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Paranormal Elements in Normed Algebra

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Abstract—For a normed algebra $\mathcal{A}$ and natural numbers k we introduce and investigate the $\|\cdot\|$-closed classes $\mathcal{P}_k(\mathcal{A})$. We show that $\mathcal{P}_1(\mathcal{A})$ is a subset of $\mathcal{P}_k(\mathcal{A})$ for all k. If T in $\mathcal{P}_1(\mathcal{A})$, then T^n lies in $\mathcal{P}_1(\mathcal{A})$ for all natural n. If $\mathcal{A}$ is unital, $U, V \in \mathcal{A}$ are such that $\|U\| = \|V\| = 1$, $VU = I$ and T lies in $\mathcal{P}_k(\mathcal{A})$, then UTV lies in $\mathcal{P}_k(\mathcal{A})$ for all natural k. Let $\mathcal{A}$ be unital, then 1) if an element T in $\mathcal{P}_1(\mathcal{A})$ is right invertible, then any right inverse element T^{-1} lies in $\mathcal{P}_1(\mathcal{A})$; 2) for $\|I\| = 1$ the class $\mathcal{P}_1(\mathcal{A})$ consists of normaloid elements; 3) if the spectrum of an element T, $T \in \mathcal{P}_1(\mathcal{A})$ lies on the unit circle, then $\|TX\| = \|X\|$ for all $X \in \mathcal{A}$. If $\mathcal{A} = \mathcal{B}(\mathcal{H})$, then the class $\mathcal{P}_1(\mathcal{A})$ coincides with the set of all paranormal operators on a Hilbert space $\mathcal{H}$.

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Introduction. Investigation of different subsets in normed algebras and in $*$-algebras of operators is an actual problem of functional analysis (see, e.g., [1–5] for classes of hyponormal, normal, idempotent, unitary operators and differences of idempotents, respectively). In this paper for a normed algebra $\mathcal{A}$ and $k \in \mathbb{N}$ we introduce and investigate $\|\cdot\|$-closed classes

$$\mathcal{P}_k(\mathcal{A}) = \{T \in \mathcal{A} : \|T^{k+1}A\| \geq \|TA\|^{k+1} \text{ for all } A \in \mathcal{A} \text{ with } \|A\| = 1\}.$$

It is shown that $\mathcal{P}_1(\mathcal{A}) \subset \mathcal{P}_k(\mathcal{A})$ for all $k \in \mathbb{N}$ (Theorem 2). If $\mathcal{A}$ is a dense subalgebra of normed algebra $\mathcal{B}$, then $\mathcal{P}_k(\mathcal{A}) \subset \mathcal{P}_k(\mathcal{B})$ for all $k \in \mathbb{N}$ (Proposition 1). If $T \in \mathcal{P}_1(\mathcal{A})$, then $T^n \in \mathcal{P}_1(\mathcal{A})$ for all $n \in \mathbb{N}$ (Theorem 5). If $\mathcal{A}$ is unital, $U, V \in \mathcal{A}$ are such that $\|U\| = \|V\| = 1$, $VU = I$ and $T \in \mathcal{P}_k(\mathcal{A})$, then $UTV \in \mathcal{P}_k(\mathcal{A})$ for all $k \in \mathbb{N}$ (Theorem 3). In particular, if $\mathcal{A}$ is a unital C^*-algebra and $T \in \mathcal{P}_k(\mathcal{A})$, then $UTU^* \in \mathcal{P}_k(\mathcal{A})$ for all isometries $U \in \mathcal{A}$ and $k \in \mathbb{N}$ (Corollary 3). If $\mathcal{A}$ is commutative and $\|T^2\| = \|T\|^2$ for all $T \in \mathcal{A}$, then $\mathcal{P}_1(\mathcal{A}) = \mathcal{A}$ (Proposition 6).

Let $\mathcal{A}$ be unital, then 1) if an element $T \in \mathcal{P}_1(\mathcal{A})$ is right invertible, then any right inverse element T^{-1} lies in $\mathcal{P}_1(\mathcal{A})$ (Theorem 4); 2) for $\|I\| = 1$ the class $\mathcal{P}_1(\mathcal{A})$ consists of normaloid elements (Corollary 1); 3) if the spectrum of an element T, $T \in \mathcal{P}_1(\mathcal{A})$ lies on the unit circle, then $\|TX\| = \|X\|$ for all $X \in \mathcal{A}$ (Corollary 4). If $\mathcal{A} = \mathcal{B}(\mathcal{H})$, then the class $\mathcal{P}_1(\mathcal{A})$ coincides with the set of all paranormal operators on a Hilbert space $\mathcal{H}$ (Corollary 6).

1. Notations and definitions. An *algebra* is a vector space $\mathcal{A}$ over the field Λ ($= \mathbb{R}$ or $\mathbb{C}$), equipped with a bilinear product such that

$$X(YZ) = (XY)Z, \quad (Y + Z)X = YX + ZX,$$

$$X(Y + Z) = XY + XZ, \quad \lambda(XY) = (\lambda X)Y = X(\lambda Y)$$

for all $X, Y, Z \in \mathcal{A}$ and $\lambda \in \Lambda$. An algebra $\mathcal{A}$ is *unital* (i.e., possesses the unity), if there exists an element $(0 \neq) I \in \mathcal{A}$ such that $IX = XI = X$ $(X \in \mathcal{A})$. An element X of algebra $\mathcal{A}$ with I is said to be *right invertible*, if there exists an element $X^{-1} \in \mathcal{A}$ such that $XX^{-1} = I$. An algebra $\mathcal{A}$ is said to

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be *normed*, if $\mathcal{A}$ admits a norm $\|\cdot\|$ such that $\|XY\| \leq \|X\|\|Y\|$ for all $X, Y \in \mathcal{A}$. Every subalgebra in $\mathcal{A}$, equipped with the induced norm, is a normed algebra. Recall that $T \in \mathcal{A}$ is *quasinilpotent*, if $\|T^n\|^{\frac{1}{n}} \to 0$ as $n \to \infty$; *normaloid*, if $\|T^n\| = \|T\|^n$ for all $n \in \mathbb{N}$. Let $\mathcal{A}$ be a normed unital algebra, then $\mathcal{A}$ admits a norm (equivalent to the initial norm) $\|\cdot\|_1$ such that $\|I\|_1 = 1$ (for example, consider an operator $\pi(X)(Y) = XY$ $(Y \in \mathcal{A})$ for every $X \in \mathcal{A}$ and put $\|X\|_1 = \|\pi(X)\|$). If $\mathcal{A}_1, \ldots, \mathcal{A}_n$ are normed algebras, then the algebra $\mathcal{A}_1 \times \cdots \times \mathcal{A}_n$, endowed with the norm

$$\|(X_i)_{i=1}^n\| = \max_{1 \leq i \leq n} \|X_i\|,$$

is a normed algebra ([6], Chap. I, § 2).

Let $\mathcal{B}(\mathcal{H})$ be the $*$-algebra of all linear bounded operators on a Hilbert space $\mathcal{H}$. An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be *paranormal*, if $\|T^2x\|_{\mathcal{H}} \geq \|Tx\|^2_{\mathcal{H}}$ for all $x \in \mathcal{H}$ with $\|x\|_{\mathcal{H}} = 1$ ([7–9]); *isometric*, if $T^*T = I$; *hyponormal*, if $T^*T \geq TT^*$. A C^*-algebra is a complex Banach $*$-algebra $\mathcal{A}$ such that $\|X^*X\| = \|X\|^2$ for any $X \in \mathcal{A}$. By Gel'fand–Naimark theorem every C^*-algebra can be realized as a C^*-subalgebra of $\mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$.

2. Main results. Let $\mathcal{A}$ be a normed algebra over a field Λ, $\mathcal{A}_1 = \{X \in \mathcal{A} : \|X\| = 1\}$ and $k \in \mathbb{N}$. We introduce the class

$$\mathcal{P}_k(\mathcal{A}) = \{T \in \mathcal{A} : \|T^{k+1}A\| \geq \|TA\|^{k+1} \text{ for all } A \in \mathcal{A}_1\}.$$

Obviously, $0 \in \mathcal{P}_k(\mathcal{A})$ and $T \in \mathcal{P}_k(\mathcal{A}) \Leftrightarrow \lambda T \in \mathcal{P}_k(\mathcal{A})$ for all $\lambda \in \Lambda \setminus \{0\}$ and $k \in \mathbb{N}$.

Theorem 1. *The class $\mathcal{P}_k(\mathcal{A})$ is $\|\cdot\|$-closed in $\mathcal{A}$.*

Proof. Let $\{T_n\}_{n=1}^\infty \subset \mathcal{P}_k(\mathcal{A})$ and $T_n \xrightarrow{\|\cdot\|} T \in \mathcal{A}$ as $n \to \infty$. Then by $\|\cdot\|$-continuity of the product operation in $\mathcal{A} \times \mathcal{A}$ we obtain $T_n^{k+1} \xrightarrow{\|\cdot\|} T^{k+1}$ and for any $A \in \mathcal{A}_1$ we have $T_nA \xrightarrow{\|\cdot\|} TA$, $T_n^{k+1}A \xrightarrow{\|\cdot\|} T^{k+1}A$ as $n \to \infty$. Continuity of the functional $\|\cdot\|$ implies that

$$\|T_nA\| \to \|TA\|, \quad \|T_n^{k+1}A\| \to \|T^{k+1}A\| \text{ as } n \to \infty$$

for every $A \in \mathcal{A}_1$. □

Proposition 1. *Let $\mathcal{A}$ be a dense subalgebra of a normed algebra $\mathcal{B}$. Then $\mathcal{P}_k(\mathcal{A}) \subset \mathcal{P}_k(\mathcal{B})$ for all $k \in \mathbb{N}$.*

Proof. Consider $T \in \mathcal{P}_1(\mathcal{A})$ and $A \in \mathcal{B}_1$. There exists a sequence $\{A_n\}_{n=1}^\infty \subset \mathcal{A} \setminus \{0\}$ such that $A_n \xrightarrow{\|\cdot\|} A$ as $n \to \infty$. Then $a_n = \|A_n\| \to 1$ as $n \to \infty$, hence by the triangle inequality we have

$$\|a_n^{-1}A_n - A\| \leq \|a_n^{-1}A_n - A_n\| + \|A_n - A\| = (a_n^{-1} - 1)a_n + \|A_n - A\| \to 0$$

as $n \to \infty$. Note that $a_n^{-1}A_n \in \mathcal{A}_1$ for all $n \in \mathbb{N}$. Now the inequality $\|T^{k+1}A\| \geq \|TA\|^{k+1}$ follows by $\|\cdot\|$-continuity of the product operation in $\mathcal{B} \times \mathcal{B}$ and via continuity of the functional $\|\cdot\|$ on $\mathcal{B}$. □

Proposition 2. *Let $\mathcal{A}, \ldots, \mathcal{A}_n$ be normed algebras, then $\mathcal{P}_k(\mathcal{A}_1) \times \cdots \times \mathcal{P}_k(\mathcal{A}_n) \subset \mathcal{P}_k(\mathcal{A}_1 \times \cdots \times \mathcal{A}_n)$ for all $k \in \mathbb{N}$.*

Proof. Let $k \in \mathbb{N}$, $T_i \in \mathcal{P}_k(\mathcal{A}_i)$ and $(0 \neq) A_i \in \mathcal{A}_i$ for all $1 \leq i \leq n$, $\max_{1 \leq i \leq n} \|A_i\| = 1$. For all $1 \leq i \leq n$ we have

$$\left\| T_i^{k+1} \frac{A_i}{\|A_i\|} \right\| \geq \left\| T_i \frac{A_i}{\|A_i\|} \right\|^{k+1},$$

hence $\|T_i^{k+1}A_i\| \geq \|T_iA_i\|^{k+1}\|A_i\|^{-k} \geq \|T_iA_i\|^{k+1}$. Thus

$$\max_{1 \leq i \leq n} \|T_i^{k+1}A_i\| \geq \max_{1 \leq i \leq n} \|T_iA_i\|^{k+1} = (\max_{1 \leq i \leq n} \|T_iA_i\|)^{k+1}$$

and the proposition is proved. □

Theorem 2. *We have $\mathcal{P}_1(\mathcal{A}) \subset \mathcal{P}_k(\mathcal{A})$ for all $k \in \mathbb{N}$.*

Proof is by induction. For $k = 1$ the assertion is evident. Let it hold for $k-1$, then for every $A \in \mathcal{A}_1$ we have

$$\|T^{k+1}A\| = \|TA\| \cdot \left\|T^k \frac{TA}{\|TA\|}\right\| \geq \|TA\| \cdot \left\|T \frac{TA}{\|TA\|}\right\|^k = \frac{\|T^2A\|^k}{\|TA\|^{k-1}} \geq \frac{\|TA\|^{2k}}{\|TA\|^{k-1}} = \|TA\|^{k+1}. \quad \square$$

Corollary 1. Let $\mathcal{A}$ be a normed unital algebra and $\|I\| = 1$. If $T \in \mathcal{P}_1(\mathcal{A})$, then T is normaloid.

Proof. We have $\|T^n\| = \|T^nI\| \geq \|TI\|^n = \|T\|^n$ for all $n \in \mathbb{N}$. $\square$

From here we obtain

Corollary 2. Let $\mathcal{A}$ be a normed unital algebra and $\|I\| = 1$. If $(0 \neq)T \in \mathcal{P}_1(\mathcal{A})$, then T cannot be quasinilpotent.

Proposition 3. *Let $\mathcal{A}$ be a normed unital algebra.*

(i) *If $T \in \mathcal{A}$ is such that $\|TX\| = \|X\|$ for all $X \in \mathcal{A}$, then $T \in \mathcal{P}_1(\mathcal{A})$.*

(ii) *If $T \in \mathcal{P}_{k-1}(\mathcal{A})$ and $T^k \in \mathcal{P}_{n-1}(\mathcal{A})$, then $T \in \mathcal{P}_{kn-1}(\mathcal{A})$ for all $k, n \geq 2$.*

Proof. (i) For any $A \in \mathcal{A}_1$ we have $1 = \|A\| = \|A\|^2$, hence $1 = \|TA\| = \|T(TA)\| = \|TA\|^2$.

(ii) For any $A \in \mathcal{A}_1$ we have $\|T^{kn}A\| = \|(T^k)^nA\| \geq \|T^kA\|^n \geq \|TA\|^{kn}$. $\square$

Proposition 4. *Let $\mathcal{A}$ be a normed algebra, $X \in \mathcal{A}_1$ and $T \in \mathcal{A}$ be such that $XTX = T$. If $k \in \mathbb{N}$ is odd and $T \in \mathcal{P}_k(\mathcal{A})$, then $XT \in \mathcal{P}_k(\mathcal{A})$.*

Proof. Obviously, $(XT)^{k+1} = (XTX \cdot T)^{\frac{k+1}{2}} = T^{k+1}$ and

$$\|(XT)^{k+1}A\| = \|T^{k+1}A\| \geq \|TA\|^{k+1} \geq \|XTA\|^{k+1}$$

for all $A \in \mathcal{A}_1$. $\square$

Proposition 5. *Let a normed algebra $\mathcal{A}$ be unital. Then $\lambda I \in \mathcal{P}_1(\mathcal{A})$ for all $\lambda \in \Lambda$ and the following assertions hold true:*

(i) *if $T \in \mathcal{A}_1$ is so that $T^{k+1} = I$, then $T \in \mathcal{P}_k(\mathcal{A})$,*

(ii) *if $T = T^{k+1} \in \mathcal{P}_k(\mathcal{A})$ and $\|I\| = 1$, then $\|T\| \in \{0, 1\}$.*

Proof. (i) We have $1 = \|T^{k+1}A\| = \|A\| \geq \|TA\| \geq \|TA\|^{k+1}$ for all $A \in \mathcal{A}_1$ and $k \in \mathbb{N}$.

(ii) For all $T = T^{k+1} \in \mathcal{A}$ we have $\|T\| = \|T^{k+1}\| \leq \|T\|^{k+1}$. So, $\|T\| \in \{0\} \cup [1, \infty)$. If $T \in \mathcal{P}_k(\mathcal{A})$, then $\|TA\| = \|T^{k+1}A\| \geq \|TA\|^{k+1}$, hence $\|TA\| \in [0, 1]$ for all $A \in \mathcal{A}_1$. In particular, $\|T\| \leq 1$ for $A = I$. Therefore $\|T\| \in \{0, 1\}$. $\square$

Proposition 6. *If $\mathcal{A}$ is commutative (i.e., $XY = YX$ for all $X, Y \in \mathcal{A}$) normed algebra and $\|T^2\| = \|T\|^2$ for all $T \in \mathcal{A}$, then $\mathcal{P}_k(\mathcal{A}) = \mathcal{A}$ for all $k \in \mathbb{N}$.*

Proof. By Theorem 2 it suffices to check the assertion for $k = 1$. For all $T \in \mathcal{A}$ and $A \in \mathcal{A}_1$ we have $T^2A = TAT$ and $\|T^2A\| = \|TAT\| \geq \|TATA\| = \|TA\|^2$. $\square$

Theorem 3. *Let $\mathcal{A}$ be a normed unital algebra and $U, V \in \mathcal{A}_1$ be such that $VU = I$. If $T \in \mathcal{P}_k(\mathcal{A})$, then $UTV \in \mathcal{P}_k(\mathcal{A})$ for all $k \in \mathbb{N}$.*

Proof. We have $(UTV)^{k+1} = UT^{k+1}V$. It is necessary to show that

$$\|(UTV)^{k+1}A\| = \|UT^{k+1}VA\| \geq \|UTVA\|^{k+1} \text{ for all } A \in \mathcal{A}_1.$$

If $VA = 0$, then the assertion is evident. Assume that $VA \neq 0$, then $0 < \|VA\| \leq 1$ and

$$\|UT^{k+1}VA\| \geq \|VUT^{k+1}VA\| = \|T^{k+1}VA\| = \left\|T^{k+1}\frac{VA}{\|VA\|}\right\|\|VA\|$$
$$\geq \left\|T\frac{VA}{\|VA\|}\right\|^{k+1}\|VA\| = \frac{\|TVA\|^{k+1}}{\|VA\|^k} \geq \|TVA\|^{k+1} \geq \|UTVA\|^{k+1}. \quad \square$$

Corollary 3. Let $\mathcal{A}$ be a unital C^*-algebra. If $T \in \mathcal{P}_k(\mathcal{A})$, then $UTU^* \in \mathcal{P}_k(\mathcal{A})$ for all isometries $U \in \mathcal{A}$ and $k \in \mathbb{N}$.

Corollary 3 for $k = 1$ generalizes assertion (ii) of theorem 2 from [10].

Theorem 4. *Let $\mathcal{A}$ be a normed unital algebra. If an element $T \in \mathcal{P}_1(\mathcal{A})$ is right invertible, then any right inverse element T^{-1} belongs to $\mathcal{P}_1(\mathcal{A})$.*

Proof. Consider $A \in \mathcal{A}_1$, $T^{-2} = (T^{-1})^2$. Let us show that $\|T^{-2}A\| \geq \|T^{-1}A\|^2$. If $T^{-2}A = 0$, then $T^{-1}A = T \cdot T^{-2}A = 0$ and the assertion holds. If $T^{-2}A \neq 0$, then

$$\left\|T^2\frac{T^{-2}A}{\|T^{-2}A\|}\right\| \geq \left\|T\frac{T^{-2}A}{\|T^{-2}A\|}\right\|^2,$$

i.e., $\frac{\|A\|}{\|T^{-2}A\|} = \frac{1}{\|T^{-2}A\|} \geq \frac{\|T^{-1}A\|^2}{\|T^{-2}A\|^2}$. $\square$

Corollary 4. Let $\mathcal{A}$ be a normed unital algebra over the field $\mathbb{C}$ and $T \in \mathcal{P}_1(\mathcal{A})$ be such that the spectrum $\sigma(T)$ lies on the unit circle, then $\|TX\| = \|X\|$ for all $X \in \mathcal{A}$.

Proof. Since $\sigma(T)$ lies on the unit circle, the relation $\|T\| = \|T^{-1}\| = 1$ holds by Corollary 1 and Theorem 4. For all $(0 \neq)X \in \mathcal{A}$ we have

$$\|X\| \geq \|TX\| = \|T^{-1}X\|\left\|T^2\frac{T^{-1}X}{\|T^{-1}X\|}\right\| \geq \|T^{-1}X\|\left\|T\frac{T^{-1}X}{\|T^{-1}X\|}\right\|^2 = \frac{\|X\|^2}{\|T^{-1}X\|} \geq \|X\|. \quad \square$$

Lemma 1. *Let $\mathcal{A}$ be a normed algebra. If $T \in \mathcal{P}_1(\mathcal{A})$, then*

$$\|T^3A\| \geq \|T^2A\| \cdot \|TA\| \textit{ for all } A \in \mathcal{A}_1. \tag{1}$$

Proof. Without loss of generality assume that $TA \neq 0$, then

$$\|T^3A\| = \|TA\| \cdot \left\|T^2\frac{TA}{\|TA\|}\right\| \geq \|TA\| \cdot \left\|T\frac{TA}{\|TA\|}\right\|^2 = \frac{\|T^2A\|^2}{\|TA\|} \geq \frac{\|T^2A\| \cdot \|TA\|^2}{\|TA\|} = \|T^2A\| \cdot \|TA\|.$$

$\square$

Lemma 2. *Let $\mathcal{A}$ be a normed algebra. If $T \in \mathcal{P}_1(\mathcal{A})$, then*

$$\|T^{k+1}A\|^2 \geq \|T^kA\|^2 \cdot \|T^2A\| \textit{ for all } A \in \mathcal{A}_1 \textit{ and } k \in \mathbb{N}. \tag{2_k}$$

Proof. We carry out the proof by induction. For $k = 1$ we have

$$\|T^2A\|^2 = \|T^2A\| \cdot \|T^2A\| \geq \|TA\|^2 \cdot \|T^2A\|$$

and (2_1) holds. Let (2_k) hold for k and $TA \neq 0$, then

$$\|T^{k+2}A\|^2 = \|TA\|^2 \cdot \left\|T^{k+1}\frac{TA}{\|TA\|}\right\|^2 \geq \|TA\|^2 \cdot \left\|T^k\frac{TA}{\|TA\|}\right\|^2 \cdot \left\|T^2\frac{TA}{\|TA\|}\right\|$$

$$= \|T^{k+1}A\|^2 \frac{\|T^3A\|}{\|TA\|} \geq \|T^{k+1}A\|^2 \cdot \|T^2A\|$$

via (1) and (2_k). Hence (2_{k+1}) holds. □

Theorem 5. *Let $\mathcal{A}$ be a normed algebra. If $T \in \mathcal{P}_1(\mathcal{A})$, then $T^n \in \mathcal{P}_1(\mathcal{A})$ for all $n \in \mathbb{N}$.*

Proof. Again the proof is by induction. It suffices to show that if $T, T^k \in \mathcal{P}_1(\mathcal{A})$, then $T^{k+1} \in \mathcal{P}_1(\mathcal{A})$. Assume that $A \in A_1$ and $T^2A \neq 0$, then

$$\|T^{2(k+1)}A\| = \left\| T^{2k} \frac{T^2A}{\|T^2A\|} \right\| \cdot \|T^2A\| \geq \left\| T^k \frac{T^2A}{\|T^2A\|} \right\|^2 \cdot \|T^2A\|$$
$$= \frac{\|T^{k+2}A\|^2}{\|T^2A\|} \geq \frac{\|T^{k+1}A\|^2 \cdot \|T^2A\|}{\|T^2A\|} = \|T^{k+1}A\|^2$$

via (2_{k+1}) of Lemma 2. □

Remark 1. Theorem 5 allows us to find another proof of Corollary 1. For elements $A = I$, $T \in \mathcal{P}_1(\mathcal{A})$ and for all $n \in \mathbb{N}$ we have

$$\|T^{2^n}\| = \|T^{2^n}A\| \geq \|T^{2^{n-1}}A\|^2 = \|T^{2^{n-1}}\|^2 \geq \|T^{2^{n-2}}A\|^{2^2} = \|T^{2^{n-2}}\|^{2^2} \geq \cdots$$
$$\geq \|TA\|^{2^n} = \|T\|^{2^n}.$$

The sequence $\{\|X^n\|^{1/n}\}$ converges as $n \to \infty$ for every $X \in \mathcal{A}$, and its limit equals $\inf_n \|X^n\|^{1/n}$ ([6], Chap. I, § 2, proposition 1). Hence $\lim_{n\to\infty} \|T^n\|^{1/n} = \lim_{n\to\infty} \|T^{2n}\|^{1/2^n} = \inf_n \|T^n\|^{1/n} \geq \|T\|$ and $\|T^n\|^{1/n} \geq \|T\|$, i.e., $\|T^n\| \geq \|T\|^n$ for all $n \in \mathbb{N}$ and normaloid T.

By Theorem 1 of [10] we have

Corollary 5. If $\mathcal{A} = \mathcal{B}(\mathcal{H})$, then the class $\mathcal{P}_1(\mathcal{A})$ coincides with the class of all paranormal operators on $\mathcal{H}$.

Since the product operation is jointly sequentially continuous in the strong operator topology in $\mathcal{B}(\mathcal{H})$ ([11], problem 93), Corollary 5 yields

Corollary 6. If $\mathcal{A} = \mathcal{B}(\mathcal{H})$, then the class $\mathcal{P}_1(\mathcal{A})$ is sequentially closed in the strong operator topology.

Corollary 7. If $\mathcal{A} = \mathcal{B}(\mathcal{H})$, then $\mathcal{P}_1(\mathcal{A})$ contains a non-hyponormal operator.

Proof. P. Halmos ([11], problem 164) presented an example of a hyponormal operator $T \in \mathcal{A}$ such that T^2 is non-hyponormal. We have $T \in \mathcal{P}_1(\mathcal{A})$ by item (i) of theorem 2 of [10], hence $T^2 \in \mathcal{P}_1(\mathcal{A})$ by Theorem 5. □

Remark 2. If $\mathcal{A} = \mathbb{M}_2(\mathbb{C})$, then the class $\mathcal{P}_1(\mathcal{A})$ is the set of all normal matrices from $\mathcal{A}$. For $\mathcal{A} = \mathcal{B}(\mathcal{H})$ Theorem 2 was established in [12] and [13], Theorem 4 (for invertible T) and Corollary 4 were proved in [12], and Lemmas 1, 2 and Theorem 5 were proved in [8]. Here we modify the corresponding proofs.

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Authorized translation