (11) *such that*

$$\lambda_- < \lambda_1 \le \ldots \le \lambda_n \le \lambda_+.$$

Moreover, $\lambda_n = \lambda_+$ if $\gamma_n^+ = \lambda_+$, *and* $\lambda_n < \lambda_+$ if $\gamma_n^+ < \lambda_+$. *Furthermore, if* $\lambda_{k-1} < \lambda_k < \lambda_{k+1}$, *then* u_k *is determined uniquely (up to a factor); if* $\lambda_{k-1} < \lambda_k = \lambda_{k+1} = \ldots = \lambda_{k+m-1} < \lambda_{k+m}$, *then* u_i, $i = k, \ldots, k+m-1$, *can be chosen as orthonormal vectors:* $(Bu_i, u_j) = \delta_{ij}$, $i, j = k, \ldots, k+m-1$. *Each* $\lambda = \lambda_k$, $k \ge 1$ *is the unique root of the equation*

$$\lambda = \min_{H_k \subset H} \max_{u \in H_k} R(\lambda, u),$$

where H_k *is a k-dimensional subspace of H with zeroth element excluded.*

Proof. Take an arbitrary constant $\lambda \in \Delta$ and consider an eigenvalue problem

$$(\gamma, u) \in \mathbb{R} \times H_0 : \quad A(\lambda)u = \gamma Bu, \tag{13}$$

linear in the spectral parameter $\gamma = \gamma(\lambda)$. As known from the spectral theory of compact operators (e.g., see [23, p. 251]), this problem has a countable set of positive eigenvalues of finite multiplicity. Arrange them in ascending order:

$$0 < \gamma_1(\lambda) \le \ldots \le \gamma_k(\lambda) \le \ldots, \quad \lim_{k\to\infty} \gamma_k(\lambda) = \infty,$$

where each number appears as many times as dictated by the corresponding eigenvalue multiplicity. The corresponding eigenvectors $u_k(\lambda)$ ($k = 1, 2, \ldots$) make up a basis in H, and we can assume that

$$(Bu_k(\lambda), u_l(\lambda)) = \delta_{kl}, \quad k, l = 1, 2, \ldots.$$

By the Courant–Weyl principle,

$$\gamma_k(\lambda) = \min_{H_k \subset H} \max_{v \in H_k} R(\lambda, v).$$

Let $\varphi_k(\lambda) = \gamma_k(\lambda) - \lambda$. Then, only the roots λ_k of the equations

$$\varphi_k(\lambda) = 0, \quad k = 1, 2, \ldots, \quad \lambda \in \Delta \tag{14}$$

are the eigenvalues of the problem (11).

Let us demonstrate that each equation in (14) has a unique solution. This will entail the first assertation of the theorem. We use the following lemma to be proved below.

Lemma 1. *The functions* $\gamma_k(\lambda)$ ($k = 1, 2, \ldots$) *are continuous, positive, and decreasing on* Δ.

Lemma 1 implies that $\varphi_k(\lambda)$ are continuous and decreasing functions on Δ. Furthermore, by the Courant–Weyl principle and Conditions 4 and 5, they have opposite signs at the endpoints of Δ: $\varphi_k(\lambda_-) > 0$ and $\varphi_k(\lambda_+) \le 0$, if $k = 1, 2, \ldots, n$. This entails the unique solvability of each of the equations (14). Note that $\gamma_k(\lambda_+) - \lambda_+ > 0$ for $k > n$ by the conditions of the theorem. Therefore, equations $\gamma_k(\lambda) = \lambda$ do not have any solutions for $k > n$. The last assertation of the theorem is obviously true.

Proof of Lemma 1. By the Courant–Weyl principle and Condition 3, every $\gamma_k(\lambda)$ is a decreasing function on the segment Δ and is obviously positive. To prove its continuity, we note that

$$|R(\lambda, v) - R(\eta, v)| \le \|A(\lambda) - A(\eta)\| \frac{\|v\|^2}{(Bv, v)}.$$

Denote by $E_k(\eta)$ the subspace spanned by the eigenvectors corresponding to the eigenvalues $\gamma_1(\eta), \ldots, \gamma_k(\eta)$ of problem (13). Then,

$$\begin{aligned}\gamma_k(\lambda) &= \min_{H_k \subset H} \max_{v \in H_k} R(\lambda, v) \le \max_{v \in E_k(\eta)} R(\lambda, v)\\ &\le \max_{v \in E_k(\eta)} R(\eta, v) + \max_{v \in E_k(\eta)} |R(\lambda, v) - R(\eta, v)|\\ &\le \gamma_k(\eta) + \|A(\lambda) - A(\eta)\| \max_{v \in H_k} \frac{\|v\|^2}{(Bv, v)}.\end{aligned}$$

This entails the continuity of $\gamma_k(\lambda)$:

$$|\gamma_k(\lambda) - \gamma_k(\eta)| \le \|A(\lambda) - A(\eta)\| \max_{v \in H_k} \frac{\|v\|^2}{(Bv, v)} \longrightarrow 0, \quad \lambda \longrightarrow \eta.$$

4. DEFINITION OF THE OPERATOR $A_\mu(\lambda)$ FOR $(\mu, \lambda) \in \bar{\Lambda}$

The operator $A_\mu(\lambda)$ in problem (P2) was defined only for $(\mu, \lambda) \in \Lambda$, where Λ is an open set. To use the results obtained in the preceding section, we must extend it to the closed set $\bar{\Lambda}$. Because only the operator $S(\sigma)$ depends on λ, we study its properties in the first place. It is convenient for this study to normalize the space $W_2^{1/2}(\Gamma)$ in the following way:

$$\|u\|^2_{1/2,\Gamma} = \sum_{n=0}^{\infty}{}'(n+1)[a_n^2(u) + b_n^2(u)]. \tag{15}$$

It is known (see, for example, [24, p. 29]) that there exists such a constant $c_{1/2}$ independent of u that

$$\|u\|_{1/2,\Gamma} \le c_{1/2}\|u\|_{1,\Omega} \quad \forall u \in W_2^1(\Omega).$$

Lemma 2. *For any $\sigma > 0$, $S(\sigma)$ is a self-adjoint, nonnegative, and compact operator and $\|S(\sigma)\| \longrightarrow 0$ at $\sigma \longrightarrow 0$. The operator-valued function $S(\sigma)$ is continuously differentiable at $\sigma > 0$.*

Proof. The symmetry and nonnegativity of $S(\sigma)$ are obvious. Let us prove that it is compact. By the properties of the Bessel functions, the functions $K_{n-1}(z)/K_n(z)$ ($n \ge 1$) are continuous and monotonically increasing from zero to unity on $[0, \infty)$. Furthermore, the function $H_0(z) = zK_1(z)/K_0(z)$ is also continuous and monotonically increasing from zero on $[0, \infty)$. Define

$$m(\sigma) = \max\{H_0(R\sigma), R\sigma\}.$$

Then, by the Parseval equality,

$$(S(\sigma)u, u) \le m(\sigma)\sum_{n=0}^{\infty}{}'[a_n^2(u) + b_n^2(u)] = m(\sigma)\|u\|_{L_2(\Gamma)}^2.$$

Since the inclusion $W_2^1(\Omega) \subset L_2(\Gamma)$ is compact, and $\|u\|_{L_2(\Gamma)} \le c_\Gamma\|u\|_{1,\Omega}$ for any $u \in W_2^1(\Omega)$, the operator $S(\sigma)$ is compact and

$$\|S(\sigma)\| \le c_\Gamma^2 m(\sigma). \tag{16}$$

Note that function $m(\sigma)$ is continuous on $[0, \infty)$ and behaves as $-1/\ln(R\sigma)$ at zero. Therefore, $\|S(\sigma)\| \longrightarrow 0$ as $\sigma \longrightarrow 0$.

Let us check the differentiability of $S(\sigma)$. Simple calculations show that when $z > 0$ and $n \ge 1$, it holds that

$$0 < H_n'(z) = \frac{H_n^2(z) - z^2 - n^2}{z} \le 2n \le 2(n+1), \quad H_0'(z) > 0.$$

Taking $c(\sigma) = R\max\{2, H_0'(R\sigma)\}$, we have

$$0 < \frac{d}{d\sigma}(S(\sigma)u, u) = \sum_{n=0}^{\infty}{}' RH_n'(R\sigma)[a_n^2(u) + b_n^2(u)]$$

$$\le c(\sigma)\sum_{n=0}^{\infty}{}'(n+1)[a_n^2(u) + b_n^2(u)] = c(\sigma)\|u\|_{1/2,\Gamma}^2 \le c_{1/2}^2 c(\sigma)\|u\|_{1,\Omega}^2.$$

The function $c(\sigma)$ is continuous in σ at $\sigma > 0$ and has the singular form $-(\sigma\ln\sigma)^{-1}$ as $\sigma \longrightarrow 0$.

It follows from Lemma 2 that the operator-valued function $S(\sigma)$ can be continued to the positive semiaxis $[0, \infty)$ by setting $S(0) = 0$:

$$\|S(\sigma) - S(\eta)\| \longrightarrow 0 \text{ as } \sigma \longrightarrow \eta, \quad \sigma, \eta \in [0, \infty).$$

In what follows, the operators $S(\sigma)$ at $\sigma \in [0, \infty)$, and $A_\mu(\lambda)$ at $(\mu, \lambda) \in \bar{\Lambda}$ are defined by this continuation.

5. CUT-OFF EQUATION AND CUT-OFF POINTS

It is obvious that the main difficulty in using Theorem 1 to prove the existence of solutions to problem (P2) is the verification of Condition 4. Denote by Δ_μ the segment $[\lambda_-(\mu), \lambda_+(\mu)]$, where $\lambda_-(\mu) = \mu/\varepsilon_+$ and $\lambda_+(\mu) = \mu/\varepsilon_\infty$. In this case, the eigenvalue problem (12) takes the form

$$(\gamma, u) \in \Delta_\mu \times V: \ A_0 u + \mu B_0 u = \gamma B u. \tag{17}$$

Indeed, at a fixed $\mu > 0$ and $\lambda = \lambda_+(\mu)$, the parameter σ vanishes and $S(\sigma) = 0$. Problem (17) plays a key role in the proof of the existence of solutions to problem (P2).

For any $\mu \ge 0$, problem (17) has a countable set of eigenvalues of finite multiplicity. Arrange them in ascending order:

$$0 \le \gamma_1(\mu) \le \ldots \le \gamma_k(\mu) \le \ldots, \quad \gamma_k(\mu) \longrightarrow \infty, \quad k \longrightarrow \infty,$$

where each number appears as many times as dictated by the corresponding eigenvalue multiplicity. These eigenvalues are associated with eigenvectors $u_k(\mu)$ that make up a basis in V and are such that

$$(Bu_i(\mu), u_j(\mu)) = \delta_{ij}, \quad i, j \geq 1.$$

By the Courant–Weyl principle, for any $k \geq 1$, it holds that

$$\gamma_k(\mu) = \min_{V_k \subset V} \max_{u \in V_k} R_\mu(u), \quad R_\mu(u) = \frac{(A_0 u + \mu B_0 u, u)}{(Bu, u)}.$$

Note the following properties of functions $\gamma_k(\mu)$.

Lemma 3. *The functions $\gamma_k(\mu)$ $(k = 1, 2, \ldots)$ are increasing and continuous functions of $\mu \geq 0$ such that*

$$0 = \gamma_1(0) < \gamma_2(0), \quad \lambda_-(\mu) < \gamma_1(\mu) < \lambda_+(\mu) \quad \forall \mu > 0,$$

$$\lim_{\mu \to \infty} \frac{\gamma_k(\mu)}{\mu} = \frac{1}{\varepsilon_+}, \quad k = 1, 2, \ldots. \tag{18}$$

Proof. The continuity of $\gamma_k(\mu)$ is proved by analogy with Lemma 1. The monotonicity of the Rayleigh ratio with respect to μ entails the increase of $\gamma_k(\mu)$. The eigenvalue $\gamma_1(0)$ is of multiplicity one because the kernel of the operator A_0 involves only functions that are constant on Ω. Next, the representation

$$\gamma_1(\mu) = \min_{u \in V} \left\{ \left[(A_0 u, u) + \mu \int_\Omega (1 - \varepsilon/\varepsilon_+) u^2 dx \right] (Bu, u)^{-1} \right\} + \frac{\mu}{\varepsilon_+}$$

entails the lower estimate $\gamma_1(\mu) > \lambda_-(\mu)$. The upper estimate for $\gamma_1(\mu)$ follows from the inequalities

$$\gamma_1(\mu) = \min_{u \in V} R_\mu(u) \leq R_\mu(1) = \mu |\Omega| \left(\int_\Omega \varepsilon dx \right)^{-1} < \lambda_+(\mu).$$

To prove validity of the equalities (18), note that, when $k \geq 1$,

$$\frac{\gamma_k(\mu)}{\mu} - \frac{1}{\varepsilon_+} \geq \frac{\gamma_1(\mu)}{\mu} - \frac{1}{\varepsilon_+} > 0.$$

Let

$$\varepsilon_+ = \lim_{x \in \Omega_i} \varepsilon(x) = \varepsilon(x_0), \quad \varepsilon_- = \min_{x \in \Omega_i} \varepsilon(x),$$

Ω_ρ be a circle of radius ρ centered at the point x_0, $\Omega_\rho \subset \Omega$,

$$\delta_\rho = \max_{x \in \Omega_\rho} \frac{\varepsilon_+ - \varepsilon(x)}{\varepsilon(x)},$$

V_ρ be the set of functions in V that vanish outside Ω_ρ. For $u \in V_\rho$, we have

$$R_\mu(u) = \left[(A_0 u, u) + \mu \int_{\Omega_\rho} (1 - \varepsilon/\varepsilon_+) u^2 dx \right] (Bu, u)^{-1} + \frac{\mu}{\varepsilon_+}$$

$$\leq \frac{(A_0 u, u)}{(Bu, u)} + \frac{\mu \delta_\rho}{\varepsilon_+} + \frac{\mu}{\varepsilon_+} \leq \left(\int_{\Omega_\rho} |\nabla u|^2 dx \right) \left(\varepsilon_- \int_{\Omega_\rho} u^2 dx \right)^{-1} + \frac{\mu \delta_\rho}{\varepsilon_+} + \frac{\mu}{\varepsilon_+}.$$

Denote by λ_k^ρ the eigenvalues of the Dirichlet problem on the circle Ω_ρ. Since $\lambda_k^\rho = \rho^{-2} \lambda_k^1$, the estimate above implies that

$$\gamma_k(\mu) = \min_{V_k \subset V} \max_{u \in V_k} R_\mu(u) \leq \min_{V_k \subset V_\rho} \max_{u \in V_k} R_\mu(u) \leq \frac{\lambda_k^1}{\rho^2 \varepsilon_-} + \frac{\mu \delta_\rho}{\varepsilon_+} + \frac{\mu}{\varepsilon_+}.$$

Therefore,

$$0 < \frac{\gamma_k(\mu)}{\mu} - \frac{1}{\varepsilon_+} \le \frac{\lambda_k^1}{\rho^2 \mu \varepsilon_-} + \frac{\delta_\rho}{\varepsilon_+}.$$

Here, the limit $\mu \longrightarrow \infty$ and $\rho \longrightarrow 0$ with $\rho^2\mu \longrightarrow \infty$ yields (18).

Lemma 4. *For every $k = 1, 2, \ldots$, the equation*

$$\gamma_k(\mu) = \lambda_+(\mu) \tag{19}$$

has a unique solution μ_k,

$$0 = \mu_1 < \mu_2 \le \mu_3 \le \ldots \le \mu_n \le \ldots, \quad \mu_n \longrightarrow \infty, \quad n \longrightarrow \infty.$$

The numbers μ_k are the eigenvalues of the problem

$$u \in V\backslash\{0\} : A_0 u = \mu C u, \quad C = \frac{1}{\varepsilon_\infty} B - B_0. \tag{20}$$

Proof. We use the previous lemma. Equation (19) at $k = 1$ has the unique solution $\mu_1 = 0$. For $k \ge 2$, suppose that

$$\nu_k(\mu) = \gamma_k(\mu)/\mu - 1/\varepsilon_\infty.$$

The function ν_k is continuous at $\mu > 0$,

$$\lim_{\mu \to 0} \nu_k(\mu) = \infty, \quad \lim_{\mu \to \infty} \nu_k(\mu) = \frac{1}{\varepsilon_+} - \frac{1}{\varepsilon_\infty} < 0.$$

Therefore, equation (19) has a solution $\mu_k > 0$ at $k \ge 2$.

Let us demonstrate that μ_k is uniquely determined. To this end, we rewrite equation (17) as

$$u \in V : A_0 u - \mu C u = \mu \nu_k(\mu) B u. \tag{21}$$

This implies that only the eigenvalues of problem (20) can be the roots of equation (19). Next, by the Courant–Weyl principle for problem (21), $\mu\nu_k(\mu)$ is a nonincreasing function (since $C \ge 0$). Therefore, ν_k is a decreasing function and has a unique root.

The qualitative behavior of $\gamma_k(\mu)$ is illustrated in Fig. 2. The eigenvalue problem (20) is called the cut-off equation, and the eigenvalues μ_k are called the cut-off points. For $\mu > 0$, we define the integer-valued

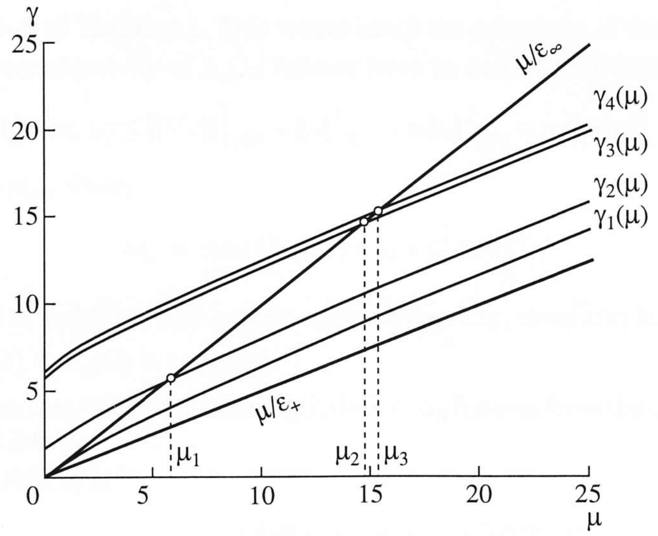

Fig. 2.

function

$$n(\mu) = \max\{k : \mu_k < \mu\}. \tag{22}$$

Lemmas 3 and 4 entail the following proposition

Theorem 2. *Let μ_k ($k = 1, 2, \ldots$) be the cut-off points and $n(\mu)$ be defined by* (22). *Then, for any $\mu > 0$, problem* (17) *has exactly $n(\mu)$ eigenvalues (with multiplicities taken into account) in the interval* $(\lambda_-(\mu), \lambda_+(\mu)]$,

$$\lambda_-(\mu) < \gamma_1(\mu) \le \ldots \le \gamma_{n(\mu)}(\mu) \le \lambda_+(\mu),$$

the corresponding eigenvectors are orthonormal:

$$(Bu_i(\mu), u_j(\mu)) = \delta_{ij}, \quad i, j = 1, 2, \ldots, n(\mu),$$

and $n(\mu) \ge 1$ for any $\mu > 0$, and $n(\mu) = 1$ for $\mu \in (0, \mu_2]$.

6. EXISTENCE OF SOLUTIONS AND THEIR PROPERTIES

Based on the above results, we are now prepared to formulate and prove the main results of this paper.

6.1. Existence Theorem

The following theorem describes the set of solutions to problem (P2).

Theorem 3. *Let μ_k ($k = 1, 2, \ldots$) be the cut-off points and function $n(\mu)$ be defined by the formula* (22). *Then, for any $\mu > 0$, problem* (P2) *has exactly $n(\mu)$ solutions* $(\mu, \lambda_i(\mu), u_i(\mu))$,

$$\lambda_-(\mu) < \lambda_1(\mu) \le \ldots \le \lambda_{n(\mu)}(\mu) < \lambda_+(\mu).$$

If $\lambda_{k-1}(\mu) < \lambda_k(\mu) < \lambda_{k+1}(\mu)$, then $u_k(\mu)$ are uniquely determined (up to a factor); otherwise, if $\lambda_{k-1}(\mu) < \lambda_k(\mu) = \lambda_{k+1}(\mu) = \ldots = \lambda_{k+m-1}(\mu) < \lambda_{k+m}(\mu)$, then vectors $u_i(\mu)$, $i = k, \ldots, k+m-1$ can be orthonormalized: $(Bu_i(\mu), u_j(\mu)) = \delta_{ij}$, $i, j = k, \ldots, k+m-1$. Each $\lambda = \lambda_k(\mu)$, $k \ge 1$ is the unique root of the equation

$$\lambda = \min_{H_k \subset V} \max_{v \in H_k} R_\mu(\lambda, u), \quad R_\mu(\lambda, u) = \frac{(A_0 u + \mu B_0 u + S(\sigma)u, u)}{(Bu, u)}.$$

Proof. At fixed $\mu > 0$, the problem of finding a nonzero function $u \in V$ and $\lambda \in \Delta_\mu$ satisfying (P2) is equivalent to the problem studied in Section 3 when

$$H = V, \quad \|\cdot\| = \|\cdot\|_{1,\Omega}, \quad A(\lambda) = A_\mu(\lambda), \quad \Delta = \Delta_\mu.$$

Let us check conditions 1–5 of Theorem 1. This would entail the assertions of the theorem being proved.

1. The symmetry and nonnegativity of $A_\mu(\lambda)$ follows from its definition. Furthermore (see (16)),

$$(A_\mu(\lambda)u, u) \le \||\nabla u|\|^2_{L_2(\Omega)} + \|u\|^2_{1/2,\Gamma} + \mu\|u\|^2_{L_2(\Omega)} + m(\sigma)\|u\|^2_{0,\Gamma}.$$

This implies that $A_\mu(\lambda) \le m_\mu I$, where

$$m_\mu = \max_{\lambda \in \Delta_\mu} [1 + \mu + c^2_{1/2} + c^2_\Gamma m(\sigma)].$$

It is also obvious that $B > 0$, and B is a self-adjoint compact operator, since $\varepsilon(x)$ is a bounded function on Ω and the embedding $W^1_2(\Omega) \subset L_2(\Omega)$ is compact.

2. Continuity of the operator-valued function $A_\mu(\lambda)$ in $\lambda \in \Delta_\mu$ follows from the continuity of the operator-valued function $S(\sigma)$ at $\sigma \ge 0$.

3. The Rayleigh ratio $R(\lambda, u)$ is

$$R(\lambda, u) = R_\mu(\lambda, u) = \frac{(A_0 u + \mu B_0 u, u)}{(Bu, u)} + \frac{(S(\sigma)u, u)}{(Bu, u)}, \quad u \ne 0.$$

Here, only the second term depends on λ. Because functions

$$zK_{n-1}(z)/K_n(z), \quad n \geq 1, \quad H_0(z), \quad z \in [0, \infty)$$

are monotonically increasing, the product $(S(\sigma)u, u)$ monotonically increases with σ and is a monotonically decreasing function of λ at $\lambda \in \Delta_\mu$. Thus, $R(\lambda, u)$ decreases as a function of λ at any fixed u.

4. The number $n(\mu)$ of the eigenvalues $\gamma_k(\mu)$ of problem (17) is determined by relation (22) (see Theorem 2).

5. The inequality

$$R_\mu(\lambda_-(\mu), u) > \lambda_-(\mu) \quad \forall u \in H\backslash\{0\}$$

is valid by Lemma 3 and nonnegativity of the operator S. Moreover, for any $\lambda \in \Delta_\mu$, we have

$$R_\mu(\lambda, u) \geq \frac{(A_0 u + \mu B_0 u, u)}{(Bu, u)} \geq \gamma_1(\mu) > \frac{\mu}{\varepsilon_+} = \lambda_-(\mu).$$

Thus, all conditions of Theorem 1 are fulfilled. By Theorem 1, the problem (P2) has exactly $n(\mu)$ solutions $(\mu, \lambda_i(\mu), u_i(\mu))$ for any $\mu > 0$:

$$\lambda_-(\mu) < \lambda_1(\mu) \leq \ldots \leq \lambda_{n(\mu)}(\mu) < \lambda_+(\mu).$$

Note that $\lambda_k(\mu)$ is defined for any $\mu > \mu_k$.

6.2. *Properties of Dispersion Curves*

By Theorem 3, the set of (μ, λ) for which problem (P2) has nontrivial solutions consists of the dispersion curves $\lambda = \lambda_k(\mu)$, where $\mu \in (\mu_k, \infty)$. Let us describe some of their properties. For $k \geq 1$, define

$$f_k(\mu, \lambda) = \min_{H_k \subset V} \max_{u \in H_k} R_\mu(\lambda, u).$$

It can readily be shown that $f_k(\mu, \lambda)$ $(k \geq 1)$ is increasing in μ, decreasing in λ, and Lipschitzian with respect to its arguments. To prove the last property, it suffices to follow the proof of Lemma 1 and to take into account the differentiability of the operator-valued function $S(\sigma)$ at $\sigma > 0$.

Lemma 5. *For every $k \geq 1$ the dispersion curve $\lambda_k = \lambda_k(\mu)$ defined on (μ_k, ∞) is a Lipschitzian increasing function and*

$$\lim_{\mu \to \infty} \frac{\lambda_k(\mu)}{\mu} = \frac{1}{\varepsilon_+}. \tag{23}$$

Proof. The function $f_k(\mu, \lambda)$ $(k \geq 1)$ increases with μ. At any $\mu \in (\mu_k, \infty)$, the value of λ_k is a solution to the equation $\lambda = f_k(\mu, \lambda)$. Consequently, $\lambda_k = \lambda_k(\mu)$ $(k \geq 1)$ is an increasing function of μ. Furthermore, let $\hat{\mu} \geq \mu$. Then,

$$0 < \lambda_k(\hat{\mu}) - \lambda_k(\mu) = [f_k(\hat{\mu}, \lambda_k(\hat{\mu})) - f_k(\hat{\mu}, \lambda_k(\mu))]$$
$$+ [f_k(\hat{\mu}, \lambda_k(\mu)) - f_k(\mu, \lambda_k(\mu))] \leq f_k(\hat{\mu}, \lambda_k(\mu)) - f_k(\mu, \lambda_k(\mu)) \leq c(\hat{\mu} - \mu)$$

and, consequently, $\mu \longrightarrow \lambda_k(\mu)$ $(k \geq 1)$ is a Lipschitzian function. Equation (23) is proved by analogy to the corresponding statement of Lemma 3.

6.3. *Properties of Eigenfunctions*

The analysis above provides insufficient information about the eigenfunctions of problem (P1). More information is readily obtained by using Theorem 3 and the equivalence of problems (P2) and (P1).

Let $(\lambda_k(\mu), u_k(\mu))$ be a solution to problem (P2) at fixed μ. Consider the metaharmonic continuation of the trace of function $u_k(\mu)$ defined on the contour Γ into Ω_∞. The resulting function on $\mathbb{R}^2$ is denoted by $u^k(\mu)$, $u^k(\mu) \in H^1$. For any $v \in H^1$, we have

$$\int_{\mathbb{R}^2} (\nabla u^k(\mu) \cdot \nabla v + \mu u^k(\mu) v) dx = \lambda_k(\mu) \int_{\mathbb{R}^2} \varepsilon u^k(\mu) v dx.$$

Based on Theorem 3 and on the uniqueness of the metaharmonic continuation, the following proposition is proved by a standard reasoning.

Theorem 4. *The eigenfunctions $u^k(\mu)$ corresponding to different eigenvalues $\lambda_k(\mu)$ are orthogonal:*

$$\int_{\mathbb{R}^2} \varepsilon u^k(\mu) u^m(\mu) dx = 0, \quad if \quad \lambda_k(\mu) \neq \lambda_m(\mu).$$

If $\lambda_{k-1}(\mu) < \lambda_k(\mu) < \lambda_{k+1}(\mu)$, then $u^k(\mu)$ is determined uniquely (up to a normalization); otherwise, if $\lambda_{k-1}(\mu) < \lambda_k(\mu) = \lambda_{k+1}(\mu) = \ldots = \lambda_{k+m-1}(\mu) < \lambda_{k+m}(\mu)$, then there exists an orthonormal set of functions $u^i(\mu)$, $i = k, \ldots, k+m-1$.

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